Partial Differential Equations

Ben Woodruff

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Preface

The following problem set goes along with Richard Haberman's *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*.

Chapter 1

The Heat Equation

We'll start this semester off by studying how heat flows. By focusing on a physical context, we can see how mathematics truly does model the world around us. Before jumping in, let's review a few concepts from calculus.

Recall that to find the mass of an object, you just need to find the mass of a small piece of the object (of width, say dx). Once you know the mass dm of a small piece of the object, you can use integrals to obtain the mass as $m = \int dm$. Here's a problem similar to something you have seen before.

Problem 1.1 Suppose you have a thin cylindrical rod lying on the x axis between $x = 0$ cm and $x = 5$ cm. The radius of the rod is a constant r cm, so the cross sectional area is $A = \pi r^2$ cm^2. The density of the rod (mass per unit volume) is known to be $\rho(x) = (x^2 + 100)$ g/m^3 (the rod is made of a blend of materials that gradually gets heavier as x gets closer to 5).

Find the mass of the rod by first finding the mass of a small piece of the rod that is dx units wide, and then integrating. As the radius of the rod is unknown, your solution should involve a constant r.

We will also need two facts about integrals that you will develop in the next problems. Recall that the fundamental theorem of calculus states that if f is a continuous function on the interval $[a, b]$, then f has an anti derivative F and $\int_a^b f(x)dx = F(b) - F(a)$. Notice that the only assumption here is that f is continuous. We need to learn how to work this theorem in reverse.

Problem 1.2 Use the fundamental theorem of calculus to rewrite the difference $g(a) - g(b)$ as an integral. Make your bounds go from a to b. What assumptions must you make about g in order for this to work.

We also need the following crucial fact.

Problem 1.3 Suppose $f(x)$ is a continuous function such that $\int_a^b f(x)dx = 0$ on every interval $[a, b]$. Show that $f(x) = 0$ for every x. Give an example of a nonzero function $g(x)$ so that $\int_0^1 g(x)dx = 0$.

We're now ready to jump into partial differential equations.

1.1 Heat in a one dimensional rod

Consider a rod of length L whose cross sectional area A is constant. For ease, we'll place the rod on the x-axis between $x = 0$ and $x = L$.

Definition 1.1. Define the thermal energy density $e(x,t)$ (or heat energy density) to be the amount of thermal energy per unit volume of the rod at position x at time t.

In general, the quantity $e(x,t)$ is unknown. However we'll find that this unknown quantity provides the theoretical foundation to our first partial differential equation. In particular, you can use the thermal energy density to find the total thermal energy in any portion of the rod at any time t, in the exact same way we use mass density to find total mass.

Problem 1.4 Suppose the thermal energy density along the entire rod at time $t=0$ is constant, say $e(x,0)=C$. The rod has constant cross sectional area A. As time moves forward, the rod uniformly looses heat with thermal energy density function $e(x,t)=\frac{C}{2^t}$. Find the total thermal energy in the entire rod at $t=0$, $t=1$, $t=2$, and then at any time t. (Feel free to solve the problem at any time t first.)

Problem 1.5 Consider a small portion of the rod on the x-axis between x and $x+\Delta x$ units. If the thermal energy density $e(x,t)$ along this small portion of the rod is constant (though it could change if t changes), and the cross sectional area A is constant, then find the total thermal energy of this small portion of the rod. Simplify any integrals.

Problem 1.6 Consider portion of the rod on the x-axis between $x=a$ and $x=b$ units. The thermal energy density $e(x,t)$ along this portion of the rod may vary, but the cross sectional area A is constant. Find the total heat energy of this portion of the rod at time t. Your answer should involve an integral and the unknown constants.

What is the change per unit time in heat energy?

As time marches forward, the heat energy inside an object can change. Heat may flow across the surface of the object. Additionally, heat could be generated (or removed) inside the object itself (a chemical reaction inside the object could easily add or subtract heat energy). This leads to the following big idea.

Observation 1.2: The Conservation of Heat Energy. The change (per unit time) in heat energy is equal to the heat energy flowing across the boundaries (per unit time) plus the heat energy generated inside (per unit time).

Along a rod, heat can flow out the left end, the right end, or along the lateral surface. If we perfectly insulate the lateral surface (or pretty close to perfect), then heat energy loss through the lateral surface can be neglected. This means we need to know how much heat is flowing out the left surface, and how much is flowing out the right.

Definition 1.3. We define the heat flux $\phi(x,t)$ to be the amount of thermal energy *per* unit time flowing to the right *per* unit surface area. If there is a heat source in an object, we let $Q(x,t)$ be the heat energy *per* unit volume generated *per* unit time.

Problem 1.7 Suppose a rod is on the x axis from $x=0$ to $x=L$. We'll consider a small portion of the rod from $x=a$ to $x=b$. Make the following assumptions: (1) the cross sectional area A is constant, (2) the thermal heat density $e(x,t)$ varies throughout the rod, (3) the lateral surface of the rod is perfectly insulated so that any heat flow occurs only across the surface at $x=a$ and $x=b$, and (4) the functions $\phi(x,t)$ and $Q(x,t)$ are as defined above.

1. What is the total heat energy (per unit time) entering the rod at $x = a$ and at $x = b$? You answer should be in terms of ϕ.
2. What is the total heat energy generated inside the rod (per unit time) between $x = a$ and $x = b$? An integral may help here.

Problem 1.8 Use the same assumptions as problem 1.7. Now use the conservation of heat energy together with problems 1.6 and 1.7 to explain why

$$\frac{d}{dt}\int_a^b e(x,t)dx = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)dx.$$

Problem 1.9 Again assume as in problem 1.7 Show that

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.$$

To do so, you'll want to use the equation in problem 1.8. Problem 1.2 will help you turn the difference ϕ into an integral. The fact that $\frac{d}{dt}\int_a^b e(x,t)dx = \int_a^b \frac{\partial}{\partial t} e(x,t)dx$ will help you get everything in terms of an integral. Finally, problem 1.3 will help you eliminate any integrals.

Once you've made it to this stage, the rest of the derivation of the heat equation requires some facts about temperature and specific heat, together with Fourier's law. Let's introduce specific heat now. Most of the time we talk about the temperature of an object, not its thermal energy. Specific heat allows us to connect the temperature of an object to it its thermal heat.

Definition 1.4.

- The specific heat c of a material is the heat energy that must be supplied to a unit mass of a substance to raise its temperature one unit.
- The function $u(x,t)$ represents the temperature of the rod on the x-axis.
- The thermal energy of an object is the energy it takes to raise the temperature of the object from a reference temperature $0°$ to the temperature $u(x,t)$.

In this class we'll assume that c depends only on the composition of the material, so that $c(x)$ is a function of x. It's useful to pay attention to the type of units of the quantities defined. Specific heat has units $\frac{\text{(heat energy)}}{\text{(mass)(degrees)}}$. You could write this in terms of Joules (energy), kg (mass), and degrees C (temperature). How much heat energy is needed to raise the temperature of an object $10°$ if that object has a mass of 7 units and specific heat c? Multiplying c by a mass and a change in temperature will result in a heat energy, so the answer is simply $c(10)(7)$ units of heat energy.

Problem 1.10 Again consider the rod as in problem 1.7. Let $c(x)$ be the specific heat, and $u(x,t)$ be the temperature. Let $\rho(x)$ be the mass density (the mass per unit volume). In problem 1.5, you showed that the total thermal energy in a small slice of a rod from x to $x + \Delta x$ equals $e(x,t)A\Delta x$. Use these results together with Definition 1.4 to explain why $e(x,t) = c(x)\rho(x)u(x,t)$. Finish by explaining why

$$c(x)\rho(x)\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.$$

To finish up, we need to connect the heat flux ϕ to the temperature u. If we can do this, then the equation above will be a partial differential equation involving the unknown function u and the independent variables x and t. This is our main goal. Fourier's Law is the key to connecting the two. Page 7 in the text has a good description of why the law is what follows. Please read it.

Observation 1.5: Fourier's Law. Fourier noticed that heat flows from hot to cold, the greater the difference in temperature, the greater the flow, and that different materials allow heat to flow at different rates. He summarized this in the equation

$$\phi(x,t) = -K_0(x)\frac{\partial u}{\partial x}(x,t).$$

The function $K_0(x)$ is a physical constant, determined from experiments, that tells us how quickly heat flows. Large values of K_0 allow heat to flow quickly.

Problem 1.11 Consider a rod on the x axis between $x = 0$ and $x = L$. Suppose the temperature at time $t = 0$ is $u(x,0) = 2x + 3$. Draw the temperature function at time $t = 0$. Based solely on intuition, will heat flow left or right? Explain. Then find $u_x(x,0)$, and explain why there is a negative sign in Fourier's law.

Problem 1.12: The Heat Equation Derive the heat equation by first showing that under the assumption of problem 1.7, together with Fourier's law, we have

$$c(x)\rho(x)\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}(x,t)\right) + Q(x,t).$$

If we assume that the rod is uniform (ρ is constant) with constant thermal properties (c and K_0 are constant), and that there are no internal heat sources, show that for some constant k we have

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad u_t = ku_{xx}.$$

What is the constant k? This last equation is call the heat equation.

Problem 1.13 (This is problem 1.2.3 on page 10.) Derive the heat equation for a rod assuming constant thermal properties with variable cross-sectional area $A(x)$ assuming no sources by considering the total thermal energy between $x = a$ and $x = b$. This basically has you review everything we did prior to now, but to not assume A is constant along the way.

Problem: 1.13.2 Suppose that f and g are continuous functions, and suppose that for every interval $[a,b]$ we know $\int_a^b f(x)dx = \int_a^b g(x)dx$. Prove that $f(x) = g(x)$.

This next problem has you change the context to how a pollutant would spread through a one dimensional region. You'll show that mathematically we can model the spread of a contaminant using the same equation as the flow of heat. There are more things modeled by the heat equation as well, this problem just shows you one additional application.

Problem 1.14 (This is problem 1.2.4b on page 11.) Derive the diffusion equation for a chemical pollutant by considering the total amount of the chemical between $x = a$ and $x = b$. You'll want to look on page 9 for the definitions of $u(x,t)$ (the concentration of the chemical per unit volume), $\phi(x,t)$ (flux of the chemical), and Fick's law of diffusion.

1.2 Initial and Boundary Conditions

The following problem is often solved in the first order ODE section of a differential equations class. It will help us introduce the material below.

Problem 1.15 A metal pan is placed into an oven. The pan was 70° F prior to being placed into the oven, which is set to 400° F (you may assume that the temperature of the oven does not change). Newton's law of cooling states that the rate of change of temperature of an object is proportional to the difference between the actual temperature of the object and the temperature of the surrounding atmosphere. So a greater temperature difference between an object and its surrounding will result in a more rapid change in temperature.

1. Write Newton's law of cooling as a differential equation. Let $T(t)$ represent the temperature (°F) of the metal pan after t minutes.
2. What are the initial conditions?
3. Solve the initial value problem above.
4. After 2 minutes, the metal pan has a temperature of 200° F. Solve for the proportionality constant to complete the solution.

Recall that when solving ODEs, we needed additional conditions to obtain solutions to the problem. These conditions came in two forms, initial values and boundary values. When solving a first order ODE, we needed one condition (as above). When solving a 2nd order ODE, we needed 2 conditions. When solving a 5th order ODE, we needed 5 conditions.

The heat equation $u_t = ku_{xx}$ involves a first partial with respect to t, and a second partial with respect to x. As such, we'll need an initial condition (when $t = 0$) and two boundary conditions (when $x = a$ and $x = b$) to find a full solution to any problem we study.

