Problems

Let's start with a question

What is a proof?

Write down your own definition. Then compare it to this footnote. [1]

Problem 1: Another way to express 2

You've seen the number $1.99\bar{9}$ before. Does this number equal 2, or is it different than 2? Prove your answer.

Let's analyze three more questions:

1. What does it mean for a proof to be logically sound?
2. What does it mean for a proof to be clear?
3. What does it mean for a proof to be concise?

Write down your own definitions first, and then compare them to those given in the footnote. [2] Our goal this semester will be to learn to produce logically sound, clear, and concise proofs of mathematical statements.

Problem 2: Between two real numbers is a real number

Suppose a and b are real numbers with $a < b$. Prove that there exists a real number c with $a < c < b$.

Here are common symbols we use for standard number systems. We use

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the natural numbers,
- $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$ for the whole numbers,
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ for the integers,
- $\mathbb{Q} = \{\frac{p}{q} \mid p \text{ and } q \text{ are integers with } q \neq 0\}$ for the rational numbers, and
- $\mathbb{R} = (-\infty, \infty)$ for the real numbers.

We'll spend most of the semester looking at collections of real numbers, and these symbols will appear quite often.

Definition: Lower and upper bounds. Let S be a collection of real numbers (written $S \subseteq \mathbb{R}$, or S is a subset of the real numbers).

- A lower bound for S is a real number m such that $m \leq x$ for every $x \in S$. We say that S is bounded below if it has a lower bound.
- An upper bound for S is a real number m such that $m \geq x$ for every $x \in S$. We say that S is bounded above if it has an upper bound.

[1] A proof of a statement is an argument that convinces your peers of the truth of that statement. Period. It might be completely flawed, it might run in circles, etc., but if it convinces your peers of the truth of the statement, then it's a proof. That said, calling something a proof depends entirely on the audience.

[2] A proof is logically sound when we can check its validity entirely with logic. To be clear means that the language we use in the proof cannot be misinterpreted, according to agreed upon standards by whatever your audience is. Being clear requires we gain precise use of correct vocabulary. A concise proof is one that provides exactly what is needed to prove the statement, and no more. Being concise does not mean being short. It is perfectly possible to have several different logically sound, clear, and concise proofs, of varying length.

- We say that S is bounded if has both a lower and upper bound.

Problem 3: Practice with bounded definitions

Consider the set

$$S = [0, 4) = \{x \in \mathbb{R} \mid 0 \leq x \text{ and } x < 4\}.$$

1. Show that S is bounded below by giving a lower bound. Prove that the number you gave is a lower bound, and then state another lower bound different than the one you gave.
2. Of all possible lower bounds, which is the greatest lower bound. In other words, produce a lower bound m so that if m' is any lower bound, then we must have $m' \leq m$. Prove your answer.
3. Show that S is bounded above by giving an upper bound. Remember to fully justify your answer.
4. Of all possible upper bounds, which is the least upper bound?

Remember to justify your answer.

Definition: Statement and open sentence.

- A statement is a sentence that can be classified as either true or false (but not both). The truth value of a statement is either "True" or "False." For a sentence to be a statement, it is not necessary that we know the truth value, but it must be clear that the value is either "True" or "False."
- Some sentences involve a variable, and the truth value of the sentence cannot be determined until the value of the variable is specified. An open sentence is a sentence involving a variable whose truth value cannot be determined until the variables in the sentence are specified, at which point the open sentence becomes a statement.

Exercise: Recognizing statements and open sentences. Classify each sentence below as a statement, an open sentence, or neither.

1. Every integer is either even or odd.
2. Today is Thursday.
3. $x^2 - 9 = 0$.
4. The second coming will occur in 2050.
5. Sunsets are beautiful.
6. Have you read the first book in the Harry Potter series?
7. $\cos^2(x) + \sin^2(x) = 1$.

Proof. We'll analyze each sentence above.

1. The sentence "Every integer is either even or odd" is a statement, as it has the truth value of "True."
2. The sentence "Today is Thursday" is an open sentence. The truth value depends on what the variable "today" is.
3. The sentence "$x^2 - 9 = 0$" is an open sentence. The variable is x, and the value x determines whether or not $x^2 - 9$ equals zero or not.

4. The sentence “The second coming will occur in 2050” is a statement. It is either true or false, however we do not have the ability to determine the truth value (as we cannot see the future). The sentence is definitely either true or false, and not both, so it is a statement.

You could classify this as an open sentence, as the location was not specified. If the second coming were to happen on Dec 31st in some places, and January 1st in others, we'd have a problem with calling this a statement.

5. The sentence “Sunsets are beautiful” is an opinion, and hence is neither a statement nor an open sentence.

6. The sentence “Have you read the first book in the Harry Potter series?” is not a statement. It has no truth value. It is a question, not a statement.

7. While “$\cos^2(x) + \sin^2(x) = 1$” has a variable x in it, we could argue that this is a statement and not an open sentence as the sentence is true for any value x that makes sense in this sentence. However, since the variable x was not specified, we could also argue that this is an open sentence. The context in which a sentence occurs may alter whether a sentence is an open sentence or statement. If the sentence had instead read “For any real number x we have $\cos^2(x) + \sin^2(x) = 1$,” then the sentence is definitely a true statement.

□

Definition: Negation, conjunction, disjunction. Let P and Q be statements or open sentences.

- The negation of P, written $\sim P$, is the statement or open sentence which is true precisely when P is false. We often read $\sim P$ as “It is not the case that P.”
- The conjunction of P and Q, written $P \wedge Q$, is the statement or open sentence “P and Q.” A conjunction is true only when both P and Q are true. So a conjunction is false unless both P and Q are true.
- The disjunction of P and Q, written $P \vee Q$, is the statement or open sentence “P or Q.” A disjunction is true when P is true, or Q is true, or both are true. So a disjunction is true unless both P and Q are false.

Definition: Truth table. Let $P_1, P_2, \ldots, P_n$ be n statements or open sentences that are used to make the compound sentence P. A truth table for this compound sentence is a table that keeps track of all possible truth values for the compound sentence based upon the possible truth values for each of $P_1, P_2, \ldots, P_n$.

Exercise: A Truth Table For A Conjunction And Its Negation. Construct a truth table for $P \wedge Q$ and $\sim (P \wedge Q)$.

Proof. There are four cases to consider when looking at the truth values of P and Q, hence our truth table has 4 rows. We know that $P \wedge Q$ is false unless both P and Q are both true. This gives us the third column in the truth table below for $P \wedge Q$. The fourth column below contains the truth values for $\sim (P \wedge Q)$ by just interchanging the T and F values from the third column.

P	Q	$P \wedge Q$	$\sim (P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

□

Definition: Logically Equivalent. We say two statements or open sentences are logically equivalent if they have the same truth value for all possible values of their component statements.

Problem 4: DeMorgan's Laws With Truth Tables

Let P and Q be statements or open sentences. Start by completing the truth table below to give the truth values for $P \vee Q$, $\sim (P \vee Q)$, $(\sim P) \vee (\sim Q)$, and $(\sim P) \wedge (\sim Q)$.

P	Q	$P \vee Q$	$\sim (P \vee Q)$	$\sim P$	$\sim Q$	$(\sim P) \vee (\sim Q)$	$(\sim P) \wedge (\sim Q)$
T	T						
T	F						
F	T						
F	F						

Then complete each of the following:

1. Use your truth table to prove that the statement$\sim (P \vee Q)$ and the statement $(\sim P) \wedge (\sim Q)$ are logically equivalent.
2. Construct a similar truth table to prove that $\sim (P \wedge Q)$ and $(\sim P) \vee (\sim Q)$ are logically equivalent .

When you have finished this problem you will have shown that the negation of a disjunction is the conjunction of the negations, and that the negation of a conjunction is the disjunction of the negations.

One skill we need to develop as mathematicians is the ability to encounter a new definition and from that definition start drawing conclusions. We'll have lots of opportunities to practice this. The beginnings of analysis and calculus are directly related to the next definition.

Definition: Limit point of a set of real numbers. Let S be a set of real numbers. We say that a point p is a limit point of S if every open interval $I = (a, b)$ that contains p also contains a point x in S with $x \neq p$.

Problem 5: A limit point of an open interval

Let $S = (0, 1)$, the open interval from 0 to 1 that does not include the end points. This set is the collection of real numbers x satisfying $0 < x$ and $x < 1$.

1. Prove that $p = 1$ is a limit point of S.
2. State another limit point of S.

Definition: Set, subset, equality of sets. A set S is a collection of elements that have been grouped together.

- We use brackets $\{$ and $\}$ to enclose elements of sets.
- We write $x \in S$ to say that x is an element of S or x is in S. Similarly, we write $x \notin S$ to say that x is not in S.
- We say that a set B is a subset of the set S, and we write $B \subseteq S$, if every element in B is also an element of S. We also read $A \subseteq B$ as "A is contained in B." We'll often write $B \supseteq A$ instead of $A \subseteq B$, and read $B \supseteq A$ as either "B is a super set of A" or "B contains A."
- We say that B is a proper subset of S if $B \subseteq S$ but there is an element of S that is not in B.

In many circles the symbols $\subset$ and $\subseteq$ are used interchangeably. In this problem set, we'll avoid the symbol $\subset$ and use $\subseteq$ always. We can use the notation $A \subsetneq B$ to express that A is a subset of B but not equal to B.

- We say that two sets A and B are equal, and write $A = B$, if and only if both $A \subseteq B$ and $B \subseteq A$.