- The initial condition at time $t = 0$ is just a statement of the initial temperature inside the rod. Often it is given in the form $u(x,0) = f(x)$, for $x \in [0,L]$. The function $f(x)$ tells the initial temperature of the rod at position x.
- The boundary conditions will come in two main types.
  - We may specify the actual temperature at the ends of the rod. This would mean we state something like

$$u(0,t) = g(t) \qquad \text{and/or} \quad u(L,t) = h(t),$$

where $g(t)$ represents the temperature on the left end of the rod, and $h(t)$ represents the temperature on the right end. If the left end of the rod is in contact with a fluid that has temperature $u_B(t)$, we'll often use a boundary condition such as $u(0,t) = u_B(t)$ to model this.

– We may specify the heat flow instead of the temperature. Recall from Fourier's law that the heat flow is given by $\phi = -K_0 u_x$. If we assume we know $\phi(0,t)$ for all t, then a boundary condition would be

$$-K_0(t)u_x(0,t) = \phi(0,t).$$

A very common boundary condition is to assume that the end of the rod is perfectly insulated. In this case, there is no heat flow through that end, so $\phi = 0$, and our boundary condition becomes

$$u_x(0,t) = 0.$$

Another boundary condition would be to assume that the heat flow in the end of the rod behaves like the metal pan in problem 1.15. This occurs when the rod is placed in a fluid whose temperature is different than the rod, and there is some insulation on the end of the rod. The next problem will help you develop this.

Problem 1.16 Suppose that the left end of a rod ($x = 0$) is placed in a fluid whose temperature at time t is $u_B(t)$. Suppose that the heat flow $\phi(0,t)$ into the rod is proportional to the difference between the temperature $u(0,t)$ of the rod at and the temperature $u_B(t)$ of the fluid.

- Under these assumptions, show that

$$-K_0(0)u_x(0,t) = -H[u(0,t) - u_B(t)],$$

where K_0 comes from Fourier's law and $H > 0$ is called the heat transfer coefficient.

- If instead the right end $x = L$ is placed in the fluid, show that

$$-K_0(L)u_x(L,t) = H[u(L,t) - u_B(t)].$$

Pay attention to the signs in this problem, and be prepared to explain why they are as given.

- The larger H is, the more quickly heat can flow between the rod and the fluid. If heat is allowed to move between the rod and fluid without any resistance, then we can model this by letting $H \to \infty$. Suppose that this is the case at the left end of the rod $x = 0$. Use your first solution, together with $H \to \infty$ to show that this is the same as prescribing the temperature $u(0,t) = u_B(t)$.

The boundary conditions described above represent the typical conditions you'll see in this class and industry. You may have a prescribed temperature at one end, and a heat flow at another end. The conditions you choose to use depend entirely on the physical problem before you. If we choose to prescribe both end temperatures, then the heat equation together with required initial and boundary value conditions would be written

$$u_t = ku_{xx}, \quad u(x,0) = f(x), \quad u(0,t) = T_1(t), \quad u(L,t) = T_2(t).$$

One of our main goals this semester will be to solve problems like the one above.

Definition 1.6. If the temperatures $T_1(t)$ and $T_2(t)$ are constant, then we would expect the temperature at each point in the rod to eventually stabilize. We define a steady-state or equilibrium solution to be a solution that does not depend on time.

To find a steady-state solution does not require PDEs, as we can remove t from the problem. The solution involves solving an ODE. The following problems will show you how to do this.

Problem 1.17 Suppose the ends of a rod are placed in fluids so that the boundary conditions are steady $u(0,t) = T_1$ and $u(L,t) = T_2$. Find a steady-state solution to the heat equation $u_t = ku_{xx}$. Show that your solution does not depend on the initial condition $u(x,0) = f(x)$.

Problem 1.18 Suppose both ends of a rod are perfectly insulated, so the boundary conditions are $u_x(0,t) = 0$ and $u_x(L,t) = 0$. Show that a steady-state solution to the heat equation $u_t = ku_{xx}$ is $u(x) = C$ (the temperature is constant). Then use the initial temperature $u(x,0) = f(x)$ to find this constant. Since the rod is perfectly insulated on all ends, how is the total energy in the rod at the beginning related to the total energy in the rod at the end?

Problem 1.19 Suppose the right end of a rod is placed in a fluid, and the left end is perfectly insulated, so that the boundary conditions are $u(0,t) = T_1$ and $u_x(L,t) = 0$. Find a steady-state solution to the heat equation $u_t = ku_{xx}$. Does your solution depend on the initial condition $u(x,0) = f(x)$?

The following problem will help remind you how to work with problems if there is heat generated inside a rod (or chemical pollutant produced inside a 1 dimensional region).

Problem 1.20 (Problem 1.2.5 from the text.) Derive an equation for the concentration $u(x,t)$ of a chemical pollutant if the chemical is produced due to chemical reaction at the rate of $\alpha(\beta - u)u$ per unit volume. This problem will will basically cause you to review the first section, but remember that $Q \neq 0$ here.

Problem 1.21 (Problem 1.4.2 from the text.) Consider the equilibrium temperature distribution for a uniform one-dimensional rod with sources $Q/K_0 = x$ of thermal energy, subject to the boundary conditions $u(0) = 0$ and $u(L) = 0$.

(a) Determine the heat energy generated per unit time inside the entire rod.

(b) Determine the heat energy flowing out of the rod per unit time at $x = 0$ and at $x = L$.

(c) What relationship should exist between the answers in parts (a) and (b)?

Problem 1.22 (Problem 1.4.12 from the text.) Suppose the concentration $u(x,t)$ of a chemical satisfies Fick's law, and the initial concentration is given $u(x,0) = f(x)$. Consider a region $0 < x < L$ in which the flow is specified at both ends by $-k\frac{\partial u}{\partial x}(0,t) = \alpha$ and $-k\frac{\partial u}{\partial x}(L,t) = \beta$. Assume α and β are constants.

(a) Express the conservation law for the entire region.

(b) Determine the total amount of chemical in the region as a function of time (using the initial conditions).

(c) Under what conditions is there an equilibrium chemical concentration and what is it?

1.3 The heat equation in 2 or 3 dimensions

We are now prepared to develop the heat equation in 2D and 3D. The solutions are essentially identical, and parallel what we did in 1D. We need a brief review of some ideas from multivariate calculus. This is perhaps the only other class for many of you where you will use these ideas as an undergraduate. If you forgot some of them, I understand completely. Well review them with some problems.

Problem 1.23 Consider the function $f(x, y) = 9x^2y^2$.

1. Construct a 3D graph of the function f.
2. Construct a contour plot of f. In other words, construct a graph in the plane of several level curves of f.
3. Compute the gradient f. On your plot contour plot, pick several points and draw the gradient at those points.
4. In which direction does the gradient point? How is the gradient related to level curves?

Problem 1.24 Let $\vec{F}(x, y, z) = \langle x, y, z\rangle$, a radial vector field. Determine the flux of $\vec{F}$ outward across the surface of the sphere $x^2 + y^2 + z^2 = 9$. Recall that the formula for flux is the surface integral

$$\int\int_S \vec{F}\cdot\vec{n}d\sigma \quad \text{or} \quad \int\int_S \vec{F}\vec{n}dS$$

where $\vec{n}$ is a unit normal vector to the surface S. The choice of notation $d\sigma$ or dS depends on the author.

The divergence theorem is often discussed in the last few days of multivariate calculus. The physics majors in our class have mostly likely used it a few times since, but the math majors may have not. The divergence theorem allows us to convert a flux integral into a volume integral. It states that for closed surface S whose interior is the 3D solid D, if $\vec{F}$ is continuously differentiable then we have

$$\int\int_S \vec{F}\cdot\vec{n}dS = \int\int\int_D \text{div}(\vec{F})dV,$$

where the divergence of $\vec{F} = \langle M, N, P\rangle$ is

$$\text{div}(\vec{F}) = \nabla\cdot\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = M_x + N_y + P_z.$$

The divergence represents an outward flux density.

Problem 1.25 Repeat problem 1.24, but this time use the divergence theorem to perform your computation.

We are now ready to develop the heat equation in 3D. Recall the law of conservation of heat (Observation 1.2) which states

> The change (per unit time) in heat energy is equal to the heat energy flowing across the boundaries (per unit time) plus the heat energy generated inside (per unit time).

We now develop these three quantities in 3D.

Definition 1.7. Consider a solid object in space occupying the region R. Let S be surface of the object.

- Let $e(x, y, z, t) = c(x, y, z)\rho(x, y, z)u(x, y, z, t)$ be the thermal energy density (energy per unit volume) of an object at (x, y, z) at time t.
- Let $\vec{\phi}(x, y, z, t)$ be the heat flux vector defined on the surface S. The magnitude of ϕ is the amount of heat energy flowing per unit time per unit surface area. The direction of the heat flux vector is the direction of heat flow.
- Let $Q(x, y, z, t)$ be the rate of heat energy generated per unit time per unit volume.

Problem 1.26 Show that the law of conservation of heat energy is mathematically modeled by the equation

$$\frac{d}{dt}\int\int\int_R c\rho u dV = -\int\int_S \vec{\phi}\cdot\vec{n}dS + \int\int\int_R QdV.$$

Compare this with problem 1.8.

Problem 1.27 Show that the equation above can be reduced to

$$c\rho\frac{\partial u}{\partial t} = -\nabla\cdot\vec{\phi} + Q.$$

Fouriers law holds in high dimensions. Heat will still flows from greatest heat to lowest heat, and is proportional to the slope of the temperature in that direction.

Problem 1.28 Explain why Fouriers law can be written in the form

$$\vec{\phi} = -K_0\nabla u.$$

Then use Fouriers law to show that the heat equation can be written as

$$c\rho\frac{\partial u}{\partial t} = \nabla\cdot(K_0\nabla u) + Q.$$

If we assume K_0 is constant and there are no heat sources, show that the heat equation can be written in the form

$$\frac{\partial u}{\partial t} = k\nabla^2 u,$$

where the symbol ∇^2 means $\nabla^2 u = \nabla\cdot\nabla u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = u_{xx} + u_{yy} + u_{zz}$.

Definition 1.8. The expression $\nabla^2 u = \nabla \cdot \nabla u$ is called the Laplacian of u.

This next problem will cause you to revisit the derivation of the heat equation, but from the context of a pollutant. Think of the pollutant as a noxious gas that is spreading throughout a region.

Problem 1.29 (Problem 1.5.1 in text.) Let $c(x, y, z, t)$ denote the concentration of a pollutant (the amount per unit volume).

(a) What is an expression for the total amount of pollutant in some region R.

(b) Suppose the heat flow $\vec{J}$ of the pollutant is proportional to the gradient of the concentration. (Is this reasonable?) Express conservation of the pollutant.

(c) Derive the partial differential equation governing the diffusion of the pollutant.

The heat equation as derived above is best suited for regions that are rectangular. If the region is spherical, we would need a way to write the Laplacian in terms of spherical coordinates. If the region is in 2D, or cylindrical, then we would need a way to write the Laplacian in polar/cylindrical coordinates. The follow problems will help you develop the formula for the Laplacian in polar coordinates. The formulas for cylindrical and spherical coordinates are on page 28 in the text.

Problem 1.30 The equations for polar coordinates are $x = r\cos\theta$ and $y = r\sin\theta$. These equations implicitly define r and θ as functions of x and y. By implicitly computing the partial derivatives of each equation above with respect to x and y, obtain 4 equations involving the partial derivatives $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}$. Solve this system of 4 equations to show that

$$\frac{\partial r}{\partial x} = \cos\theta, \frac{\partial r}{\partial y} = \sin\theta, \frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r}, \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}.$$

When working the plane, the most common set of basis vectors is $\hat{\mathbf{i}} = (1, 0)$ and $\hat{\mathbf{j}} = (0, 1)$. These vectors are sometimes called $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ as they represent motion in the x and y directions. If you are working with a problem that has radial symmetry, then the directions of interest are the outward r direction, and rotational θ direction. The corresponding vectors are

$$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \hat{\theta} = \frac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{\sqrt{x^2 + y^2}}.$$

Notice that the dot product of these two vectors is 0, so the vectors are orthogonal.

Problem 1.31 Show that $\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$ and $\hat{\theta} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}$. Then use the chain rule, together with the previous problem, to show that

$$\nabla u = \frac{\partial u}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\theta}.$$

The previous problem shows that in polar coordinates, we have $\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial}{\partial \theta}$. For a vector field in polar form, such as $\vec{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\theta}$, we compute

$$\begin{aligned}\nabla \cdot \vec{A} &= \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial}{\partial \theta}\right) \cdot \left(A_r\hat{\mathbf{r}} + A_\theta\hat{\theta}\right) \\ &= \hat{\mathbf{r}} \cdot \frac{\partial}{\partial r}\left(A_r\hat{\mathbf{r}}\right) + \frac{1}{r}\hat{\theta} \cdot \frac{\partial}{\partial \theta}\left(A_r\hat{\mathbf{r}}\right) + \hat{\mathbf{r}} \cdot \frac{\partial}{\partial r}\left(A_\theta\hat{\theta}\right) + \frac{1}{r}\hat{\theta} \cdot \frac{\partial}{\partial \theta}\left(A_\theta\hat{\theta}\right).\end{aligned}$$

Problem 1.32 Explain (either geometrically or computationally) why

$$\frac{\partial \hat{\mathbf{r}}}{\partial r} = \vec{0},\ \frac{\partial \hat{\theta}}{\partial r} = \vec{0},\ \frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \hat{\theta},\ \text{and}\ \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{\mathbf{r}}.$$

Then use this information together with the fact that $\hat{\theta}$ and $\hat{\mathbf{r}}$ are an orthogonal set of unit vectors to explain why

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta).$$

[Hint: it might be easiest to first show why $\nabla\cdot\vec{A} = \frac{1}{r}\left(A_r + r\frac{\partial}{\partial r}(A_r)\right) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta)$.]

Using problem 1.32, we can replace $\vec{A}$ with $\nabla u = \frac{\partial u}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\theta}$ (from problem 1.31). This shows the Laplacian in polar coordinates is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

We'll finish this chapter with 2 final problems. These problems will help you learn to work with the heat equation in 3D.

Problem 1.33 (Problem 1.5.12 from the text). I'll call on multiple people in class to put up different parts.

Assume that the temperature is spherically symmetric, $u = u(r, t)$, where r is the distance from a fixed point ($r^2 = x^2 + y^2 + z^2$). Consider the heat flow (without sources) between any two concentric spheres of radii a and b.

(a) Show that the total heat energy is $4\pi \int_a^b c\rho u r^2 dr$.

(b) Show that the flow of heat energy per unit time out of the spherical shell at $r = b$ is $-4\pi b^2 K_0 \frac{\partial u}{\partial r}\big|_{r=b}$. A similar result holds at $r = a$.

(c) Use parts (a) and (b) to derive the spherically symmetric heat equation

$$\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right).$$

Here's some hints, if you want them. For part (a), just set up the triple integral and compute two of the integrals. Don't forget the Jacobian in spherical coordinates. For (b), you'll want to use Fourier's law to get this in terms of the gradient of u. Note that $\nabla u = \frac{\partial u}{\partial r}\hat{\mathbf{r}}$ since the temperature is spherically symmetric. For (c), you'll want to repeat what we have done with the laws of conservation of heat energy, and then use the integral cancellation laws to remove some integrals.

Problem 1.34 (Problem 1.5.13 from the text.)

Determine the steady-state temperature distribution between two concentric spheres with radii 1 and 4 units, respectively, if the temperature of the outer sphere is maintained at 80° and the inner sphere at 0°. [Hint: Use part (c) from the previous problem. You should be able to reduce this to a second order ODE (not time dependent). If you let $z = \frac{du}{dx}$, you can reduce the problem to solving 2 first order ODEs, both of which are separable.]

Chapter 2

Separation of Variables

In this chapter, we will solve the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ subject to various boundary conditions. In order to do so, we will need to first introduce the concepts of linearity and the principle of superposition. We'll develop our solution to the heat equation by finding multiple solutions, and then using the principle of superposition to add the solutions together.

2.1 Linearity

An operator is a function whose domain is functions. The gradient operator ∇ is an operator on differentiable functions, as once you know u, the operator $\nabla(u)$ gives you the vector (u_x, u_y, u_z).

Definition 2.1. A linear operator L is an operator so that if c_1 and c_2 are scalars and u_1 and u_2 are functions to be operated on, then we have

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2).$$

Problem 2.1 Use the definition of a linear operator to show that the heat operator, given by

$$L(u) = \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2},$$

is a linear operator. You should assume that any function placed inside the operator L is a function of x and t.

Definition 2.2. If L is a linear operator, then a linear equation for u is an equation of the form $L(u) = f$, where f is a known function. If the linear operator involves partial derivatives, then the equation $L(u) = f$ is called a linear partial differential equation. In the special case $f = 0$, we say that the equation $L(u) = 0$ is a linear homogeneous equation.

Problem 2.2: Principle of Superposition Suppose that u_1 and u_2 both satisfy the linear equation $L(u) = 0$. Show that any linear combination of u_1 and u_2, written $c_1u_1 + c_2u_2$ where c_1 and c_2 are scalars, also satisfies the linear equation. This is called the principle of superposition.

Problem 2.3 (Exercise 2.2.4 from the text.) In this exercise we derive superposition principles for non homogeneous problems.

(a) Consider the linear equation $L(u) = f$. If u_p is a particular solution, so $L(u_p) = f$, and if u_1 and u_2 are homogeneous solutions, so $L(u_i) = 0$, show that $u = u_p + c_1 u_1 + c_2 u_2$ is another particular solution.

(b) Suppose u_{p1} is a particular solution of $L(u) = f_1$, and suppose u_{p2} is a particular solution of $L(u) = f_2$. What is a particular solution of $L(u) = f_1 + f_2$? Make sure you justify your answer.

2.2 Separation of Variables

We now focus on the solution to the heat equation in one dimension, namely $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$. Recall that to solve the heat equation fully, we need an initial condition $u(x, 0) = f(x)$ and then some kind of boundary conditions at $x = 0$ and $x = L$. For the time being, we'll use the boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$, so the ends of the rod are kept at 0 degrees. This gives us linear homogeneous boundary conditions. We'll change these boundary conditions later.

The following problem uses a technique invented by Fourier called separation of variables. The idea is to assume that the function $u(x, t)$ is just the product of two functions, namely $u(x, t) = \phi(x)G(t)$. This effectively splits the solution up as the product of two disjoint pieces that we can then find independently. Some people write $u(x, t) = X(x)T(t)$ to simplify the book keeping. I'll use the notation from the book as it turns out to be helpful later on when we study Sturm-Liouville theory. The next problem shows you the value of separation of variables.

Problem 2.4 Use separation of variables (so suppose that $u(x, t)$ is the product of two functions, namely $u(x, t) = \phi(x)G(t)$) to show that you can reduce the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ to the two single variable ODEs

$$\frac{d\phi}{dx^2} = \lambda\phi \quad \text{and} \quad \frac{dG}{dt} = \lambda kG,$$

where the unknown constant λ is the same in both equations. [Hint: try separating the heat equation so that time dependence is on one side, and spacial dependence is on another. Oh, and I left off the negative sign that you see in the book on purpose.]

Problem 2.5 Find a general solution to the time dependent ODE

$$\frac{dG}{dt} = \lambda kG.$$

Once you obtain your solution, discuss what happens to your solution as $t \to \infty$, based upon different value of λ. If this solution is to model a real world situation, what must be true about the unknown constant λ?

Remark 2.3. The problem above suggest that if we use the ODEs $\frac{d\phi}{dx^2} = \lambda\phi$ and $\frac{dG}{dt} = \lambda kG$, then the unknown constant λ is a negative number. It's often easier to assume that constants are positive. To simplify our work later on, we'll make a change to our notation and replace each λ with $-\lambda$. This gives us the two ODEs

$$\frac{d\phi}{dx^2} = -\lambda\phi \quad \text{and} \quad \frac{dG}{dt} = -\lambda kG,$$

We'll show shortly that using these new ODEs, the constant λ must now be positive.

The following is a review problem from your ODE class.

Problem 2.6 Find a general solution to the following ODEs. Assume that y is a function of x.

- $y'' - 9y = 0$.
- $y'' = 0$.
- $y'' + 9y = 0$.

Problem 2.7 Find a general solution to the spacial dependent ODE from remark 2.3.

$$\frac{d^2\phi}{dx^2} = -\lambda\phi.$$

There are three cases to consider, namely $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Make sure you show what the solution is in all three cases.

It's now time to use our boundary conditions to simplify the general solutions obtained above. Suppose that $u(x,t) = \phi(x)G(t)$ satisfies the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$. Also suppose that the boundary conditions are $u(0,t) = 0$ and $u(L,t) = 0$. At $x = 0$, we have $0 = \phi(0)G(t)$. If $G(t)$ is identically zero, then $u(x,t) = \phi(x)\cdot 0 = 0$ is the trivial solution, and not very useful. So we'll assume $G(t)$ is not identically zero which means that $0 = phi(0)$ after dividing by $G(t)$. Similarly we have $0 = \phi(L)$.

Problem 2.8 In problem 2.7 we found the general solution to the x dependent ODE given by $\dfrac{d^2\phi}{dx^2} = -\lambda\phi$. The solution split into 3 cases. Apply the boundary conditions $\phi(0) = 0$ and $\phi(L) = 0$ to this solution.

1. Show that if $\lambda = 0$ or $\lambda < 0$, then $\phi(x) = 0$ for all x. This is the trivial solution and is not useful.
2. If $\lambda > 0$, show that the nontrivial solutions to the ODE are

   $$\phi_n(x) = c_2 \sin(\sqrt{\lambda_n}\ x)$$

   where we have $\lambda_n = \left(\dfrac{n\pi}{L}\right)^2$ for $n = 1, 2, 3, 4 \ldots$.

Definition 2.4. In the solution of the boundary value problem (BVP)

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad \phi(0) = 0, \quad \phi(L) = 0,$$

we call the scalars $\lambda_n = \left(\dfrac{n\pi}{L}\right)^2$ the eigenvalues of the BVP. For each λ_n, we call the corresponding nonzero function $\phi_n(x) = c_2 \sin(\sqrt{\lambda_n}\ x)$ the eigenfunction corresponding to λ_n.

Problem 2.9 In problem 2.5, we showed that $G(t) = Ae^{-k\lambda t}$ (where we replaced λ with $-\lambda$ because of remark 2.3). In problem 2.8, we found that λ must be specific values, and also obtained multiple solutions for $\phi(x)$.

Explain why for each $n = 1, 2, 3, \ldots$, the function

$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}$$

is a solution to the heat equation $u_t = ku_{xx}$ with boundary conditions $u(0,t) = 0 = u(L,t)$. The numbers B_n represent arbitrary constants, and could be different for each n. Then explain why

$$\sum_{n=1}^{M} u_n(x,t) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}$$

is also a solution for any positive integer M. If we let $t = 0$, how does this change the solution?

Since the solution above is valid for any M, we will consider the infinite series solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}.$$

We have not shown this series converges, nor will we till next chapter. Now, notice that in our solution above, we have not yet included the initial condition $u(x,0) = f(x)$. If we blindly insert this initial condition into our series solution, we obtain the equation

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 0} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

Definition 2.5. Suppose f is a function defined on $[0, L]$. Then the Fourier sine series for f is defined to be the infinite series

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

The series above will converge to $f(x)$ for each x with reasonable assumptions on f (like f is continuous). We'll determine the constants B_n after we introduce orthogonality.