There are two general ways to express elements of a set. We often use the roster method where we list the elements of a set, as in $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. The roster method requires the reader to guess the remaining elements of the set, and hence can sometimes lead to unclear proofs. To avoid this potential confusion, we use set builder notation. With set builder notation, we express how to obtain the elements instead. For example the collection of numbers satisfying $0 < x < 1$ is given by the interval notaion $(0, 1)$, which we give by the set builder notation

$$(0, 1) = \{x \in \mathbb{R} \mid 0 < x \text{ and } x < 1\}.$$

The set of fractions of the form $\frac{1}{n}$ for some positive integer n is give by $\{\frac{1}{n} \mid \in \mathbb{N}\}$. Any time we see $\{x \mid P(x)\}$, we read this as the set of x such that statement $P(x)$ holds. We will also use $\{e(x) \mid P(x)\}$ to describe the collection of elements $e(x)$ such that statement $P(x)$ holds.

There are many different ways we can use set builder notation to describe the exact same set, and we need to be able to show when two different ways are equal. For an example, consider the interval $I = (5, 9]$. Let S be the collection of upper bounds of I, which we can write in set builder notation as

$$S = \{x \mid x \text{ is an upper bound of } I\}.$$

I claim that $S = \{x \mid x \geq 9\}$. Halt. This is a claim that two sets, namely S and the set $A = \{x \mid x \geq 9\}$ are equal. From the definition above about equality of sets, to prove this claim is true we must prove that $S \subseteq A$ and that $A \subseteq S$. That's precisely what the next problem has us prove.

Problem 6: First proof that two sets are equal

Let $I = (5, 9]$. Consider the sets $S = \{x \mid x \text{ is an upper bound of } I\}$ and $A = \{x \mid x \geq 9\}$. Prove that $S = A$.

1. Start by proving that $S \subseteq A$. Let $x \in S$, and then prove that $x \in A$.
2. Now prove that $A \subseteq S$.

We have seen that the truth value of many sentences cannot be determined without an appropriate context. This requires that we quantify any possible variable in the sentence. Do we consider all possible values of the variable over some range, or should we consider just one possible instance of the variable. Consider the open sentence "$x^2 + 5x + 6 = 0$." Two ways we can quantify what the variable x represents are given below.

- For all real numbers x, we have $x^2 + 5x + 6 = 0$.
- There exists a real number x such that $x^2 + 5x + 6 = 0$.

The first sentence is false (since when $x = 0$ we do not have $6 = 0$), and the second is true (let $x = -2$). We'll see the phrases "for every" and "there exists" quite often in mathematical writing. Because they occur so often, mathematicians have agreed upon some shorthand symbols for writing these phrases.

Definition: The Quantifiers $\forall$ and $\exists$.

- We'll use $\forall$ as shorthand in place of the phrases "for every," "for all," "for each," or any equivalent expressions that suggest for every possible case. We call $\forall$ the universal quantifier.

- We'll use $\exists$ as shorthand in place of the phrases "there exists," "there is at least one," or any equivalent expression that suggest there is at least one possible case. We call $\exists$ the existential quantifier.

We use these symbols often in open discussions, presentations, and informal work. However, when publishing formal papers, we avoid using these symbols and instead just write the words.

Problem 7: The Order Of Quantifiers Matters

Translate each of the following into an English sentence. Then determine the truth value of each statement. Prove your claims.

1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $y + 1 > x$.

2. $\exists y \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$, we have $y + 1 > x$.

Problem 8: Associativity Laws With Truth Tables

Let P, Q, and R be statements or open sentences. Use truth tables to prove each of the following.

1. Prove that $(P \wedge Q) \wedge R$ is equivalent to $P \wedge (Q \wedge R)$.

2. Prove that $(P \vee Q) \vee R$ is equivalent to $P \vee (Q \vee R)$.

3. Is $(P \vee Q) \wedge R$ equivalent to $P \vee (Q \wedge R)$.

Problem 9: Distributive Laws With Truth Tables

Let P, Q, and R be statements or open sentences. Use truth tables to prove each of the following.

1. Prove that $(P \wedge Q) \vee R$ is equivalent to $(P \vee R) \wedge (Q \vee R)$.

2. Prove that $(P \vee Q) \wedge R$ is equivalent to $(P \wedge R) \vee (Q \wedge R)$.

Definition: Infimum and supremum of a set. When a set S is bounded below, there are infinitely many lower bounds. The infimum of S is the greatest lower bound, which we write as $\inf S$. So if S is a set, then we write $m = \inf S$ if and only if both

- m is a lower bound for S, and

- m is the greatest lower bound for S (if m' is a lower bound, then $m \geq m'$).

When a set S is bounded above, there are infinitely many upper bounds. The supremum of S is the least upper bound, which we write as $\sup S$. So if S is a set, then we write $m = \sup S$ if and only if both

- m is an upper bound for S, and

- m is the least upper bound for S (if m' is a upper bound, then $m \leq m'$).

In mathematics there are some ground rules that we cannot prove, rather they are just accepted as true without proof. We call these axioms. The following axiom just gives words to a fact that you may think is pretty obvious (something axioms tend to be).

Axiom: Archimedean property. Given a real number x, there exists a natural number n that is larger than x. In other words, the set of natural numbers is not bounded above.

You may need this axiom to justify some of your work in the next problem.

Problem 10: Practice with bounded definitions 2

Let S be the set

$$S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}.$$

1. Give two different upper bounds for S. Then give the supremum of S.
2. Give two different lower bounds for S. Then give $\inf S$.

Remember, any time you make a claim, you must prove your claim.

Here is another axiom that we'll come back to several times this semester.

Axiom: The completeness axiom. Every nonempty set of real numbers that is bounded above has a least upper bound. Equivalently, every nonempty set of real numbers that is bounded below has a greatest lower bound.

Problem 11: Using the completeness axiom

Consider the rational solutions to the inequality $x^2 < 2$. We can write this using set builder notation as

$$S = \{x \mid x \in \mathbb{R} \text{ and } x^2 < 2\} \quad \text{or just} \quad S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

1. Is S bounded above? Prove your claim.
2. What does the completeness axiom say about this set?
3. Give $\sup S$, or explain why there is no supremum of S.

Remember, any time you make a claim, you must prove your claim.

Problem 12: Creating A Truth Table For An Implication

Suppose your teacher tells you the following "If you pass the final, then you pass the class." This sentence contains a construction of the form "If P, then Q."

1. There are four different scenarios that might occur as you may or may not pass the final, and you may or may not pass the class. List the four scenarios and decide in each scenario if the teacher lied.
2. Suppose that P and Q are statements. Use your answer to the previous part to construct a truth table for the statement "If P then Q."

Definition: Implication. If P and Q are statements or open sentences, then an implication, written symbolically as $P \implies Q$, is the sentence "If P, then Q" or equivalently "P implies Q." There are several equivalent ways to express this sentence such as "Q if P" or "P only if Q." We will define the implication $P \implies Q$ to be true unless P is true and Q is false.

Definition: Converse, Inverse, And Contrapositive. Consider the implication $P \implies Q$. From this implication we can define 3 other implications.

- The converse of $P \implies Q$ is the implication $Q \implies P$.

- The inverse of $P \implies Q$ is the implication $(\sim P) \implies (\sim Q)$.
- The contrapositive of $P \implies Q$ is the implication $(\sim Q) \implies (\sim P)$.

Problem 13: Converse, Inverse, And Contrapositive Practice

Let $A = [3, 7)$. Consider, "If $x \geq 8$ then x is an upper bound of A."

1. Is this implication true or false?
2. Write the converse of this implication and determine the truth value of the converse.
3. Write the inverse of this implication and determine the truth value of the inverse.
4. Write the contrapositive of this implication and determine the truth value of the contrapositive.

Remember to always justify any claims you make.

Problem 14: A set with one limit point

Let $S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$, the collection of fractions of the form $\frac{1}{n}$ where n is a natural number. Prove that $p = 0$ is a limit point of S.

Definition: Union And Intersection. Let A and B be sets.

A Venn diagram is a great way to visual the intersection and union of finite sets, and even a great way to conceptualize relationships for infinite sets.

- The intersection of A and B, written $A \cap B$, is a new set whose elements are those that are in A and in B. Using set builder notation, we can write the intersection as

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

- The union of A and B, written $A \cup B$, is a new set whose elements are those that are in A or in B (or both). Using set builder notation, we can write the union as

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Problem 15: Intersection Of Two Intervals

Suppose that $a, b, c, d \in \mathbb{R}$ and that $a < b < c < d$. Consider the intervals $A = (a, c)$ and $B = [b, d]$. Prove that $A \cap B = [b, c)$ by doing the following:

1. First, prove that $A \cap B \subseteq [b, c)$. Your proof could start with "Let $x \in A \cap B$." After writing what this means, and organizing these facts, you will need to eventually end with, "This shows $x \in [b, c)$."
2. Second, prove that $[b, c) \subseteq A \cap B$.

The next theorem lists several facts about set unions and intersections that we would like to refer to from here on out.

Theorem 1 (Union And Intersection Properties)**.** *Let A, B, C be sets. Then the following facts are true.*

1. *$A \subseteq A$ (Every set is a subset of itself.)*
2. *$A \subseteq A \cup B$ (A set is a subset of the union of itself and another set.)*
3. *$A \cap B \subseteq A$ (The intersection of two sets is a subset of the first set.)*
4. *$A \cup B = B \cup A$ (Set unions are commutative.)*
5. *$A \cap B = B \cap A$ (Set intersections are commutative.)*
6. *$A \cup (B \cup C) = (A \cup B) \cup C$ (Set unions are associative.)*
7. *$A \cap (B \cap C) = (A \cap B) \cap C$ (Set intersections are associative.)*
8. *$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$*
9. *$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$*

These last two we call the distributive laws for unions and intersections.

Take some time to prove the theorem above is true. The next exercise includes several of the proofs. I'll leave some for you to prepare for class.