Definition 2.6: Orthogonal Functions. Suppose f and g are continuous functions. We say that f and g are orthogonal over the interval $[a, b]$ if

$$\int_a^b f(x)g(x)dx = 0.$$

See the first paragraph on page 58 for a good explanation of how this definition is similar to how we talk about orthogonal vectors.

Theorem 2.7 (Orthogonality of trig functions.)**.** *The following functions are orthogonal over the intervals given. Assume that n and m are non-negative integers with $n \neq m$.*

1. $\sin\left(\frac{n\pi x}{L}\right)$ *and* $\sin\left(\frac{m\pi x}{L}\right)$ *over* $[0, L]$.
2. $\cos\left(\frac{n\pi x}{L}\right)$ *and* $\cos\left(\frac{m\pi x}{L}\right)$ *over* $[0, L]$.
3. $\sin\left(\frac{n\pi x}{L}\right)$ *and* $\sin\left(\frac{m\pi x}{L}\right)$ *over* $[-L, L]$.
4. $\cos\left(\frac{n\pi x}{L}\right)$ *and* $\cos\left(\frac{m\pi x}{L}\right)$ *over* $[-L, L]$.
5. $\sin\left(\frac{n\pi x}{L}\right)$ *and* $\cos\left(\frac{m\pi x}{L}\right)$ *over* $[-L, L]$ *(including if $n = m$).*

If $n = m$ above, then on $[0, L]$ we have

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)^2 dx = \begin{cases} \frac{L}{2} & n \neq 0 \\ 0 & n = 0 \end{cases} \quad \textit{and} \quad \int_0^L \cos\left(\frac{n\pi x}{L}\right)^2 dx = \begin{cases} \frac{L}{2} & n \neq 0 \\ L & n = 0 \end{cases}.$$

On the interval $[-L, L]$ we have

$$\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right)^2 dx = \begin{cases} L & n \neq 0 \\ 0 & n = 0 \end{cases} \quad \textit{and} \quad \int_{-L}^L \cos\left(\frac{n\pi x}{L}\right)^2 dx = \begin{cases} L & n \neq 0 \\ 2L & n = 0 \end{cases}.$$

To prove the entire theorem above, we would just need to compute lots and lots of integrals. We will use the theorem above whenever, but only prove a small part in class.

Problem 2.10 Prove that for non-negative integers $n \neq m$, the functions $\sin\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$ are orthogonal over $[0, L]$. Then show that if $n = m$, we have

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \begin{cases} \frac{L}{2} & n \neq 0 \\ 0 & n = 0 \end{cases}.$$

Problem 2.11 Suppose that the set of functions $g_n(x)$ for $n = 1, 2, 3, \ldots$ are orthogonal to each other on the interval $[a, b]$ (so $\int_a^b g_n(x) g_m(x) dx$ if $n \neq m$). Suppose that

$$f(x) = \sum_{n=1}^{\infty} B_n g_n(x).$$

- Let m be a specific integer. Multiply both sides of the equation above by $g_m(x)$ and integrate over the interval $[a, b]$.
- Swap the sum and the integral, and then simplify the integrals inside.
- Solve for the unknown constant B_m.

[Hint: you should get $B_m = \dfrac{1}{\int_a^b g_m^2(x)dx} \displaystyle\int_a^b f(x) g_m(x) dx$.]

Problem 2.12 Suppose f is a function defined on $[0, L]$, and that

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

Determine the constants B_n. Compute any integrals that can be computed exactly.

Remark 2.8. You have now solved the heat equation with homogeneous boundary values $u(0, t) = 0$ and $u(L, 0) = 0$, and initial value $u(x, 0) = f(x)$. The solution is

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t},$$

where the coefficients B_n are determined from the Fourier sine series of $f(x)$. The results are summarized in the first column on the front end page of your text. The remaining problems in this chapter have you solve the heat equation with various other boundary conditions.

Let's start by first practicing the separation of variables technique used in Problem 2.4

Problem 2.13 (Exercise 2.3.1(ac) in the book.) For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

(a) $\dfrac{\partial u}{\partial t} = \dfrac{k}{r}\dfrac{\partial}{\partial r}\left(\dfrac{\partial u}{\partial r}\right)$

(c) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Problem 2.14 (Exercise 2.3.2(d) on page 55 in the book.) Consider the differential equation $\dfrac{d^2\phi}{dx^2}+\lambda\phi = 0$. Determine the eigenvalues and corresponding eigenfunctions if ϕ satisfies the boundary conditions

$$\phi(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0.$$

Analyze all three cases $(\lambda > 0, \lambda = 0, \lambda < 0)$. You may assume that the eigenvalues are real (Sturm-Liouville will give us this for free eventually).

The next two problems walk you through solving the heat equation with insulated ends. The solution to this problem shows up as the middle column on the front end cover of your book.

Problem 2.15 (Exercise 2.3.7(ab) on page 56 in the book.) Consider the boundary value problem given by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

(a) Give a one-sentence physical description of this problem.

(b) Solve by the method of separation of variables. First show that there are no separated solutions which exponentially grow in time. You should obtain

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda n k t} \cos\frac{n\pi x}{L}.$$

What is λ_n?

Problem 2.16 (Exercise 2.3.7(cde) on page 57 in the book.) Continue from the above.

(c) Show that the initial condition, $u(x,0) = f(x)$, is satisfied if

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}.$$

(d) Using the theorem about the orthogonality of trig functions, solve for A_0 and A_n $(n \geq 1)$.

(e) What happens to the temperature distribution as $t \to \infty$? Show that it approaches the steady-state temperature distribution.

The next two problems have you use the general solution above to give specific solutions for specified initial temperature distribution $f(x)$. Actually compute any integrals. Be prepared to show how to compute the integrals.

Problem 2.17 (Exercise 2.4.1(a) on page 69 in the book.) Solve the heat equation $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ (for $t > 0$) subject to the boundary conditions $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$, together with the initial condition $u(x,0) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$.

Problem 2.18 (Exercise 2.4.1(d) on page 69 in the book.) Solve the heat equation $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ (for $t > 0$) subject to the boundary conditions $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$, together with the initial condition $u(x,0) = -3\cos\frac{8\pi x}{L}$.

We now turn our attention to solving the heat equation on a thin circular ring. We'll assume that the rod is insulated perfectly on the lateral sides, the material has constant thermal properties and uniform density, and that the length of the rod is $2L$. To mathematically model this problem, let x represent the angle from the x-axis up to the point on the rod. We'll consider $x \in [-L, L]$. The function $u(x,t)$ represents the temperature of the rod if you rotate x radians up from the x-axis. We'll assume the temperature is a continuous function, which means that $u(-L,t) = u(L,t)$. We'll also assume that the heat flux is continuous, which means that $\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$. The initial temperature along the ring is $u(x,0)$ for $x \in [-L, L]$. Mathematically, we'll summarize this as the boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad u(-L,t) = u(L,t), \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t), \quad u(x,0) = f(x).$$

Problem 2.19 Consider the boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad u(-L,t) = u(L,t), \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t), \quad u(x,0) = f(x).$$

Use the method of separation of variables $u(x,t) = \phi(x)G(t)$ to show that the eigenfunctions are $\phi_n(x) = c_1\cos\frac{n\pi x}{L} + c_2\sin\frac{n\pi x}{L}$ for $n = 0, 1, 2, 3, \ldots$.

Problem 2.20 Using the same set up as problem 2.19, show that the general solution is

$$u(x,t) = \sum_{n=0}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)e^{-(n\pi/L)^2 kt},$$

if you neglect the initial condition $u(x,0) = f(x)$. Then use the orthogonality of cosines and sines to give formulas for a_n and b_n when you apply the initial condition $u(x,0) = f(x)$. [The case $n = 0$ is different than all the others.]

Problem 2.21 Complete problem 2.4.6.

This is all we'll do before the first exam.

Chapter 3

Fourier Series

3.1 Graphs and Computations

Definition 3.1: Piecewise Smooth. We say that function $f(x)$ is smooth on an interval $[a, b]$ if the function and its derivative are both bounded and continuous on (a, b). We say that a function $f(x)$ is piecewise smooth on an interval (a, b) if the interval can be partitioned into a finite number of pieces and on each piece the function $f(x)$ is smooth (so $f'(x)$ may not exist at finitely many points).

Problem 3.1 Complete each of the following. See page 90.

1. Give an example of a function that is not continuous on $[0, 1]$, but is piecewise smooth.
2. Give an example of a function that is continuous on $[0, 1]$ and is piecewise smooth, but is not differentiable on $[0, 1]$.
3. Give an example of a function that is continuous on $[0, 1]$, but is not piecewise smooth.

Definition 3.2. Let $f(x)$ be a function defined on $[-L, L]$ such that the Fourier coefficients

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

exist. We define the Fourier series of f over the interval $[-L, L]$ to be the formal infinite series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Regardless of whether or not the series converges, we will write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Remark 3.3. In the definition above, we do not require the Fourier series to actually converge to f. The following theorem, which we will give without proof, provides the needed conditions for the series to converge. We will use this theorem (which requires some real analysis) to prove other facts about Fourier series throughout this chapter.

Theorem 3.4 (Fourier's Theorem)**.** *Suppose $f(x)$ is piecewise smooth on the interval $-L \leq x \leq L$. Then the Fourier series of $f(x)$ converges to the periodic extension of f at the points where f is continuous. If the periodic extension of f is not continuous at a point x, then the Fourier series converges to the average of the left and right limits at x, namely* $\dfrac{f(x+)+(f(x-)}{2}$.

Problem 3.2 For each function below, consider the Fourier series of f over the interval $-L \leq x \leq L$. Sketch a graph of the Fourier series over the interval $-3L \leq x \leq 3L$. At points of discontinuity, place an $\times$ on the graph at the point to which the Fourier series converges.

1. $f(x) = 1 + x$
2. $f(x) = \begin{cases} x & x < 0 \\ x^2 & x > 0 \end{cases}$
3. $f(x) = \begin{cases} x & x < L/2 \\ 0 & x > L/2 \end{cases}$

Problem 3.3 For the function $f(x) = x$ over $[-L, L]$, determine the Fourier coefficients and sketch a graph of the Fourier series (your graph should use the bounds $[-3L, 3L]$). Be prepared to show the integration steps.

Problem 3.4 For the function $f(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$ over $[-L, L]$, determine the Fourier coefficients and sketch a graph of the Fourier series (your graph should use the bounds $[-3L, 3L]$).

Definition 3.5. Let $f(x)$ be a function defined on $[0, L]$. We define the Fourier sine series and Fourier cosine series of f (on $[-L, L]$) to be the series

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right),$$

respectively. The coefficients above are given by the formulas

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{and} \quad \begin{aligned} A_0 &= \frac{1}{L}\int_0^L f(x)dx, \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \end{aligned}$$

Definition 3.6. We say that a function $f(x)$ is odd if $f(-x) = -f(x)$ for all x in the domain of f. We say that a function is even if $f(-x) = f(x)$ for all x in the domain of f.

Problem 3.5 Suppose that f is an odd function defined on $[-L, L]$. Prove that the Fourier series of f and the Fourier sine series of f are the same (start with the Fourier series definition, and then show that $a_n = 0$ and $b_n = B_n$). Then suppose f is an even function and prove that the Fourier series of f is equal to the Fourier cosine series.

Be prepared to explain why the products of even and odd functions are either even and/or odd.

Remark 3.7. Let f be defined on $[0, L]$. The problem above shows that the Fourier sine series of f is the Fourier series of the odd extension of f (extend f as an odd function on $[-L, L]$). Similarly, the Fourier cosine series of f is the Fourier series of the even extension of f (extend f as an even function on $[-L, L]$). We can then use Fourier's theorem to graph both the Fourier sine series and Fourier cosine series of a function.

Problem 3.6 Consider the function $f(x) = \begin{cases} x & x < 0 \\ x^2 & x > 0 \end{cases}$ for $x \in [-L, L]$. Construct a graph of the Fourier series of f, the Fourier sine series of f, and the Fourier cosine series of f. Graph the functions over the interval $[-3L, 3L]$.

Problem 3.7 Consider the function $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$ for $x \in [-L, L]$.

1. Construct a graph of the Fourier series of f, the Fourier sine series of f, and the Fourier cosine series of f. Graph the functions over the interval $[-3L, 3L]$.
2. Compute the coefficients of the Fourier series of f.
3. Compute the coefficients of the Fourier sine series of f, and the Fourier cosine series of f.

Problem 3.8 Consider the function $f(x) = x$ for $x \in [-L, L]$. In problem 3.3, we showed the Fourier coefficients are $a_n = 0$ and $b_n = \frac{2L}{n\pi}(-1)^{n+1}$.

1. Construct a graph of the Fourier series of f, the Fourier sine series of f, and the Fourier cosine series of f. Graph the functions over the interval $[-3L, 3L]$.
2. What are the coefficients of the Fourier sine series and Fourier cosine series of f? Only perform a computation if necessary.

Problem 3.9 Consider the function $f(x) = \cos\left(\frac{\pi x}{L}\right)$ on $[0, L]$. Sketch the graph of the Fourier sine series of f for $-3L \leq x \leq 3L$. Then find the coefficients of the Fourier sine series of f. What does your result say about the orthogonality of $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ over the interval $[0, L]$? Without doing any computations at all, what are the coefficients of the Fourier cosine series of f?

Problem 3.10 Suppose f is a continuous function defined on $[-L, L]$.

1. Under what conditions does the Fourier series converge to $f(x)$ for every x in $[-L, L]$?
2. Under what conditions does the Fourier sine series converge to $f(x)$ for every x in $[-L, L]$?
3. Under what conditions does the Fourier cosine series converge to $f(x)$ for every x in $[-L, L]$?

Note that this says *every* $x \in [-L, L]$, which includes the endpoints.

3.2 Term-by-Term Differentiation

In our solution to the heat equation in chapter 2 (see the comments after problem 2.9 on page 15), we blindly inserted an infinity symbol on the top of our sum, and said that adding together infinitely many solutions to the heat equation would still produce a solution to the heat equation. In many physics an engineering environments, jumps are made from finite sums (using the principle of superposition) to an infinite sum, without ever checking if such a change is valid. In this section, we'll show that replacing a finite sum with an infinite one does not always produce a solution. We'll also provide conditions where such a replacement is valid. My hope, after you finish this section, is that you remember to always ask yourself "Does this infinity symbol cause problems with my ability to interchange the order of operations."

Problem 3.11 Consider the heat equation $u_t = ku_{xx}$. We would like to know if the infinite sum

$$u(x,t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 kt}$$

is a solution to the heat equation. In this problem, we'll provide a partial proof that $u(x,t)$ is a solution.

1. By interchanging the derivative and the sum, compute
$$\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 kt}\right).$$
2. By interchanging the derivative and the sum, compute
$$\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 kt}\right).$$
3. Using your results above, show that $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial t^2}$.

The problem above would be a proof that the infinite series is a solution to the heat equation, if it were true that you can always interchange a derivative and an infinite sum. Beware. Any time you want to swap the order of two operations, and one has an infinity in it, you may not be able to without completely changing the meaning. If you do interchange the order, and then proceed without verifying you can, you may be surprised later on when your solution doesn't make sense (your solution may not even converge anywhere). The following problem illustrates this issue.

Problem 3.12 Consider the function $f(x) = x$ on $[0, L]$. We computed the Fourier sine series in problem 3.8. The Fourier sine series is

$$x \sim \sum_{n=1}^{\infty}\frac{2L}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L}.$$

1. We know the derivative of x is 1. What is the cosine series of $f'(x) = 1$?
2. Compute the derivative of the Fourier sine series (incorrectly) by interchanging the derivative and infinite sum (so compute the derivative of each term). Compare with (a).

3. The nth term test states that if the nth term of a series does not approach 0, then the series does not converge. Does the nth term of the series in (b) approach 0? Why or why not?

Theorem 3.8. *Let f be a piecewise smooth function defined on $[-L, L]$ with f' piecewise smooth (so that f' has a Fourier series). If the Fourier series of f is continuous, then the Fourier series can be differentiated term-by-term.*

We'll prove this theorem shortly. The next problem provides some quick conditions to check prior to using term-by-term differentiation. Problem 3.10 is quite similar.

Problem 3.13 Suppose f is a continuous function and that f' is piecewise smooth. We know that we can perform term-by-term differentiation if the Fourier series is continuous. Give relevant boundary conditions to answer each of the following.

1. Under what conditions is the Fourier series of f continuous?
2. Under what conditions is the Fourier sine series of f continuous?
3. Under what conditions is the Fourier cosine series of f continuous?
4. Why did term-by-term differentiation of the series in problem 3.12 fail?

Problem 3.14 Let f and f' be piecewise smooth functions (with f being continuous to validate integration by parts). Suppose that f has the Fourier sine series

$$f(x) \sim \sum_{n=1}^{\infty} \left(B_n \sin \frac{n\pi x}{L} \right).$$

Then the derivative of f should be give the cosine series

$$f'(x) \sim A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} \right).$$

In this problem you will prove when it is valid to differentiate a sine series term by term.

1. If term-by-term differentiation is to be valid, what relationship must exist between B_n and A_n?
2. Show that $A_0 = \frac{1}{L}[f(L) - f(0)]$? Under what conditions is $A_0 = 0$?
3. Use integration by parts to show that $A_n = \frac{n\pi}{L} B_n + \frac{2}{L}[(-1)^n f(L) - f(0)]$.
4. Under what conditions is term-by-term differentiation valid?

In the problem above, we did more than prove when term-by-term differentiation is valid. We also showed how to find the Fourier cosine series of f', if we know the Fourier sine series of f. If we know $f \sim \sum_{n=1}^{\infty} \left(B_n \sin \frac{n\pi x}{L} \right)$, then we can say

$$f'(x) \sim \frac{1}{L}[f(L) - f(0)] + \sum_{n=1}^{\infty} \left(\left(\frac{n\pi}{L} B_n + \frac{2}{L}[(-1)^n f(L) - f(0)] \right) \cos \frac{n\pi x}{L} \right).$$

Problem 3.15 Complete problem 3.4.5. We already know the Fourier sine series of $\cos(\pi x/L)$. So use the results above to rapidly find the Fourier cosine series of $\sin(\pi x/L)$. See page 104, the bottom left corner, for the facts you need to complete this problem.

Problem 3.16 Let f and f' be piecewise smooth functions (with f being continuous to validate integration by parts). Suppose that f has the Fourier cosine series

$$f(x) \sim A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} \right).$$

Then the derivative of f should be give the sine series

$$f'(x) \sim \sum_{n=1}^{\infty} \left(B_n \sin \frac{n\pi x}{L} \right).$$

Prove that term-by-term differentiation is always valid. [First note the relationship that must exist between A_n and B_n, and then use integration by parts on B_n to show that this condition is always satisfied.]

Problem 3.17 Complete problem 3.4.7 on page 125. See the top of page 124 for the mathematical statement of this theorem. This will justify why we can term-by-term differentiate with respect to t in a Fourier series.

Problem 3.18 Consider the initial value boundary value problem (IVBP) for $x \in [-L, L]$ and $t > 0$ given by

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, u(L,t) = 0, \quad u(x,0) = f(x).$$

We showed in problem 3.11 that term-by-term differentiation shows that

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \sin \frac{n\pi x}{L} \right) e^{-(n\pi/L)^2 kt},$$

is a solution to this IVBP, where the B_n are the Fourier coefficients of the sine series of $f(x)$. The only missing piece was why term-by-term differentiation is valid. Explain why term-by-term differentiation is valid in this IVBP. See page 123.

Problem 3.19 Consider the initial value boundary value problem (IVBP) for $x \in [-L, L]$ and $t > 0$ given by

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0, \quad u(x,0) = f(x).$$

We showed in problem 3.11 that term-by-term differentiation shows that

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} \right) e^{-(n\pi/L)^2 kt},$$

is a solution to this IVBP, where the A_n are the Fourier coefficients of the cosine series of $f(x)$. Explain why term-by-term differentiation is valid in this IVBP.

Problem 3.20 Consider the initial value boundary value problem (IVBP) for $x \in [-L, L]$ and $t > 0$ given by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad u(-L,t) = u(L,t), \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t), \quad u(x,0) = f(x).$$

We showed in problem 3.11 that term-by-term differentiation shows that

$$u(x,t) = \sum_{n=0}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 kt},$$

is a solution to this IVBP. where a_n and b_n are the Fourier coefficients of $f(x)$. Explain why term-by-term differentiation is valid in this IVBP.

Chapter 4

The Wave Equation

4.1 Developing the Wave Equation

We now turn our attention to the one dimensional wave equation, namely

$$\frac{\partial^2 u}{\partial t^2} = k\frac{\partial^2 u}{\partial x^2}.$$

Let's start by describing a physical context which is modeled by this equation. Then we will derive the wave equation and then solve it.

Start with a violin, that has been placed sideways on the x-axis. One of the violin strings begin at $x = 0$ and end at $x = L$ (where at $x = L$ is a nob for tightening the string). The violin string is plucked and starts to vibrate. Our goal is to understand the vibration of the string based on the initial pluck. At time t, let $u(x,t)$ represent the deflection of the string above (or below) the x coordinate of the x-axis. To solve this problem, we'll start by making the following assumptions.

- The initial deflection of the string, given by $u(x,0) = f(x)$, is small.
- The string has uniform density, so that $\rho(x)$ is constant.
- The tension in the string is so great that we can ignore gravity.
- As the string vibrates, the portion of the string above spot x will only move vertically. There is no left/right movement, rather only vertical movement.
- The string is perfectly elastic. It can expand and contract as needed without introducing any additional forces other than the tension.

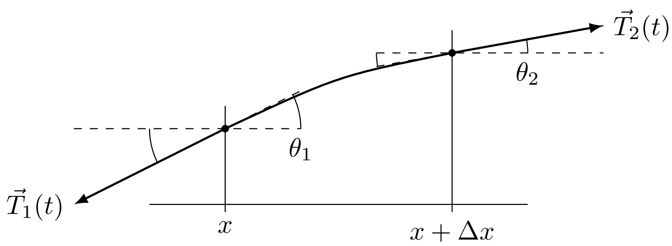
Under these assumptions, we will show that $\frac{\partial^2 u}{\partial t^2} = k\frac{\partial^2 u}{\partial x^2}$.

We need to review a tiny bit of physics before proceeding. Newton's second law of motion states that the $\vec{F} = m\vec{a}$, or rather that the total force acting on an object equals the mass of the object multiplied by the acceleration of the object. Forces causes masses to accelerate ($\vec{F} = m\vec{a}$). The total force on an object is found by summing all the forces acting on the object.

Consider the portion of the violin string between x and $x + \Delta x$. Let's consider the external forces acting on this tiny segment of the string.

- At x, the tension in the string from the left end, which we'll call $\vec{T_1}(t)$, wants to pull this little segment left. The tension $\vec{T_1}$ has both a vertical and horizontal component, and we know the horizontal component acts left.

- At $x + \Delta x$, the tension in the string from the right end, which we'll call $\vec{T}_2(t)$, wants to pull this little segment right. There is again both a vertical and horizontal component, and we know the horizontal component acts right.
- Gravity acts on the string, but we assumed the tension was so great that we'll neglect it's effect. If we want to model the vibrations of electrical wires, or cables in a suspension bridge, then we shouldn't neglect gravity.
- We assumed the string was perfectly elastic. If it is not, then as the string stretches or contracts, we would have additional external forces acting on this bit of string.