Exercise: Union And Intersection Properties. Prove the statements in the Union and Intersection Properties theorem.

Proof.

1. To prove $A \subseteq A$, we let $a \in A$ (the first set). Then we know $a \in A$ (the second set) which means $A \subseteq A$ as desired.
2. We now prove $A \subseteq A \cup B$. Let $a \in A$. Then clearly $a \in A$ or $a \in B$. This means that $a \in A \cup B$, which proves that $A \subseteq A \cup B$.
3. We now prove $A \cap B \subseteq A$. Let $x \in A \cap B$. This means $x \in A$ and $x \in B$. In particular, notice that we know $x \in A$. This completes the proof that $A \cap B \subseteq A$.
4. We now prove $A \cup B = B \cup A$. Let $y \in A \cup B$. Then we know $y \in A$ or $y \in B$. This is equivalent to $y \in B$ or $y \in A$, which means $y \in B \cup A$. Hence we've shown $A \cup B \subseteq B \cup A$. The proof that $B \cup A \subseteq A \cup B$ is similar. Thus $A \cup B = B \cup A$.
5. The proof that $A \cap B = B \cap A$ is almost identical to the previous.
6. This is the next problem.
7. This is the next problem.
8. We now prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. We first show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Let $x \in A \cup (B \cap C)$. This means that $x \in A$ or $x \in B \cap C$. We must show that $x \in A \cup B$ and that $x \in A \cup C$. There are two cases, namely $x \in A$ or $x \notin A$. Suppose first that $x \in A$. Then clearly $x \in A \cup B$ and $x \in A \cup C$ as x is a member of the first set in each union. This shows that $x \in (A \cup B) \cap (A \cup C)$. Now suppose $x \notin A$. We then know that $x \in B \cap C$. Hence we know that $x \in B$ and $x \in C$. Since $x \in B$, we know $x \in A \cup B$. Since $x \in C$, we know $x \in A \cup C$. This shows that $x \in (A \cup B) \cap (A \cup C)$ as desired. This completes the proof that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

   To finish, we must prove $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Let $y \in (A \cup B) \cap (A \cup C)$. We again use two cases. Suppose $y \in A$. Then clearly $y \in A \cup (B \cap C)$ by definition of union. The only other option is $y \notin A$. Recall we assumed that $y \in (A \cup B) \cap (A \cup C)$, which means $y \in (A \cup B)$ and $y \in (A \cup C)$. This means $y \in A$ or $y \in B$, and it means $y \in A$ or

$y \in C$. Since we have assumed that $y \notin A$, this means that $y \in B$, and it means that $y \in C$. Together, this gives $y \in B \cap C$, which shows that $y \in A \cup (B \cap C)$.

9. Your proof should be very similar to the previous.

□

Problem 16: Associative Laws For Set Unions And Intersections

Let A, B, and C be sets.

1. Prove that $A \cup (B \cup C) = (A \cup B) \cup C$.
2. Prove that $A \cap (B \cap C) = (A \cap B) \cap C$.

Problem 17: Negating Quantifiers

Rewrite each statement below using the quantifiers $\forall$ and/or $\exists$. Then write the negation of each statement.

1. For each $x \in \mathbb{N}$ we have $x > 4$.
2. There exists $y \in \mathbb{R}$ such that $y \in (-3, 4)$.
3. For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$.

An important set that will show up often throughout the semester is the set with nothing in it, which we call the empty set.

Definition: The empty set. The empty set is the set $\emptyset = \{\}$ that contains no elements. If we think of a set as a box with elements in it, then the empty set is a box with nothing in it.

Here is another axiom that you have probably used many times in your life without ever realizing it.

Axiom: Well-ordering principle. Every nonempty subset S of the natural numbers has a least element. By least element, we mean that there is a natural number m which is an element of S such that $m \leq x$ for every x in S.

Problem 18: Which dominoes remain standing

Suppose that Jon has set up an infinite number of dominoes, with the dominoes numbered $1, 2, 3, \ldots$. The dominoes are set up so that if the kth domino falls, then the $(k+1)$st domino will also fall. So if the 7th domino falls, then the 8th must fall as well. Jon knocks down the first domino, which starts causing other dominos to fall. Which dominos fall? Which dominoes remain standing? Make sure you prove your result. The well ordering principle will come in handy.

Suggestion: Use set builder notation to help you. Let $F = \{n \in \mathbb{N} \mid \text{domino } n \text{ fell}\}$ and $S = \{n \in \mathbb{N} \mid \text{domino } n \text{ stands}\}$. Then make some claims and prove they are correct.

Problem 19: What Is Logically Equivalent To An Implication

Consider the implication $P \implies Q$.

1. Construct a truth table that contains the possible values for this implication, the converse, the inverse, and the contrapositive. Feel free to use the table at the end of this problem to complete your work.

2. Which of these statements are logically equivalent?

P	Q	$P \implies Q$	$Q \implies P$	$\sim P$	$\sim Q$	$(\sim P) \implies (\sim Q)$	$(\sim Q) \implies (\sim P)$
T	T						
T	F						
F	T						
F	F						

Problem 20: Creating Examples Of Implications

Give an example of each of the following, or explain why it cannot be done. Make sure you justify your claims (as always).

1. An implication that is true, and the converse is true.
2. An implication that is false, but the converse is true.
3. An implication that is true, but the contrapositive is false.

Problem 21: The Negation Of An Implication Is A Conjunction

Let P and Q be statements, or open sentences. In this problem we discover a logically equivalent expression for the negation of $P \implies Q$.

1. In a truth table for the implication $P \implies Q$, how many of the 4 rows contain the truth value T?
2. In a truth table for the negation of the implication $P \implies Q$, how many of the 4 rows contain the truth value T? Based off this answer, explain why we expect the negation of an implication to be a conjunction.
3. Complete the truth table below, and use your answer to determine which statement is logically equivalent to $\sim (P \implies Q)$.

P	Q	$P \implies Q$	$\sim (P \implies Q)$	$P \wedge Q$	$P \wedge (\sim Q)$	$(\sim P) \wedge Q$	$(\sim P) \wedge (\sim Q)$
T	T						
T	F						
F	T						
F	F						

Problem 22: Second proof that two sets are equal

Let $I = (5, 9]$. Consider the sets $T = \{x \mid x \text{ is a lower bound of } I\}$ and $B = \{x \mid x \leq 5\}$. Prove that $T = B$.

Problem 23: Union Of Two Intervals

Suppose that $a, b, c, d \in \mathbb{R}$ and that $a < b < c < d$. Consider the intervals $A = (a, c)$ and $B = [b, d]$. Prove that $A \cup B = (a, d]$.

Did you show that $A \cup B \subseteq (a, d]$ and then that $(a, d] \subseteq A \cup B$?

Problem 24: Finding Truth Values With Universal Quantifiers

Determine the truth value of each statement below. Be prepared to justify your claim.

1. $\forall x \in \mathbb{R}$ and $\forall y \in \mathbb{R}$, $\exists z \in \mathbb{R}$ such that $x + y = z$.
2. $\forall x \in \mathbb{R}$ and $\forall y \in \mathbb{R}$, $\exists z \in \mathbb{R}$ such that $xz = y$.
3. $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $\forall z \in \mathbb{R}$, $z > y$ implies $z > x + y$.
4. $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, $\exists z \in \mathbb{R}$ such that $z > y$ implies $z > x + y$.

Definition: Maximum And Minimum. Let $S \subseteq \mathbb{R}$.

- We say m is a minimum of S if m is a lower bound for S and $m \in S$. We write $\min S$ for the minimum of S.
- We say m is a maximum of S if m is an upper bound for S and $m \in S$. We write $\max S$ for the maximum of S.

Problem 25: Minimums And Maximums Are Unique

Prove that a minimum of set S, if it exists, is unique. In other words, prove that if m_1 and m_2 are both minimums of S, then we must have $m_1 = m_2$.

A similar proof will show that maximums are unique.

Problem 26: Limit Points Of A Singleton Set

Let $a \in \mathbb{R}$. Suppose $S = \{a\}$, so a set with a single number. What are the limit points of S? Prove your claim.

Problem 27: The Empty Set Is A Subset Of Every Set

Prove or disprove that the empty set is a subset of every set. In other words, is the statement, "If S is a set, then we have $\emptyset \subseteq S$," true or false.

We have already talked about the union and intersection of sets. Here are two more set operations that we'll need. They are the set complement and the Cartesian product. The next 4 problems have you prove some facts about these set operations.

Definition: Set Complement And Cartesian Product. Let A and B be sets.

- The complement of B in A is the set of elements in A that are not in B. We can write this in set builder notation as
$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$
- The Cartesian product (or cross product, or product) of A and B is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$. We can write the product in set builder notation as
$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Theorem 2 (Rules For Set Complements)**.** *Let A, B, and C be sets. Then the following statements are true.*

1. $A \setminus A = \emptyset$

2. $A \setminus \emptyset = A$

3. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

4. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

5. $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$

Problem 28: Set Complements Rules 1 And 2

Prove rules 1 and 2 of the *Rules For Set Complements* theorem. So let A, B, and C be sets. Prove the following are true.

1. $A \setminus A = \emptyset$
2. $A \setminus \emptyset = A$

Here is an exercise, with a solution, to help you with limit points. The problem following this exercise will give you some more practice with limit points.

Exercise: Points In An Interval Are Limit Points. Let $a, b \in \mathbb{R}$ with $a < b$. Let $M = [a, b]$. Prove that if $p \in M$, then p is a limit point of M.

Proof. We'll examine two solutions below. The first solution uses the supremum and infimum. The second solution doesn't use these words at all, but rather uses the same ideas needed to prove facts about the infimum and supremum of a set. Please read both proofs. Come with questions if you have any.