Problem 4.1 Let $\vec{T}_1(t)$ and $\vec{T}_2(t)$ be defined as above. Let $T_1(t)$ and $T_2(t)$ be the magnitudes of these vectors. Let $\theta_1(t)$ and $\theta_2(t)$ be the angles between the x-axis and the tension vectors. Choose the angles to be between $-\pi/2$ and $\pi/2$, where we measure a positive angle as a counter-clockwise rotation from the x-axis.

1. Explain why $0 = -T_1 \cos\theta_1 + T_2 \cos\theta_2$. In what follows, let $T = T_1 \cos\theta_1 = T_2 \cos\theta_2$.
2. Explain why $\rho \Delta x \dfrac{\partial^2 u}{\partial t^2} \approx -T_1 \sin\theta_1 + T_2 \sin\theta_2$.
3. Show that $\rho \Delta x \dfrac{\partial^2 u}{\partial t^2} \approx T(\tan\theta_2 - \tan\theta_1)$.

Problem 4.2: Approximate derivation of the wave equation Continue from problem 4.1

1. Explain why we can write
$$\tan\theta_2 - \tan\theta_1 = \frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t).$$
2. Explain why we can write
$$\rho \Delta x \frac{\partial^2 u}{\partial t^2}(x, t) \approx T\left(\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right).$$
3. By considering the limit at $\Delta x \to 0$, explain why
$$\frac{\partial^2 u}{\partial t^2}(x, t) \approx c^2 \frac{\partial^2 u}{\partial x^2}(x, t)$$
for some constant c. What is the constant c? Show that the units of c are those of speed.

The solution above suffers from the fact that it contains the symbol $\approx$ in the solution. Most physics and engineering applications will turn the symbol $\approx$ to an $=$ in the limit process. This is justified, provided an appropriate mean value theorem applies. The following problem shows you how this is done.

Problem 4.3: Exact derivation of the wave equation Continue from problem 4.2.1.

1. The mean value theorem states that (under reasonable assumptions) there exists some k in the interval $[a, b]$ with $f'(k)(b-a) = (f(b) - f(a))$. Use this theorem to rewrite the expression $\left(\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right)$ as a derivative evaluated at some point k, multiplied by a distance. What are the bounds for k?

2. The force acting on the small segment from x to $x+\Delta x$ is the mass of the segment multiplied by an unknown acceleration which we'll call $a(x, t)$. Explain why
$$\rho \Delta x a(x,t) = T\Delta x \frac{\partial^2 u}{\partial x^2}(k,t).$$
The equal sign here is emphasized. We no longer have an approximation.

3. By considering the limit as $\Delta x \to 0$ of each side of the equation above, explain why
$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}(x,t).$$
[Why can we replace $a(x, t)$ with $\frac{\partial^2 u}{\partial t^2}$, and why did the k turn into an x?]

4.2 Solving the Wave Equation with Fourier Series

We've now got another partial differential equation. In order to give a full solution to the wave equation, we need some boundary and initial conditions. We'll consider many of the same boundary conditions as we saw in the solution to the heat equation. The initial conditions are the initial displacement of the string, given by $u(x, 0) = f(x)$, and the initial velocity of the string, given by $\frac{\partial u}{\partial t}(x, 0) = g(x)$. If the string is displaced, and then released with no initial velocity, we know that $g(x) = 0$.

Problem 4.4 Consider the initial value boundary problem (IVBP) for $0 \le x \le L$ and $t > 0$ given by
$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad u(0,t)=0, u(L,t)=0, \quad u(x,0)=f(x), \frac{\partial u}{\partial t}(x,0)=g(x).$$
We wish to solve this IVBP.

1. Use the method of separation of variables to obtain two second order ODEs. Remember to use $\phi(x)$ as the name for the function dependent only on x.

2. What are the boundary conditions on ϕ? In other words, explain how the boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$ are used to obtain conditions on ϕ.

3. Obtain the eigenvalues λ_n and eigenfunctions $\phi_n(x)$.

Problem 4.5 Continue from the previous problem.

1. For each λ_n, solve the time dependent ODE.
2. For each λ_n, provide a solution u_n to the wave equation that satisfies the boundary conditions.
3. Give an infinite series solution to the wave equation. You should find that in your series you have two sets of coefficients (the book calls them A_n and B_n). Use the end cover of your book to determine the coefficients.

You have now solved the wave equation using Fourier series.

Problem 4.6 In part 3 of the previous problem, we summed up infinitely many solutions to the wave equation to obtain a full solution. The principle of superposition suggests that we should be able to do this, but superposition technically only works when we sum finitely many solutions. Justify why we can add together infinitely many terms. In other words, please justify why term-by-term differentiation of

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}$$

is justified. Show that this infinite sum satisfies the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$. This should be a review of chapter 3.

Problem 4.7 Consider the initial value boundary problem (IVBP) for $0 \le x \le L$ and $t > 0$ given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0, \quad u(x,0) = f(x), \frac{\partial u}{\partial t}(x,0) = g(x).$$

Solve this IVBP. Reuse as much of the previous problem as you can. In other words, what does the new boundary condition in this problem cause to change from our previous problems.

Problem 4.8 (Exercise 4.4.1 on page 147 of the book.) Consider the vibrating string of uniform density ρ_0 and tension T_0.

(a) What are the natural frequencies of a vibrating string of length L fixed at both ends.

(b) What are the natural frequencies of a vibrating string of length H, which is fixed at $x = 0$ and "free" at the other end [i.e., $\partial u/\partial x(H,t) = 0$]? Sketch a few modes of vibration as in Fig. 4.4.1.

(c) Show that the modes of vibration for the odd harmonics (i.e. $n = 1, 3, 5, \ldots$) of part (a) are identical to the modes of part (b) if $H = L/2$. Verify that their natural frequencies are the same. Briefly explain using symmetry arguments.

Problem 4.9 (Exercise 4.4.2 on page 147 of the book.) In section 4.2 it was shown that the displacement u of a nonuniform string satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial t^x} + Q,$$

where Q represents the vertical component of the body force per unit length. If $Q = 0$, the partial differential equation is homogeneous. A slightly different homogeneous equation occurs if $Q = \alpha u$.

(a) Show that if $\alpha < 0$, the body force is restoring (toward $u = 0$). Show that if $\alpha > 0$, the body force tends to push the string further away from its unperturbed position $u = 0$.

(b) Separate variables if $\rho_(x)$ and $\alpha(x)$ but T_0 is constant for physical reasons. Analyze the time-dependent ordinary differential equations.

(c) Specialize part (b) to the constant coefficient case. Solve the initial value problem (if $\alpha < 0$) given by the boundary and initial conditions

$$u(0,t) = u(L,t) = 0, \quad u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = f(x).$$

What are the frequencies of vibration.

Problem 4.10 (Exercise 4.4.7 on page 148 of the book.) If a vibrating string satisfying (4.4.1)-(4.4.3) is initially at rest, so $g(x) = 0$, show that

$$u(x,t) = \frac{1}{2}[F(x-ct) + F(x+ct)],$$

where $F(x)$ is the odd periodic extension of $f(x)$.. Hints:

1. For all x, we have $F(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$.
2. Use the trig identity $\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$.

Problem 4.11 Consider the initial value boundary problem (IVBP) for $0 \leq x \leq L$ and $t > 0$ given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, u(L,t) = 0, \quad u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = 0.$$

1. Give a physical explanation as to why the only solution to this equation is the trivial solution $u(x,t) = 0$. What do all the initial values and boundary conditions imply?

2. Prove that the solution of the IVBP

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, u(L,t) = 0, \quad u(x,0) = f(x), \frac{\partial u}{\partial t}(x,0) = g(x)$$

is unique.

In other words, we found a solution in problems 4.4 and 4.5. If someone finds a solution in another way (using some other crazy method), then you are going to show that this crazy solution must be the same as ours. [As a hint, let u_1 and u_2 be two solutions, possibly arrived at in different ways. What can you say about the difference $u_2 - u_1$? What boundary and initial conditions does the difference $u_2 - u_1$ satisfy.]

4.3 The Method of Characteristics

In the previous section, we solved the wave equation using Fourier series. We'll find that another, more powerful, technique exists to solve the heat equation. This new method will allow us to solve every problem that the Fourier series method gave, together with additional problems. The idea relates back to level curves, and is called the method of characteristics. The material in this section is found in chapter 12 of your textbook. You may want to review problem 1.23 before proceeding.

Problem 4.12: Level Curve Theory Review Recall that a level curve of a function $z = f(x,y)$ is a curve in the xy-plane obtained by letting the output z be constant. So a curve C is a level curve if and only if the output z does not change at all along the curve C.

- Suppose $\vec{r}(t) = (x(t), y(t))$ is a parametrization of a level curve. Explain why $\frac{df}{dt} = 0$. Then use the chain rule to find the angle between the vectors

$$\frac{d\vec{r}}{dt}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)\Big|_{t=t} \quad \text{and} \quad \vec{\nabla} f(\vec{r}(t)) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\Big|_{(x,y)=\vec{r}(t)}.$$

- Suppose that $\vec{r}(t) = (x(t), y(t))$ is the parametrization of a curve. Suppose also that for all t, we know that zero equals the dot product

$$0 = \vec{\nabla} f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}(t).$$

  Explain why this means $\vec{r}$ is a parametrization of a level curve, and that $\frac{df}{dt} = 0$.

Problem 4.13 Show that we can write the one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ in the two factored forms

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0.$$

If we let $w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}$ and $v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}$, rewrite the above equations as two first order partial differential equations

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0.$$

Problem 4.14 Consider the PDE $\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$. Suppose $\vec{r}(t) = (x(t), t)$ is a parametrization of a curve in the xt plane. We are interested in the composite function $w(\vec{r}(t)) = w(x(t), t)$. You can think of the parametrization as a person moving along the x-axis at time t and watching what happens to w as they move. We currently have two function, namely $w(x,t)$ and $r(t) = (x(t), t)$.

1. Explain why
$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{dx}{dt}\frac{\partial w}{\partial x}.$$
What rule are you using to perform your computation. Be prepared to explain the difference between between $\frac{\partial w}{\partial t}$ and $\frac{dw}{dt}$?

2. If $dx/dt = c$ (so the observer is moving right at speed c), show that $dw/dt = 0$.

3. Explain why $\vec{r}(t) = (x(t), t)$ is a level curve of w.

Problem 4.15 Continue from the previous problem. For this problem assume $dx/dt = c$, which means we know that $dw/dt = 0$.

1. Solve the ODE $dx/dt = c$ for $x(t)$. Your solution is a collection of parallel lines, which we call a family of characteristic of the PDE $\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$.

2. If we have the initial condition $w(x, 0) = P(x)$, and we move along the path $x = x(t)$ given above, explain why
$$w(x, t) = P(x - ct).$$

Problem 4.16 Consider the PDE $\frac{\partial w}{\partial t} + 3\frac{\partial w}{\partial x} = 0$, subject to the initial condition
$$w(x, 0) = \begin{cases} 0 & x < 0 \\ 2x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}.$$

1. In this problem, what is c?

2. Recall that if $\vec{r}(t) = (x(t), t)$ is level curve of w, then we know
$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{dx}{dt}\frac{\partial w}{\partial x} = 0.$$
Find a family of characteristics of the PDE, by solving $\frac{dx}{dt} = c$.

3. Recall that the general solution to the PDE $\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$, given the initial condition $w(x, 0) = P(x)$, is
$$w(x, t) = P(x - ct).$$
State the general solution to the PDE given in this problem. [Hint: replace x with $x - ct$ in $P(x)$.]

4. Provide a sketch in the wx plane of $w(x, 0)$, $w(x, 1)$, and $w(x, 2)$. We'll add a 3D surface plot in class of $w(x, t)$, though you're welcome to create it on your own and present it in class.

Problem 4.17 Complete problems 12.2.2 and 12.2.3, which ask you to solve the following two PDEs.