1. (Solution using infimum and supremum of (a, b).) Pick $p \in M = [a, b]$. There are three cases to consider, namely $p = a$, $p = b$, and $p \in (a, b)$. We first suppose $p = a$. Let $I = (c, d)$ be an open interval that contains $p = a$. Since a is the infimum of (a, b), we know that any number larger than a cannot be a lower bound of (a, b). This means that d is not a lower bound for (a, b), so we can pick a number x between a and d such that $x \in (a, b)$. Since $a < x < d$, we know $x \in I$ and $x \neq a$. Since $x \in (a, b) \subseteq [a, b]$, we know that $x \in M$. This completes that proof that $p = a$ is a limit point of M.

   To prove that $p = b$ is a limit point of M, we use similar reasoning as above. Given an interval $I = (c, d)$ that contains b, we use the fact that b is the supremum of (a, b) to obtain a number $x \in (a, b)$ that lies between c and b (possible since c is not an upper bound of (a, b). We know $x \in M$ since $x \in (a, b)$. We also know $c < x < b$ which means $x \in I$ and $x \neq b$. This proves $p = b$ is a limit point of M.

   To finish the proof, we now assume that $p \in (a, b)$ and must prove that p is a limit point of M. Let $I = (c, d)$ be an open interval that contains p. We must produce a number x such that $x \in I$, $x \in M$, and $x \neq p$. There are lots of ways to proceed, so what follows is not the only option. Let's look to the right of p. We know that both b and d are greater than p. All we need to do is pick a value for x that is larger than p but less than both b and d. How do we do this? We use the fact that between any two real numbers, there is another real number. If $b < d$, then we pick $x \in (p, b)$. Otherwise, we know $d \leq p$ and we pick $x \in (p, d)$. So basically, we chose a number x between p and the smaller of b and d. In either case, we have $p < x < b$ (hence $x \in M$) and $p < x < d$ (hence $x \in I$). Since $p < x$, we know $p \neq x$. This produces the needed value of x to finish the proof that p is a limit point of M.

2. (Solution without infimum or supremum of (a, b).) Pick $p \in M = [a, b]$. There are two cases to consider, namely $p \in [a, b)$ and $p = b$. We first let $p \in [a, b)$. Let $I = (c, d)$ be an open interval that contains p. Since both d and b are greater than p, we pick a number x that is greater than p and less than the smaller of d and b. Since $c < p < x < d$, we know $x \in I$. Since $a \leq p < x < b$, we know $x \in M$. Since $p < x$, we know $p \neq x$. This completes that proof that $p \in [a, b)$ is a limit point of M.

   To prove that $p = b$ is a limit point of M, we use similar reasoning as above, but this time pick a point left of $p = b$ rather than above it. Given an interval $I = (c, d)$ that contains b, we pick a number x that is less than b and greater than the larger of a and c. Since $c < x < b$, we know $x \in I$. Since $a < x < b$, we know $x \in M$. Since $x < b$, we know $b \neq x$. This proves $p = b$ is a limit point of M.

□

Problem 29: Closed Interval And Non Limit Points

Let $a, b \in \mathbb{R}$, with $a < b$. Let $M = [a, b]$. Prove that if $p \notin M$, then p is not a limit point of M.

Problem 30: Periodic Functions And Practice With Quantifiers

Consider the following definition:

> We say that a function $y = f(x)$ is periodic over the reals if there exists a positive real number k such that $f(x + k) = f(x)$ for every real number x.

1. Rewrite the definition above using the symbols $\exists$ and $\forall$.
2. Prove that $y = \sin(x)$ is periodic over the reals.
3. Finish the following: “We say that a function $y = f(x)$ is not periodic over the reals if ...”
4. Prove that $y = x^2$ is not periodic over the reals.

Problem 31: First Induction Problem

For each $n \in \mathbb{N}$ consider the statement

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

This is infinitely many statements. How do we prove infinitely many statements are true? What follows is a proof that all of these statements are true. The proof relies on the domino problem done earlier in the semester. Part way through the proof are several different versions of how to proceed. Your job is to read the proof, with the various different versions, and then decide for each version if it is logically sound and clear.

Option	Logically Sound (Y/N)	Clear (Y/N)
1		
2		
3		
4		
5		
6		

We'll have a discussion in class about your answers. Now for the proof.

To simplify our writing below, for each $n \in \mathbb{N}$ let S_n be the statement

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

We will prove that for each $n \in \mathbb{N}$, we know that S_n is a true statement. To do this, we'll refer to the domino problem we proved earlier in the semester. Imagine for a minute that we have an infinite collection of dominos numbered $1, 2, 3, \ldots$. The first domino has the statement $1 = \frac{1(2)}{2}$ on it, the second domino has the statement $1+2 = \frac{2(3)}{2}$ on it, the third domino has the statement $1+2+3 = \frac{3(4)}{2}$ on it, and so on. For each $n \in \mathbb{N}$, domino n has the statement S_n on it. We now have an infinite collection of dominos, one for each natural number n. We will now knock over each domino when we know the statement written on that domino is true. If we can knock over every domino, then S_n is true for every $n \in \mathbb{N}$.

In the domino problem, two important things happened. First, note that Jon knocked over the first domino. Second, we knew that "if the kth domino falls, then it knocks over the $(k+1)$st domino." We will verify these two facts are true about our dominos.

The first domino clearly falls, as the statement on the first domino is $1 = \frac{1(2)}{2}$, which is true. This proves that S_1 is a true statement. We now must prove the second condition is true, namely "if the kth domino falls, then it knocks over the $(k+1)$st domino."

Consider each of the following 6 options (starting on the next page). Note that only 2 of them are logically sound, clear, and concise. Three of them are not logically sound, for various reasons. One of the options is very close to correct, however is not clear. This unclear option may or may not be logically sound, and without more information it is impossible to determine.

Option 1: For each $k \in \mathbb{N}$ suppose domino k fell. This means statement S_k is true, which means $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ is true. We then compute

$$\begin{aligned} \underbrace{1 + 2 + 3 + \cdots + k} + (k+1) &= \frac{k(k+1)}{2} + (k+1) && \text{(replace using the assumption)} \\ &= (k+1)\left[\frac{k}{2} + 1\right] && \text{(factor)} \\ &= \frac{(k+1)(k+2)}{2} && \text{(get a common denominator).} \end{aligned}$$

This proves that $1 + 2 + \cdots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$, which means S_{k+1} is a true statement. This shows that if statement S_k is true, then statement S_{k+1} must be true as well.

Option 2: Suppose for some $k \in \mathbb{N}$ that domino k fell. This means statement S_k is true, which means $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ is true. We then compute

$$\begin{aligned} \underbrace{1 + 2 + 3 + \cdots + k} + (k+1) &= \frac{k(k+1)}{2} + (k+1) && \text{(replace using the assumption)} \\ &= \frac{k^2 + 3k + 2}{2} && \text{(get a common denominator)} \\ &= \frac{(k+1)(k+2)}{2} && \text{(factor).} \end{aligned}$$

This proves that $1 + 2 + \cdots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$, which means S_{k+1} is a true statement. This shows that if statement S_k is true, then statement S_{k+1} must be true as well.

Option 3: Suppose that domino k fell where $k \in \mathbb{N}$. This means statement S_k is true, which means $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ is true. We then compute

$$\begin{aligned} \underbrace{1 + 2 + 3 + \cdots + k} + (k+1) &= \frac{k(k+1)}{2} + (k+1) && \text{(replace using the assumption)} \\ &= (k+1)\left[\frac{k}{2} + 1\right] && \text{(factor)} \\ &= \frac{(k+1)(k+2)}{2} && \text{(get a common denominator).} \end{aligned}$$

This proves that $1 + 2 + \cdots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$, which means S_{k+1} is a true statement. This shows that if statement S_k is true, then statement S_{k+1} must be true as well.

Option 4: For each $k \in \mathbb{N}$ suppose domino k fell. This means statement S_k is true, which means $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ is true. Since this assumption is true for every $k \in \mathbb{N}$, it must also be true when we look at the integer $k+1$. This means that statement S_{k+1} is true, which means domino $k+1$ fell as needed. This shows that if statement S_k is true, then statement S_{k+1} must be true as well.

Option 5: Suppose for some $k \in \mathbb{N}$ that domino k fell. This means statement S_k is true, which means $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ is true. We must prove that statement S_{k+1} is true, which means we must prove

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}.$$

We can replace the $1 + 2 + \cdots + k$ with $\frac{k(k+1)}{2}$ in the above equation to get

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)((k+1)+1)}{2}.$$

The left side simplifies to $\frac{k(k+1)}{2} + (k+1) = \frac{k^2+k+2k+2}{2} = \frac{k^2+3k+2}{2}$. The right side simplifies to $\frac{(k+1)((k+1)+1)}{2} = \frac{(k+1)(k+2)}{2} = \frac{k^2+3k+2}{2}$. Since the left and right side both simplify to the same thing, then the original statement S_{k+1} is true. This shows that if statement S_k is true, then statement S_{k+1} must be true as well.

Option 6: Suppose for some $k \in \mathbb{N}$ that domino k fell. This means statement S_k is true, which means $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ is true. We must prove that statement S_{k+1} is true, which means we must prove

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}.$$

Because we know $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ (from our assumption that S_k is true), substitution shows that statement S_{k+1} is equivalent to the statement

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)((k+1)+1)}{2}.$$

The left side of the above equation is equal to $\frac{k(k+1)}{2} + (k+1) = \frac{k^2+k+2k+2}{2} = \frac{k^2+3k+2}{2}$. This means that statement S_{k+1} is equivalent to

$$\frac{k^2+3k+2}{2} = \frac{(k+1)((k+1)+1)}{2}.$$

This last statement is clearly true as $(k+1)((k+1)+1) = (k+1)(k+2) = k^2 + 3k + 2$. This completes the proof that statement S_{k+1} is true. This shows that if statement S_k is true, then statement S_{k+1} must be true.

We conclude by referring to the domino problem. Since the first domino was knocked over, and since the kth domino falling implies the $(k+1)$st domino falls, then we know that every domino falls (the conclusion of the domino problem). This means that S_n is a true statement for every $n \in \mathbb{N}$.

We are ready to make a formal statement of what we call "The principle of mathematical induction." The next problem asks you to prove this. If your proof looks very similar to the domino problem, then you are doing this correctly.

Theorem 3 (The Principle Of Mathematical Induction)**.** *If S is a subset of the natural numbers such that*

- *1 is an element of S, and*
- *if $k \in S$, then $k+1 \in S$*

then we must have $S = \mathbb{N}$.

Problem 32: Proof Of Mathematical Induction

Use the well ordering principle to prove the principle of mathematical induction.

Problem 33: Distribution With Cartesian Products

Let A, B, and C be sets. Prove or disprove each statement.

1. $(A \cap B) \times C = (A \times C) \cap (B \times C)$
2. $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Problem 34: Relation Between Minimums And Infimums

Suppose S is a set of real numbers.

- If m is the minimum of S, must m be the infimum of S? Prove your claim.
- If m is the infimum of S, must m be the minimum of S? Prove your claim.

Similar facts hold true for the maximum and supremum of a set.

The next two problems have you prove the exact same thing, but in two different ways. The first asks you to prove a statement is true by proving the contrapositive is true. The second asks you to prove that the statement is true by proving that the negation is false. The two proofs will be quite similar, with subtle differences.

Problem 35: Proof By Contrapositive

Let a and b be real numbers with $a < b$. Consider the set $S = (a, b) = \{x \in \mathbb{R} \mid a < x < b\}$. We know b is an upper bound for this set as if $x \in S$, then by definition of S we have $x < b$, which clearly implies $x \leq b$. To show that b is the supremum of S, we must show "If m is an upper bound of S, then $b \leq m$." This is an implication of the form "If P, then Q."

1. State the contrapositive of this implication.
2. Prove that the implication is true by proving that the contrapositive is true.

Problem 36: Proof By Contradiction

Let a and b be real numbers with $a < b$. Consider the set $S = (a, b) = \{x \in \mathbb{R} \mid a < x < b\}$. We know b is an upper bound for this set as if $x \in S$, then by definition of S we have $x < b$, which clearly implies $x \leq b$. To show that b is the supremum of S, we must show "If m is an upper bound of S, then $b \leq m$." This is an implication of the form "If P, then Q."

1. State the negation of this implication.
2. Prove that the implication is true by proving that the negation is false.

As a corollary to the previous problems, similar reasoning shows that if S is an interval (of the form (a, b), $[a, b)$, $(a, b]$, or $[a, b]$), then we have $\sup S = b$ and $\inf S = a$. You may use these facts now without proof.

Problem 37: Set Complements Rules 3 And 4

Prove rules 3 and 4 of the *Rules For Set Complements* theorem. So let A, B, and C be sets. Prove the following are true.

3. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
4. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

We use Cartesian products all the time without realizing it. When we look for relationships between two groups, we are looking at a subset of a Cartesian product. Family history involves looking at a Cartesian product. For example, we could let A and B both be the set of all humans that have lived on the planet. In our family tree, we often care to know when person x descends from person y. One goal of family history is to determine all the pairs (x, y) where person x descends from person y. This is a subset of $A \times B$. Similarly, any time we start grouping objects together from two sets, we're formally stating a relationship between the two sets and looking for a subset of the Cartesian product. We call these subsets relations.

Definition: Relation Between Two Sets. Let A and B be sets. A relation R between A and B is a subset of $A \times B$, so $\mathrm{R} \subseteq A \times B$. Given $a \in A$ and $b \in B$, we write $(a, b) \in \mathrm{R}$ precisely when a is related to b.

A function is one important kind of relation that we've been studying since middle school. Here's a formal definition.

Definition: Function. Let A and B be sets. A function f from A into B, written $f : A \to B$, is a relation between A and B (so $f \subseteq A \times B$) such that for every $x \in A$, there exists a unique $y \in B$ such that $(x, y) \in f$. When f is a function from A into B, we'll use the notation $y = f(x)$ or $f(x) = y$ rather than the more cumbersome notation $(x, y) \in f$ used for sets.

Problem 38: Which Relations Are Functions

In each number below, you are given a relation R between a set A and a set B. Prove that the relation is a function from A into B or give a counter example to show that the relation is not a function from A into B.

1. Let R be the relation between $\mathbb{N}$ and $\mathbb{Z}$ given by $(n, m) \in \mathrm{R}$ if and only if $n^2 = m$.
2. Let R be the relation between $\mathbb{N}$ and $\mathbb{Z}$ given by $(n, m) \in \mathrm{R}$ if and only if $n = m^2$.
3. Let R be the relation between $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$ given by $((x, y), z) \in \mathrm{R}$ if and only if $x + y = z$.
4. Let R be the relation between $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$ given by $((x, y), z) \in \mathrm{R}$ if and only if $x/y = z$.

Problem 39: Practice With Universal Quantifiers Take 2

For each statement below, first write the negation. Then determine whether the statement or the negation is true, and justify your claim. In your work below, assume that $f(x) = x^2$.

1. $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $\forall z \in \mathbb{R}$ we have $x + y = z$.

2. $\forall x, y \in \mathbb{R}$, $\exists z \in \mathbb{R}$ such that $x + z = y$.
3. $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{Z}$, $x < y$ implies $f(x) < f(y)$.
4. $\forall x, y \in \mathbb{R}$, if $f(x) = f(y)$ then $x = y$.

Definition: Epsilon Neighborhoods And Deleted Neighborhoods. Given $\varepsilon > 0$, an ε-neighborhood of the real number x is the interval

$$N_\varepsilon(x) = (x - \varepsilon, x + \varepsilon) = \{y \in \mathbb{R} \colon |x - y| < \varepsilon\}.$$

A deleted ε-neighborhood of x is the same interval minus the point x, which we'll write as

$$N^*_\varepsilon(x) = N_\varepsilon(x) \setminus \{x\} = (x - \varepsilon, x) \cup (x, x + \varepsilon) = \{y \in \mathbb{R} \colon 0 < |x - y| < \varepsilon\}.$$

Problem 40: Practice With Neighborhoods

Consider the interval $S = (1, 8)$.

1. Find real numbers x and ε so that $S = N_\varepsilon(x)$.
2. Find real numbers x and ε so that $N_\varepsilon(x)$ is a proper subset of S.
3. For every $x \in S$, give a formula for ε so that $N_\varepsilon(x) \subseteq S$. Your choice of ε will depend on x.

Problem 41: Limit Points And Subsets

Suppose that A and B are subsets of the real numbers. Prove that if $A \subseteq B$ and p is a limit point of A, then p is a limit point of B.

Problem 42: Induction With The Sum Of Squares

Use induction to prove that $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for every $n \in \mathbb{N}$. Feel free to use the hint below this problem if needed.

The Standard Induction Hint - this is the same hint for all induction problems.

For each $n \in \mathbb{N}$, let S_n be a statement. Let S be the set of natural numbers for which statement S_n is true. So in the previous problem, we let S_n be the statement $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$, and then let

$$S = \{n \in \mathbb{N} \mid S_n \text{ is true}\}.$$

We want to show that $S = \mathbb{N}$.

1. We first show that S_1 is true, which shows that $1 \in S$.
2. We then assume that S_k is true for some $k \in \mathbb{N}$.
3. We use this assumption to then prove S_{k+1} is true. (This shows if $k \in S$, then $k + 1 \in S$.)

The principle of mathematical induction now gives $S = \mathbb{N}$.

Problem 43: More Practice With Universal Quantifiers

For each $n \in \mathbb{N}$, define $a_n = \dfrac{n-1}{n}$. Only one of the statements below is true. Rewrite each statement using the quantifiers $\forall$ and $\exists$. Then determine which statement is true, prove that statement is true, and then prove the other statement is false.

1. For each real number $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n > N$ then $|a_n - 0| < \varepsilon$.

2. For each real number $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n > N$ then $|a_n - 1| < \varepsilon$.

Definition: Domain And Codomain. Let A and B be sets and let f be a function from A into B, so we have $f : A \to B$.

- We call the set A the domain of f.
- We call the set B the codomain of f.

Definition: Injective, Surjective, And Bijective. Let D and R be sets and let f be a function from D into R, so we have $f : D \to R$.

- We say that f is injective (or one-to-one) if and only if for every $a, b \in D$ we have $f(a) = f(b)$ implies $a = b$.
- We say that f is surjective (or onto) if and only if for every $y \in R$ there exists an $x \in D$ such that $y = f(x)$.
- We say that f is bijective if and only if the function f is both injective and surjective.

Problem 44: Practice With Injective And Surjective

For each function below, state the domain and codomain, determine if the function is injective, and then determine if the function is surjective.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$.
2. Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^2$.
3. Let $f : \mathbb{R} \to [0, \infty)$ be defined by $f(x) = x^2$.
4. Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = x^2$.

As always, remember to justify each claim you make.

Problem 45: Proving A Set Has No Minimum

Let $S = (3, 7]$. Prove that S has no minimum.

Problem 46: Set Complements Rule 5

Prove rule 5 of the *Rules For Set Complements* theorem. So let A, B, and C be sets. Prove the following is true.