1. Solve $\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0$ with $w(x,0) = \cos(x)$.
2. Solve $\frac{\partial w}{\partial t} + 4\frac{\partial w}{\partial x} = 0$ with $w(0,t) = \sin(3t)$.

Problem 4.18 We've reduced solving $\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$ to solving

$$\frac{dx}{dt} = c \quad \text{and} \quad \frac{dw}{dt} = 0.$$

Now solve $\frac{\partial w}{\partial t} + 3t^2\frac{\partial w}{\partial x} = 2tw$ by solving

$$\frac{dx}{dt} = 3t^2 \quad \text{and} \quad \frac{dw}{dt} = 2tw.$$

Assume that $w(x,0) = P(x)$.

Problem 4.19 Solve both $\frac{\partial w}{\partial t} + x\frac{\partial w}{\partial x} = 1$ and $\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 1$ where $w(x,0) = f(x)$. (This is problems 12.2.5 (b) and (c).)

4.4 Solving the Wave Equation with the Method of Characteristics

Problem 4.20 Let $P(x)$ and $Q(x)$ be two arbitrary functions (where we assume as much differentiability as needed). Verify (by taking derivatives) that $w(x,t) = P(x-ct)$ is a solution to $\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$, and that $v(x,t) = Q(x+ct)$ is a solution to $\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0$.

Combining problems 4.13 and the previous problem, we've shown that

$$w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = P(x-ct) \quad \text{and} \quad v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = Q(x+ct).$$

The solution for w and v are just standing waves moving right (for w) and left (for v) at speed c.

Problem 4.21 We now solve the wave equation.

1. Explain why

$$2\frac{\partial u}{\partial t} = P(x-ct) + Q(x+ct) \quad \text{and} \quad 2c\frac{\partial u}{\partial x} = Q(x+ct) - P(x-ct).$$

2. Pick either equation above and anti-differentiate to obtain $u(x,t)$. Then anti-differentiate the other. In both cases, what is $u(x,t)$?

3. Explain why we can write the solution to $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ in the form

$$u(x,t) = F(x-ct) + G(x+ct),$$

where F and G are arbitrary functions. What relationship exists between the functions F and G and the functions P and Q.

We've now shown that the solution to the wave equation is precisely the sum of two standing waves, F and G, where one wave moves right and the other moves left at a constant speed c (whose units are precisely the units of speed).

4.5 Applying Initial and Boundary Conditions

We now need turn our attention to find the function F and G when we are provided with the initial displacement and initial velocity given by

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

We'll start by assuming no boundary conditions, which means we know $f(x)$ and $g(x)$ for all $x \in (-\infty, \infty)$. Once we've obtained the solution to an infinitely long wave, we'll be able to quickly obtain solutions for a semi-infinite wave, and finally a wave in the interval $[0, L]$. Since the solution to the wave equation is unique (which you showed in problem 4.11), the solution we obtain here must be identical to the solution we obtained using Fourier series.

4.5.1 Infinite Wave

Problem 4.22: Infinite Wave We know the general solution to the wave equation is $u(x,t) = F(x-ct) + G(x+ct)$. Assume the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for} \quad -\infty < x < \infty.$$

1. Apply the initial conditions to obtain a system of ordinary differential equations involving the unknown functions F and G. The functions f and g are assumed to be known.

2. Show that
$$\frac{dG}{dx} = \frac{1}{2}\left(\frac{df}{dx} + \frac{g(x)}{c}\right).$$
Obtain a similar equation for $\frac{dF}{dx}$.

3. Integrate the equations above to show that
$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(y)dy + k \quad \text{and}$$
$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(y)dy - k.$$
The constant k can be ignored, because the solution to the wave equation requires that you add $F(x-ct)$ and $G(x+ct)$ together.

Problem 4.23 Consider the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ for $-\infty < x < \infty$, with $c = 2$ and initial values

$$u(x,0) = f(x) = \begin{cases} 1 & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x) = 0.$$

Graph the solution $u(x,t)$ in the xu-plane for $t = 1, 2, 3, 4$. Combine your solutions into a 3D plot using xtu coordinates (see page 547). At what time do the waves separate?

Problem 4.24 Consider the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ for $-\infty < x < \infty$, with $c = 2$ and initial values

$$u(x,0) = f(x) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x) = \begin{cases} 1 & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}.$$

Graph the solution $u(x,t)$ for $t = 1, 2, 3, 4$ in the xu-plane. Combine your solutions into a 3D plot using xtu coordinates (see page 547).

Problem 4.25 Complete 12.3.1. Provide a sketch of the solution for times $ct = 1$, $ct = 2$, $ct = 3$, and $ct = 4$. Combine your results into a 3D surface plot, as in the previous two problems. I'll provide an animated graph of the entire solution in class (or if you want the challenge, try creating such an animation yourself).

Problem 4.26 Complete problem 12.3.5. Your solution should look something like the diagram on page 548.

Problem 4.27: D'alembert's Solution Prove that the solution to the infinite wave equation with initial displacement $f(x)$ and velocity $g(x)$ is

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\bar{x}) d\bar{x}.$$

4.5.2 Semi-Infinite Wave

We now turn our attention to a semi-infinite wave. Assume the initial conditions for the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $x > 0$, are

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for} \quad x > 0.$$

We now make one additional assumption. We fix the left end of the string at

$$u(0,t) = 0.$$

From our previous work, we know that $u(x,t) = F(x - ct) + G(x + ct)$ where

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(y) dy \quad \text{for } x > 0 \quad \text{and}$$
$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(y) dy \quad \text{for } x > 0,$$

provided the arguments of F and G are both positive.

Problem 4.28: Semi-infinite Wave with fixed end Use the set up from the previous paragraph. Namely, we want to solve the IBVP

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x), \frac{\partial u}{\partial t}(x,0) = g(x), \quad u(0,t) = 0, \quad \text{for} \quad x > 0.$$

1. Explain why $G(x+ct)$ is well-defined for all x and t in this problem. For which x and t is $F(x-ct)$ not defined?
2. If $x > ct$, explain why the solution is simply $u(x,t) = F(x-ct)+G(x+ct)$.
3. Let $x = 0$ in the equation $u(x,t) = F(x-ct) + G(x+ct)$. Use this to develop a formula for $F(x)$ if x is negative.
4. Show that if $x < ct$, then the solution to the wave equation for a semi-infinite string is $u(x,t) = G(x+ct) - G(ct-x)$.

Problem 4.29 Consider a semi-infinite string $x > 0$ with a fixed end $u(0,t) = 0$ which is initially at rest (so $\frac{\partial u}{\partial t}(x,0) = 0$). The initial shape of the wave is $f(x) = 2$ if $x \in [3,4]$ and $f(x) = 0$ otherwise. The constant c is $1/5$. Determine formulas for $F(x)$ and $G(x)$ if $x > 0$. Find a formula for $u(x,t)$, and then construct a 3D graph of your solution for $0 < t < 40$.

Problem 4.30 Consider the same set up as in problem 4.28. Show that if we had extended f and g as odd functions, then we could have just written our solution as $u(x,t) = F(x-ct) + G(x+ct)$. In other words, show that $F(x) = -G(-x)$ precisely when f and g are extended as odd functions. [Hint: Write the definitions of $F(x)$ and $G(-x)$, and use the definitions of even and odd.]

The previous problem shows that if we use the boundary condition $u(0,t) = 0$, then we can obtain a solution to the semi-infinite wave equation by just extending the initial conditions as odd functions to the entire real line, and then using the solution to the infinite wave equation. We'll see this type of symmetry occurring again and again, though the boundary conditions will determine if we should extend our function as an even or an odd function.

Problem 4.31: Semi-infinite Wave with free end Complete problem 12.4.4. This asks you to change the boundary condition from $u(0,t) = 0$ to $\frac{\partial u}{\partial x}(0,t) = 0$ and then solve. I'll have one person present each part of this problem in class. Part (a) asks you to solve in general. Part (b) asks you to show that the solution is the same as extending f and g as even functions. Part (c) asks you to give a graphical solution for a specific set of initial conditions. The previous 3 problems serve as a model for what you should do here.

Problem 4.32 Solve 12.4.1. This problem starts with a semi-infinite string at rest. The left end is no longer fixed, rather has position function $u(0,t) = h(t)$. You'll show how moving the left endpoint changes the shape of the wave.

4.5.3 Vibrating String of Fixed Length

We showed in the previous section that we can represent the fixed boundary condition $u(x, 0) = 0$ by just extending the initial conditions $f(x)$ and $g(x)$ as odd functions. Similarly, the free boundary condition $\frac{\partial u}{\partial t}(x, 0) = 0$ requires that we extend $f(x)$ and $g(x)$ as even functions. We will now use these facts to solve the wave equation for vibrating strings with a fix length $0 \leq x \leq L$. Whenever you encounter a fixed boundary, make sure you extend the waves across that boundary by using an odd extension. Whenever you encounter a free boundary, make sure you extend the waves across that boundary by using an even extension.

Problem 4.33 Complete 12.5.3 (a) and (c). Here you have a fixed end at $x = 0$ and a free end at $x = L$.

Problem 4.34 Complete 12.5.4. Here you have a free end at $x = 0$ and a fixed end at $x = L$.

Chapter 5

Sturm-Liouville Eigenvalue Problems

5.1 Definition and Theorem

Definition 5.1. A regular Sturm-Liouville eigenvalue problem consists of a Sturm-Liouville differential equation,

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0 \quad a < x < b,$$

subject to boundary conditions of the form

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0 \quad \text{and} \quad \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,$$

where β_i are real constants. The coefficients p, q, and σ must be real and continuous everywhere, including the end points, with $p(x) > 0$ and $\sigma(x) > 0$ for all $x \in [a, b]$.

Problem 5.1 Start by showing that the differential equation $\phi'' = -\lambda\phi$ is a Sturm-Liouville differential equation (state the coefficients p, q, and σ). Then, for each collection of boundary values below, determine if these boundary values provide us with a regular Sturm-Liouville eigenvalue problem. If so, state the constants β_i. If not, explain why.

1. $\phi(0) = 0$ and $\phi(L) = 0$.
2. $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$.
3. $\phi(-L) = \phi(L)$ and $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$.

Problem 5.2 Consider the PDE $c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q$. We discussed this PDE in chapter 1, and showed that it modeled the flow of heat in a rod with constant cross-sectional area. Assume that the heat sources are proportional to the temperature, so $Q = \alpha u$ for some function $\alpha(x)$. Also assume that c, ρ, and K_0 could depend on x.

1. Use separation of variables with $u(x,t) = \phi(x)h(t)$ to obtain differential equations for $\phi(x)$ and $h(t)$.

2. Show that the ODE for ϕ is a Sturm-Liouville differential equation.

Problem 5.3 Consider the PDE $\dfrac{\partial u}{\partial t} = k\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)$. This PDE from chapter models the flow of heat on a disc, where the temperature is assumed to be circularly symmetric (only dependent on time and distance r to the origin), where all the thermal properties are constant.

1. Use separation of variables with $u(r,t) = \phi(r)h(t)$ to obtain differential equations for $\phi(r)$ and $h(t)$.

2. Show that the ODE for ϕ is a Sturm-Liouville differential equation.

Problem 5.4 Consider the operator $L(u) = \dfrac{d}{dx}\left(p(x)\dfrac{du}{dx}\right) + q(x)u$.

1. Prove that L is a linear operator.

2. Rewrite the Sturm-Liouville differential equation in terms of the operator L (you should then see why we call this an eigenvalue problem). We'll use this notation in later problems.

3. Show that $uL(v) - vL(u) = \dfrac{d}{dx}\left[p\left(u\dfrac{dv}{dx} - v\dfrac{du}{dx}\right)\right]$. [Expand both sides.]

The following theorem (which we will use without proof), provides a lot of information about any function ϕ which satisfies a regular Sturm-Liouville eigenvalue problem.