5. $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$

Definition: Interior Point, Open Set, and Closed Set. Let $S \subseteq \mathbb{R}$.

- We say that x is an interior point of S if and only if there exists an $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq S$.
- The interior of S is the collection of interior points of S.
- We say that S is an open set if and only if for every $x \in S$ there exists an $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq S$ (so every point in S is an interior point, or equivalently S equals the interior of S).
- We say that S is a closed set if and only if the complement $\mathbb{R} \setminus S$ is open.

Problem 47: Open Intervals Are Open Sets

Prove that if a and b are real numbers such that $a < b$, then the interval $S = (a, b)$ is an open set.

Exercise: Induction With The Sum Of Cubes. Prove that for every $n \in \mathbb{N}$, we have

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Proof. Consider the set

$$S = \left\{ n \in \mathbb{N} \mid 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \right\}.$$

We need to prove that $S = \mathbb{N}$. We proceed by induction. We know $1 \in S$ because $1^3 = 1$ and $\left(\frac{1(1+1)}{2}\right)^2 = (1)^2 = 1$. Assume for some $k \in \mathbb{N}$ that $k \in S$, which means we've assumed that

$$1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2.$$

We need to prove that $k + 1 \in S$, which means we need to prove that

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{(k+1)((k+1)+1)}{2}\right)^2.$$

We'll start with the expression $1^3 + 2^3 + \cdots + k^3 + (k+1)^3$ and modify it to obtain the right hand side of the equation above. We now compute

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \left(1^3 + 2^3 + \cdots + k^3\right) + (k+1)^3 && \text{(group the first } k \text{ terms)} \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{(substitute using our assumption)} \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} && \text{(prepare to combine fractions)} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} && \text{(combine and factor)} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} && \text{(expand)} \\
&= \frac{(k+1)^2(k+2)^2}{4} && \text{(factor)} \\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2 && \text{(note } k+2 = (k+1)+1\text{).}
\end{aligned}$$

This shows that if $k \in S$, then $k+1 \in S$. By mathematical induction, we know that $S = \mathbb{N}$. This prove that for every $n \in \mathbb{N}$, we have

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

□

Problem 48: Induction With Sum Of Odds

Notice that

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \\ 1+3+5+7+9 &= 25. \end{aligned}$$

Make a conjecture about the sum of odd numbers by giving a general formula for what you see above. Use mathematical induction to prove your result.

Problem 49: The Integers Have No Limit Points

There are three parts to this problem.

1. Start by writing the definition of a limit point p of a set S using the quantifiers $\forall$ and $\exists$. Feel free to use set operations $\cup$ and/or $\cap$ in your definitions.
2. Then write, using these quantifiers, what it means to not be a limit point.
3. Finish by proving that if $p \in \mathbb{R}$, then p is not a limit point of $\mathbb{Z}$. In other words, prove that $\mathbb{Z}$ has no limit points.

Problem 50: Subsets, Infimums, And Supremums

Suppose that S and T are nonempty bounded subsets of $\mathbb{R}$ and that $S \subseteq T$. Prove that

$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

Definition: Sequence. A sequence of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$.

- We'll often use the notation a_n rather than $a(n)$ to denote the value of the sequence at n. We call a_n the nth term of the sequence.
- We may refer to the sequence by writing (a_n) or by writing $(a_1, a_2, \ldots)$.
- We may change the domain from $\mathbb{N}$ to $\mathbb{N} \cup \{0\}$ by adding in 0, or we might remove several of the first entries from the sequence. When we do this, we'll use the notation $(a_n)_{n=m}^{\infty}$ where m is the first term we want to consider.

Definition: Convergent Sequence. Let $(a_n) = (a_1, a_2, \ldots)$ be a sequence of real numbers.

- We say that (a_n) converges to L and write $(a_n) \to L$ if and only if for every $\varepsilon > 0$, there exists a real number M such that for every $n \in \mathbb{N}$ we have $n > M$ implies $|a_n - L| < \varepsilon$.
- When (a_n) converges to L, we call L the limit of (a_n).
- We say that (a_n) is a convergent sequence if and only if (a_n) converges to L for some real number L.
- We say that (a_n) is a divergent sequence if and only if (a_n) is not a convergent sequence.

Problem 51: Showing A Sequence Converges

Consider the sequence $(a_n) = \left(\dfrac{n+1}{n}\right)$. Prove that (a_n) converges to $L = 1$.

Definition: Accumulation Point Of A Set. Let S be a set of real numbers. We say that a real number a is an accumulation point of S if for every $\varepsilon > 0$, we have $N^*_\varepsilon(a) \cap S \neq \emptyset$.

The definition of accumulation point should look very similar to the definition of limit points. The next problem has you make this precise. You'll see a statement of the for "P if and only if Q" in the problem below. This is shorthand for writing both "P if Q" (so $Q \implies P$) and "P only if Q" (so $P \implies Q$), which is the same as proving the two statements are logically equivalent.

Exercise: Accumulation Points And Limit Points. Let S be a set. Prove that if a is an accumulation point of S, then a is a limit point of S.

Proof. Suppose that a is an accumulation point of S. To prove that a is a limit point of S, we pick an interval $I = (c, d)$ such that $c < a < d$. Let d_1 be the distance from a to c, and let d_2 be the distance from a to d. Let ε be the smaller of d_1 and d_2, which means that $N^*_\varepsilon(a) \subseteq I$. Since a is an accumulation point of S, then we know that $N^*_\varepsilon(a) \cap S \neq \emptyset$. Pick $x \in N^*_\varepsilon(a) \cap S$. By definition of an intersection, this means that $x \in S$ and $x \in N^*_\varepsilon(a)$. Since $x \in N^*_\varepsilon(a)$, we know that $x \neq a$ (the * guarantees this). Because $x \in N^*_\varepsilon(a)$ and $N^*_\varepsilon(a) \subseteq I$, we know that $x \in I$. We've produced an $x \in I$ with $x \in S$ and $x \neq a$. Since I is arbitrary, this proves that a is a limit point of S. □

The converse of the above exercise is also true.

Problem 52: Limit points are Accumulation Points

Let S be a set. Prove that if a is is a limit point of S, then a an accumulation point of S.

Let's now examine a problem related to limits of sequences. Notice in the definition of a limit of a sequence that we used the word "a" instead of "the." What we will prove now is that when a sequence converges, we can talk about "the" limit of the sequence instead of just "a" limit of the sequence.

Exercise: A Convergent Sequence Has A Unique Limit. Let (a_n) be a convergent sequence of real numbers. Prove that (a_n) converges to a unique real number.

Proof. The general way to prove something is unique is to suppose that there are two of those things, and then prove they must be equal. We need to prove that if (a_n) converges to both L_1 and L_2, then we must have $L_1 = L_2$. What follows is one version of a proof of this fact. We will see another later in the semester.

Suppose by way of contradiction that (a_n) converges to both L_1 and L_2, but we have $L_1 \neq L_2$. Let $d = |L_1 - L_2|$, so the distance between the two different limits. We've assumed that $d \neq 0$, so we can use this distance to create an epsilon. Let $\varepsilon = \frac{d}{2}$.

- Since we know (a_n) converges to L_1, then pick $N_1 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n > L_1$ then $|a_n - L_1| < \varepsilon$.
- Since we know (a_n) converges to L_2, then pick $N_2 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n > L_2$ then $|a_n - L_2| < \varepsilon$.

We now pick $n \in \mathbb{N}$ such that $n > \max\{N_1, N_2\}$. Since $n > N_1$, we know $|a_n - L_1| < \varepsilon$, which is equivalent to $a_n \in N_\varepsilon L_1 = (L_1 - \varepsilon, L_1 + \varepsilon)$. Similarly, since $n > N_2$, we know $|a_n - L_2| < \varepsilon$, which means $a_n \in N_\varepsilon L_2 = (L_2 - \varepsilon, L_2 + \varepsilon)$. But we now have a problem. Note that the two intervals $N_\varepsilon L_1$ and $N_\varepsilon L_2$ have no points in common, precisely because the value ε is half the distance between L_1 and L_2. However, we have produced a point a_n that lies inside both intervals, impossible. □

The formal principle of induction allows us to show that a set S equals $\mathbb{N}$. This requires that we show $1 \in S$ and if $k \in S$, then $k + 1 \in S$. What if the statement we wish to show is not true for every natural number, but is true for every natural number past some base case. So suppose we know that $4 \in S$ and if $k \in S$ with $k \geq 4$, then $k + 1 \in S$. Can we conclude that S contains every natural number greater than or equal to 4? If you examine the proof we used for induction, you'll find the quick answer to this question is "yes." So we now have a new form of induction.

Theorem 4 (An Alternate Form of Induction)**.** *Suppose S is a set of integers such that*

- *$b \in S$ and*
- *if $k \in S$ with $k \geq b$, then $k + 1 \in S$.*

Then S contains every integer greater than or equal to b.

Problem 53: Induction And An Inequality

Use induction to prove that for every $n \in \mathbb{N}$ except for $n = 3$ we have $2^n \geq n^2$.

One of the statements in the next problem is false. I'll let you find it.

Problem 54: Additional Properties Of Cartesian Products

Let A, B, C, and D be sets. Prove or disprove each statement.

1. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
2. $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$

Let's now return to studying functions.

Definition: Image And Preimage Of A Set. Consider the function $f : X \to Y$. Let A be a subset of the domain X and let B be a subset of the codomain Y.

- The image of A under f is a subset of Y defined by

  $$\begin{aligned} f(A) &= \{y \in Y \mid y = f(a) \text{ for some } a \in A\} \\ &= \{f(a) \mid a \in A\}. \end{aligned}$$

  This means that $y \in f(A)$ if and only if $y = f(a)$ for some $a \in A$.

- The preimage (or inverse image) of B under f is a subset of X defined by

  $$\begin{aligned} f^{-1}(B) &= \{x \in X \mid f(x) = b \text{ for some } b \in B\} \\ &= \{x \in X \mid f(x) \in B\}. \end{aligned}$$

  This means that $x \in f^{-1}(B)$ if and only if $f(x) = b$ for some $b \in B$, if and only if $f(x) \in B$. Note that when the set B contains a single element, then we often write $f^{-1}(b)$ rather than $f^{-1}(\{b\})$.

We were just introduced to a new definition. We should try to apply this definition to a function we understand. The next problem has you do this.

Problem 55: Function Notation With Sine

Consider the function $f : [0, 4\pi] \to \mathbb{R}$ defined by $f(x) = \sin x$.

1. What are the sets $f(\mathbb{R})$ and $f([0, \pi/2))$? Justify your claim.
2. What are the sets $f^{-1}(\{0\})$ and $f^{-1}([0, 2))$? No justification needed.
3. Find a set $A \subseteq [0, 4\pi]$ so that $g : A \to \mathbb{R}$ defined by $g(x) = \sin(x)$ is injective.
4. Find a set $B \subseteq \mathbb{R}$ so that $h : A \to B$ defined by $h(x) = \sin(x)$ is surjective.

The following theorem lists many facts about the image and preimage of a set under a function. We'll take some time to prove a few of them, but not all of them in class. Be prepared to prove any of them for the final exam.

Theorem 5 (Image/Preimage Properties)**.** *Consider the function $f : X \to Y$.*

1. *If $A \subseteq X$, then we have $A \subseteq f^{-1}(f(A))$.*
2. *If $B \subseteq Y$, then we have $f(f^{-1}(B)) \subseteq B$.*
3. *If $A_1 \subseteq X$ and $A_2 \subseteq X$, then we have $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.*
4. *If $A_1 \subseteq X$ and $A_2 \subseteq X$, then we have $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.*
5. *If $B_1 \subseteq Y$ and $B_2 \subseteq Y$, then we have $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.*
6. *If $B_1 \subseteq Y$ and $B_2 \subseteq Y$, then we have $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.*
7. *We have $f(A) \subseteq B$ if and only if $A \subseteq f^{-1}(B)$.*
8. *If $A_1 \subseteq A_2 \subseteq X$, then we have $f(A_1) \subseteq f(A_2)$.*
9. *If $B_1 \subseteq B_2 \subseteq Y$, then we have $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.*
10. *If $B \subseteq Y$, then $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$.*

Exercise: Image And Preimage Properties 1 And 2. Prove properties 1 and 2 for images and preimages. So for a function $f : X \to Y$, prove that

1. If $A \subseteq X$, then $A \subseteq f^{-1}(f(A))$.
2. If $B \subseteq Y$, then $f(f^{-1}(B)) \subseteq B$. Then give an example of a function $f : X \to Y$ and subsets $A \subseteq X$ and $B \subseteq Y$ where $A \neq f^{-1}(f(A))$ and $B \neq f(f^{-1}(B))$.

Proof. 1. We'll give two proofs of the implication "If $A \subseteq X$, then $A \subseteq f^{-1}(f(A))$." The first proof includes a few more steps and reminders about what each definition means, whereas the second proof shows how to immediately use the image and preimage definitions.

   (a) Let $A \subseteq X$. To prove $A \subseteq f^{-1}(f(A))$, we first pick $x \in A$. Let $y = f(x)$. Since $x \in A$, we know that $y = f(x)$ for some $x \in A$, which means $y \in f(A)$ (from the definition of the image of A). Let $B = f(A)$, and note that $f(x) \in B$. From the definition of the preimage of B, the fact that $f(x) \in B$ means that $x \in f^{-1}(B)$. Substitution of $B = f(A)$ yields $x \in f^{-1}(f(A))$. This completes the proof that $A \subseteq f^{-1}(f(A))$.

   (b) Let $A \subseteq X$. Let $x \in A$. By definition of $f(A)$, the fact that $x \in A$ means that $f(x) \in f(A)$. By definition of preimage of $f(A)$, the fact that $f(x) \in f(A)$ means that $x \in f^{-1}(f(A))$. This proves that $A \subseteq f^{-1}(f(A))$.

2. We now prove "If $B \subseteq Y$, then $f(f^{-1}(B)) \subseteq B$." Let $B \subseteq Y$. Let $y \in f(f^{-1}(B))$. By the definition of the image of $f^{-1}(B)$, the fact that $y \in f(f^{-1}(B))$ means that $y = f(x)$ for some $x \in f^{-1}(B)$. By the definition of the preimage of B, the fact that $x \in f^{-1}(B)$ means that $f(x) \in B$. Since we know that $y = f(x)$, substitution gives $y \in B$. This proves that $f(f^{-1}(B)) \subseteq B$.

□

Problem 56: Image And Preimage Property 3

Prove property 3 for images. So let $f : X \to Y$. Then prove that

- If $A_1 \subseteq X$ and $A_2 \subseteq X$, then we have $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

Optionally, finish by proving that if f is injective, then equality holds.

Problem 57: Proving A Quotient Of Two Linear Sequences Converges

Prove that $\left(\frac{2n+1}{3n+4}\right)$ converges to $\frac{2}{3}$.

Hint: Given $\varepsilon > 0$, solve the equality $|a_M - L| = \varepsilon$ for M, which should help you find a value M you can choose to satisfy the definition of converges. So start by solving $\left|\frac{2M+1}{3M+4} - \frac{2}{3}\right| = \varepsilon$ for M.

Definition: Function Composition. Consider the functions $f : A \to B$ and $g : C \to D$. When $B \subseteq C$, then we know for each $a \in A$ that $f(a) \in B \subseteq C$. Since $f(a) \in C$, we can compute the quantity $g(f(a))$. If $B \subseteq C$, then we define the composition of g and f to be the new function $g \circ f : A \to D$ defined by

$$(g \circ f)(a) = g(f(a)).$$

Problem 58: The Composition Of Surjective Functions Is Surjective

Let A, B, and C be sets, and consider the functions $f : A \to B$ and $g : B \to C$. Prove that if both f and g are surjective, then $g \circ f$ is surjective.

Problem 59: The Composition Of Injective Functions Is Injective

Let A, B, and C be sets, and consider the functions $f : A \to B$ and $g : B \to C$. Prove that if both f and g are injective, then $g \circ f$ is injective.

Problem 60: Proving A Quotient Of Two Quadratic Sequences Converges

Prove that $\left(\frac{n^2}{n^2+1}\right)$ converges.

Hint: Given $\varepsilon > 0$, solve the equality $|a_M - L| = \varepsilon$ to find a value M you can choose to satisfy the definition of converges.

Problem 61: Triangle Inequality

For any real numbers u and v, let $d(u, v)$ be the distance between u and v, which means $d(u, v) = |u - v| = |v - u|$.

1. Let $a, b, c \in \mathbb{R}$. Prove that $d(a, b) \leq d(a, c) + d(c, b)$.
2. Let $x, y \in \mathbb{R}$. Use the previous result to prove that $|x + y| \leq |x| + |y|$.

Both facts above we call the triangle inequality. Both facts basically state that the distance from point A to point B is less than or equal to the distance traveled if you take the shortest route from A to B that must also pass through a third point C. Equality holds if C is already on the shortest path from A to B, otherwise the distance must increase.

Exercise: A Convergent Sequence Has A Unique Limit - Using the triangle inequality. Let (a_n) be a convergent sequence of real numbers. Prove that (a_n) converges to a unique real number.

Proof. We already proved this earlier, without the triangle inequality. Here is a different option.

Suppose that (a_n) converges to both L_1 and L_2. Let $\varepsilon > 0$. We will show that $L_1 = L_2$, by proving that the distance between L_1 and L_2 cannot be any positive number (hence must equal 0 as distance is either positive or zero). Note that the distance between L_1 and L_2 is given by $|L_1 - L_2|$. Let d be a positive distance, and let $\varepsilon = \frac{d}{2}$.

- Since we know (a_n) converges to L_1, then pick $N_1 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n > L_1$ then $|a_n - L_1| < \varepsilon$.
- Since we know (a_n) converges to L_2, then pick $N_2 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n > L_2$ then $|a_n - L_2| < \varepsilon$.

We now pick $n \in \mathbb{N}$ such that $n > \max\{N_1, N_2\}$. Since $n > N_1$, we know $|a_n - L_1| < \varepsilon$. Similarly, since $n > N_2$, we know $|a_n - L_2| < \varepsilon$. We now compute

$$\begin{aligned}
|L_1 - L_2| &= |L_1 + (0) - L_2| \\
&= |L_1 + (-a_n + a_n) - L_2| \\
&= |(L_1 - a_n) + (a_n - L_2)| \\
&\leq |a_n - L_1| + |a_n - L_2| \\
&< \varepsilon + \varepsilon \\
&= \frac{d}{2} + \frac{d}{2} \\
&= d.
\end{aligned}$$

This proves that $|L_1 - L_2| < d$ for any distance $d > 0$. The distance between L_1 and L_2 cannot be positive, hence must equal zero, which proves $L_1 = L_2$. □

Problem 62: Limit Of A Sum Equals Sum Of Limits

Suppose (a_n) converges to A and (b_n) converges to B. Prove that $(a_n + b_n)$ converges to $A + B$. The triangle inequality should help.

Problem 63: Image And Preimage Property 6

Prove property 6 for preimages. So let $f : X \to Y$. Then prove that

- If $B_1 \subseteq Y$ and $B_2 \subseteq Y$, then we have $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

Definition: Image Of A Function. Let $f : X \to Y$. The image of f is the set $f(X)$.[3]

Exercise: A Function Is Surjective Iff Codomain And Image Are Equal. Prove that a function is surjective if and only if the codomain of f and the image of f are equal.