Theorem 5.2. *Suppose that ϕ satisfies a regular Sturm-Liouville eigenvalue problem, meaning ϕ satisfies the ODE*

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0 \quad a < x < b,$$

and has appropriate boundary conditions. Then the following statements all hold.

1. *All the eigenvalues are real.*

2. *There exists an infinite number of eigenvalues with*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots .$$

   (a) *There is a smallest eigenvalue which we'll call λ_1.*

   (b) *There is no largest eigenvalue, as $\lambda_n \to \infty$ as $n \to \infty$.*

3. *For each n, corresponding to λ_n there is an eigenfunction $\phi_n(x)$. This eigenfunction is unique up to multiplying by a constant. In addition, the eigenfunction $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.*

4. *The eigenfunctions form a complete set in the vector space of piecewise smooth functions. This means that we can represent any piecewise smooth function $f(x)$ with a generalized Fourier series*

$$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x).$$

   Furthermore, the infinite series converges to $\frac{f(x+)+f(x-)}{2}$ for $a < x < b$.

5. *The eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$, which means*

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)dx = 0 \quad \text{if } n \neq m.$$

6. *Any eigenvalue can be related to its eigenfunction by the Rayleigh quotient*

$$\lambda = \frac{(-p\phi\frac{d\phi}{dx})|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\ dx}{\int_a^b \phi^2\sigma\ dx}.$$

The boundary conditions may greatly simplify the Rayleigh quotient.

The following problem justifies why we never needed to consider the case $\lambda < 0$ in our work with the heat equation and wave equation.

Problem 5.5 Suppose ϕ satisfies a regular Sturm-Liouville eigenvalue problem, with $q \leq 0$. [Hint: use the Rayleigh quotient.]

1. Explain why there cannot be negative eigenvalues if the boundary conditions are $\phi(a) = 0$ and $\phi(b) = 0$. Furthermore, show that $\lambda \neq 0$ with these boundary conditions.

2. Explain why there cannot be negative eigenvalues if the boundary conditions are $\frac{d\phi}{dx}(a) = 0$ and $\frac{d\phi}{dx}(b) = 0$.

3. Explain why $\lambda > 0$ if $q \leq 0$ and $(-p\phi\frac{d\phi}{dx})|_a^b \geq 0$.

Problem 5.6 Suppose ϕ satisfies a regular Sturm-Liouville eigenvalue problem. Suppose that the eigenvalues λ_n and eigenfunctions ϕ_n have already been determined. Suppose that $f(x)$ has a generalized Fourier series given by

$$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x).$$

Explain why we can compute the coefficients as

$$a_m = \frac{\int_a^b f(x)\phi_m(x)\sigma(x)dx}{\int_a^b \phi_m^2(x)\sigma(x)dx}.$$

[Hint: what did we learn about orthogonality earlier in the semester?]

Problem 5.7 Solve problem 5.3.5.

Problem 5.8: Optional Solve problem 5.3.9. [Hint: on part 3, you'll want to guess the solution is of the form x^r for some r and then find what the constant r must equal.]

Problem 5.9 Suppose ϕ satisfies the Sturm-Liouville differential equation

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0 \quad a < x < b,$$

where the coefficients p, q, and σ must be real and continuous everywhere, including the end points, with $p(x) > 0$ and $\sigma(x) > 0$ for all $x \in [a, b]$. Obtain the Rayleigh quotient. In other words, show that

$$\lambda = \frac{(-p\phi\frac{d\phi}{dx})|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\ dx}{\int_a^b \phi^2\sigma\ dx}.$$

[Hint: Multiply both sides of the ODE by ϕ, and then integrate both side from a to b. You'll need to use integration-by-parts along the way. This is basically the idea used to develop the finite element method.]

Theorem 5.3. *The smallest eigenvalue in a Sturm-Liouville eigenvalue problem is the minimum value of the Rayleigh quotient for all continuous functions satisfying the boundary conditions (not necessarily the differential equation). We can write this as*

$$\lambda_1 = \min_{u \in C[a,b]} \frac{(-pu\frac{du}{dx})|_a^b + \int_a^b [p(du/dx)^2 - qu^2]\ dx}{\int_a^b u^2\sigma\ dx}.$$

A partial proof of this fact is on page 193. It's a partial proof because they apply linearity to an infinite sum, which is not justified until a much later time in the text. We'll skip the proof.

We will not prove this fact, but let's use it to show how you can estimate the smallest eigenvalue.

Problem 5.10 Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad \phi(0) = 0, \quad \phi(2) = 0.$$

Show this is a regular Sturm-Liouville eigenvalue problem, and state the values of $p, q, \sigma, \beta_i, a, b$.) We already know the solution to this problem, so start by stating λ_1. Then, for each function u below, compute the quotient

$$\frac{(-pu\frac{du}{dx})|_a^b + \int_a^b [p(du/dx)^2 - qu^2]\ dx}{\int_a^b u^2\sigma\ dx},$$

and show that λ_1 is smaller than this quotient.

1. $u = \begin{cases} x & x \in [0,1] \\ 2-x & x \in [1,2] \end{cases}$
2. $u = x(x-2)$
3. $u = \sin(\pi x/2)$

Problem 5.11 Let $L(u)$ be the Sturm-Liouville operator introduced in problem 5.4. Explain why

$$\int_a^b uL(v) - vL(u)dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b.$$

Then, if u and v both satisfy the homogeneous boundary conditions

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0 \quad \text{and} \quad \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,$$

show that $\int_a^b uL(v) - vL(u)dx = 0$. You may assume that $\beta_1 \neq 0$ and $\beta_4 \neq 0$.

Problem 5.12 Show that if λ_n and λ_m are different eigenvalues, with corresponding eigenfunctions $\phi_n(x)$ and $\phi_m(x)$, of a regular Sturm-Liouville eigenvalue problem, then the eigenfunctions are orthogonal with the weight function $\sigma(x)$. In other words, prove that $\int_a^b \phi_n\phi_m\sigma\ dx = 0$. [Hint: In problem 5.4, we wrote the Sturm-Liouville differential equation in the form $L(u) = -\lambda\sigma u$. Use the integral equation from the previous problem. You'll need to use the fact that eigenvalues are different at some point.]

Problem 5.13: Optional Show that if ϕ_1 and ϕ_2 are two eigenfunctions corresponding to the same eigenvalue λ, then ϕ_2 is a multiple of ϕ_1. [Hint: Problem 5.4 is the key. You'll also have to notice a quotient rule along the way.]

Problem 5.14: Optional Suppose that we know the function f solves a Sturm-Liouville differential equation with orthogonal eigenfunctions ϕ_n, and that we know

$$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x).$$

If we had decided instead to approximate f with a finite sum of the form

$$f(x) \sim \sum_{n=1}^{M} \alpha_n\phi_n(x),$$

we may discover that using coefficients α_n different than a_n could produce a better approximation. In this problem, your job is to prove that the mean-square deviation, given by

$$E = \int_a^b \left(f(x) - \sum_{n=1}^{M} \alpha_n\phi_n(x)\right)^2 \sigma\ dx,$$

is minimized precisely when

$$\alpha_n = a_n = \frac{\int_a^b f\phi_n\sigma\ dx}{\int_a^b \phi_n^2\sigma\ dx}.$$

In other words, not only do the Fourier coefficients a_n provide the appropriate coefficients needed for the infinite series to converge to f, they also provide the best approximation using any finite sum.

5.2 Non homogeneous Problems

The key idea in this section is to show that if the boundary conditions are not homogeneous, then we can always modify the PDE in some way to force the boundary conditions to be homogeneous. We can't guarantee that the PDE will be be homogeneous, but we can ALWAYS force the boundary conditions to be homogeneous.

Problem 5.15 Consider the initial value boundary problem (IVBP)

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = A, u(L,t) = B, \quad u(x,0) = f(x).$$

The boundary conditions are not homogeneous.

1. Find a steady-state solution, which we'll call $u_E(x)$.
2. If u is a solution to the $IVBP$, show that $v(x,t) = u(x,t) - u_E(x)$ satisfies the same PDE, but now has homogeneous boundary conditions $v(0,t) = 0$ and $v(L,t) = 0$. What is the initial temperature distribution $v(x,0)$?
3. Since $v(x,t)$ satisfies the heat equation with homogeneous boundary conditions, state the solution using Fourier series (just look up the solution from chapter 2). Make sure you state what the Fourier coefficients are, as they depend on the modified initial temperature distribution you obtained in part 2.
4. Now give the solution $u(x,t)$ to the non homogeneous IVBP. [See part 2 for the connection between u and v.]

In the problem above, we showed how to solve a homogeneous PDE with non homogeneous constant boundary conditions. We started by changing the problem so that the boundary conditions were homogeneous. We would like to be able to solve non homogeneous PDEs with non-constant boundary conditions. The following problem shows that we can always make a modification to force the boundary conditions to be homogeneous.

Problem 5.16 Consider the initial value boundary problem (IVBP)

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t), \quad u(0,t) = A(t), u(L,t) = B(t), \quad u(x,0) = f(x).$$

The PDE has a non homogeneous term $Q(x,t)$ and the boundary conditions are not homogeneous (and not constant). Separation of variables will not work on this problem.

1. Find a function $r(x,t)$ that has the same boundary conditions as a solution $u(x,t)$ to the PDE. Any function will do, as long as $r(0,t) = A(t)$ and $r(L,t) = B(t)$. [Hint: for each t, give an equation of a line between the two end points.]
2. Let $v(x,t) = u(x,t) - r(x,t)$. What are the boundary conditions for $v(x,t)$?
3. We know that $u(x,t) = v(x,t) + r(x,t)$ satisfies the PDE above. What PDE does $v(x,t)$ satisfy? Show that $v(x,t)$ satisfies the IVBP

   $$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \bar{Q}(x,t), \quad v(0,t) = 0, u(L,t) = 0, \quad u(x,0) = g(x),$$

   where the functions $\bar{Q}$ and g can be written in terms of the functions Q, r and f.

We now examine how to solve the IVBP given by

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \bar{Q}(x,t), \quad v(0,t) = 0, u(L,t) = 0, \quad u(x,0) = g(x).$$

We'll use a method called eigenfunction expansion. This method assumes that we can write

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x),$$

where the coefficients a_n are functions of t and the eigenfunctions ϕ_n come from the eigenfunctions of the corresponding homogeneous PDE (which in this problem are $\phi_n(x) = \sin\left(\frac{n\pi}{L}\right)^2$). We would like to simply state that

$$a_n(t) = \frac{\int_0^L v(x,t)\phi_n(x)dx}{\int_0^L \phi_n^2(x)dx} \qquad \text{(not O.K.)},$$

however this is not appropriate because we do not know $v(x,t)$. To solve the IVBP for $v(x,t)$, we need to determine the coefficients $a_n(t)$. The next few problems will help you learn how this is accomplished.

Problem 5.17 Find the general solution to the differential equation $\frac{dy}{dt}(t) + ky(t) = q(t)$, where k is constant, but q is a function of t. [Hint: if you find an appropriate integrating factor, this ODE can be written as an exact ODE. This is a review problem from differential equations. As a side note, you can actually use this solution, in matrix form, to solve almost every ODE from a traditional differential equations class. Come see me if you're interested in knowing how.]

Problem 5.18 Consider the IVBP given by

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \bar{Q}(x,t), \quad v(0,t) = 0, u(L,t) = 0, \quad u(x,0) = g(x).$$

Assume that we can write

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x),$$

where the coefficients a_n are functions of t and the eigenfunctions ϕ_n come from the eigenfunctions of the corresponding PDE with homogeneous boundary conditions.

Problem 5.19 Solve problem 8.2.2(b).

Problem 5.20 Solve problem 8.3.1(c). You'll want to use your solution to the previous problem.

5.3 To be continued...