Proof. Let $f : X \to Y$. Suppose f is surjective. Clearly the image of f is a subset of the codomain by definition. Let y be an element of the codomain. Since f is surjective, this means we can pick x in the domain such that $f(x) = y$. This shows that y is in the image of f, which completes the proof that if f is surjective, then the image of f equals the codomain of f.

Now suppose that the image of f equals the codomain of f. We need to show that f is surjective. Pick y in the codomain of f. Since the image equals the codomain, this means y is in the image of f. This means that we can pick x in the domain such that $f(x) = y$. This completes the proof that f is surjective when the image equals the codomain. □

Exercise: The Composition Of Surjective Functions Is Surjective Take 2. Suppose both $f : A \to B$ and $g : B \to C$ are surjective. Prove that $g \circ f : A \to C$ is surjective.

Proof. First note that a function is surjective if and only if the image of the domain equals the codomain. We will use both the "if" and "only if" parts of that statment. We now begin the proof. Because the two functions are surjective, we know $f(A) = B$ and $g(B) = C$ (we used the "only if", or implies, part). This means that $(g \circ f)(A) = g(f(A)) = g(B) = C$. Since we know $(g \circ f)(A) = C$, this proves that $g \circ f$ is surjective (we used the "if" part). □

[3] Some people call the image of f the "range of f", while other people use the word "range" to denote the codomain. The word "range" is unfortunately dependent on author. For this reason, we'll stick to the words "codomain" and "image" to separate these two different sets.

Definition: The Image Of A Sequence Is A Set. Given a sequence (a_n) of real numbers, note that the image of the sequence, namely $a(\mathbb{N}) = \{a_n \mid n \in \mathbb{N}\}$, is a subset of the real numbers. Because the image of the sequence is a set of real numbers, we can use any of our previous words that we defined on sets of real numbers, and now apply them to a sequence. Here are some examples:

- We say a sequence is bounded if the image of the sequence is a bounded set.
- A lower bound for a sequence is a lower bound for the image of the sequence.
- The supremum of a sequence is the supremum of the image of the sequence.
- A limit point of a sequence is a limit point of the image of the sequence.

Problem 64: Convergent Sequences Are Bounded

Prove that if a sequence of real numbers converges, then the sequence is bounded.

Problem 65: Limit Of Product Equals Product Of Limits

Suppose (a_n) converges to A and (b_n) converges to B. Prove that (a_nb_n) converges to AB. The triangle inequality should help, along with the fact that convergent sequences are bounded.

For a hint, see this footnote.[4]

Exercise: Converges To L versus Converges To 0. Let (a_n) be a sequence of real numbers, and A a real number. Prove that (a_n) converges to A if and only if the sequence $(a_n - A)$ converges to 0. This will simplify proving that some sequences converge.

Proof. Suppose that (a_n) converges to A. We must prove that $(a_n - A)$ converges to 0. Let $\varepsilon > 0$ be given. Since (a_n) converges to A we can pick $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n > N$ then $|a_n - A| < \varepsilon$. Since $|a_n - A| = |(a_n - A) - 0|$, we know that we can pick $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n > N$ then $|(a_n - A) - 0| < \varepsilon$. This proves that $(a_n - A)$ converges to 0. This completes the proof that if (a_n) converges to A then the sequence $(a_n - A)$ converges to 0.

We now prove that if the sequence $(a_n - A)$ converges to 0, then (a_n) converges to A. Suppose that $(a_n - A)$ converges to 0. We must prove that (a_n) converges to A. Let $\varepsilon > 0$. Since $(a_n - A)$ converges to 0, we can pick $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n > N$ then $|(a_n - A) - 0| < \varepsilon$. Note that $|(a_n - A) - 0| = |a_n - A|$. Hence we can pick $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n > N$ then $|a_n - A| < \varepsilon$. This prove that (a_n) converges to A, as needed. □

Definition: Increasing, Decreasing, Monotonic Sequences. Let (a_n) be a sequence of real numbers.

[4] The key here is to rewrite $|a_nb_n - AB|$ in a new way so that the quantities $a_n - A$ and $b_n - B$ appear. There are lots of ways to do this. One is to just force them to appear by replacing a_n with $a_n - A + A$, giving us

$$|a_nb_n - AB| = |(a_n - A + A)b_n - AB| = |(a_n - A)b_n + Ab_n - AB| = |(a_n - A)b_n + A(b_n - B)|.$$

At this point, the triangle inequality should help to separate things. Then you'll need to pick M large enough so that both $|(a_n - A)b_n| \leq \frac{\varepsilon}{2}$ and $|A(b_n - B)| < \frac{\varepsilon}{2}$. You'll need to guarantee $|a_n - A||b_n| \leq \frac{\varepsilon}{2}$ and $|b_n - B||A| < \frac{\varepsilon}{2}$. You will need an estimate for $|b_n|$ (did you see the bounded part), and use facts about (a_n) and (b_n) converging to get the needed M.

- We say that (a_n) is (strictly) increasing if $a_n < a_{n+1}$ for every $n \in \mathbb{N}$.
- We say that (a_n) is (strictly) decreasing if $a_n > a_{n+1}$ for every $n \in \mathbb{N}$.
- We say that (a_n) is nonincreasing if $a_n \geq a_{n+1}$ for every $n \in \mathbb{N}$.
- We say that (a_n) is nondecreasing if $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$.
- We say that (a_n) is monotonic if (a_n) is either nonincreasing or nondecreasing.

Notice that a strictly decreasing sequence is nonincreasing, and a strictly increasing sequence is nondecreasing.

Problem 66: Monotonic Bounded Sequences Converge

Let (a_n) be a monotonic sequence. Prove that (a_n) converges if and only if (a_n) is bounded.

Definition: Diverges To Infinity. Let (a_n) be a sequence of real numbers. We say that (a_n) diverges to infinity if for every $V \in \mathbb{R}$, there exists $N \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ we know $n > N$ implies $a_n > V$. When (a_n) diverges to infinity, we write $(a_n) \to \infty$.

Problem 67: A Sequence Diverges To Infinity

Prove that the sequence (n^2) diverges to $+\infty$.

Problem 68: Diverges To Negative Infinity

Construct a definition of what it means to diverge to $-\infty$, by appropriately modifying the definition of diverges to ∞. Then prove that the sequence $(-n^3 + 2n)$ diverges to $-\infty$.

Problem 69: Image And Preimage Property 7

Prove property 7 for preimages. So let $f : X \to Y$. Then prove that

- We have $f(A) \subseteq B$ if and only if $A \subseteq f^{-1}(B)$.

Problem 70: Proving A Rational Sequence Converges

Consider the sequence $(s_n) = \left(\dfrac{4n^3 - 2n^2 - 7n}{5n^3 - 3n^2 + 2n - 1}\right)$. In this problem, your job is to prove that $s_n \to 4/5$. You may assume that for all natural numbers, we have $5n^3 - 3n^2 + 2n - 1 > 0$.

1. Show that $(s_n - 4/5) = \left(\dfrac{2n^2 - 43n + 4}{5\left(5n^3 - 3n^2 + 2n - 1\right)}\right)$.
2. Find a $k_1 > 0$ and M_1 so that $|2n^2 - 43n + 4| \leq k_1 n^2$ for all natural numbers $n > M_1$.
3. Find a $k_2 > 0$ and M_2 so that $|5\left(5n^3 - 3n^2 + 2n - 1\right)| \geq k_2 n^3$ for all natural numbers $n > M_2$.
4. Prove that (s_n) converges to $4/5$.

Problem 71: Closed Intervals Are Closed Sets

Prove that if a and b are real numbers such that $a < b$, then the interval $S = [a, b]$ is a closed set.

Problem 72: Unions And Intersections Of Two Opens Sets

Let U_1 and U_2 be open sets. Show that $U_1 \cup U_2$ and $U_1 \cap U_2$ are open sets.

Now that we've seen that unions and intersections of two open sets are open, what happens if we have more than two open sets? We need a formal definition of how to create a union and intersection of any collection of sets.

Definition: Unions And Intersections Of Arbitrarily Many Sets. Suppose we have a large collection of sets. Each set has been given a name of the form A_j where j is an element of some nonempty set J. We call the collection $\mathrm{A} = \{A_j \mid j \in J\}$ an indexed family of sets with index set J. The union and intersection of all the sets in A are the sets given by

$$\bigcup_{j \in J} A_j = \{x \mid x \in A_j \text{ for some } j \in J\} \text{ and } \bigcap_{j \in J} A_j = \{x \mid x \in A_j \text{ for every } j \in J\}.$$

When $J = \mathbb{N}$, then we'll often write the union and intersection using the notation

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \cdots \text{ and } \bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap A_3 \cap \cdots.$$

Exercise: Nested Sets. For each $n \in \mathbb{N}$ let $A_n = \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$.

1. Show that $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \cdots$.
2. What is $\bigcup_{n \in \mathbb{N}} A_n$?
3. What is $\bigcap_{n \in \mathbb{N}} A_n$?

Problem 73: Unions And Intersections Of Nested Sets

Let n be a natural number and suppose that $A_1, A_2, \ldots, A_n$ are sets. Suppose also that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \supseteq A_n$.

1. Prove that $\bigcup_{i=1}^{n} A_i = A_1$.
2. Make a conjecture that simplifies $\bigcap_{i=1}^{n} A_i$. Then prove your conjecture.

It's never a bad idea to start looking at a problem that involves arbitrary things by first considering specific examples. Make up some examples with 3 or 4 sets. What do you notice happening? Then make your conjecture and prove it.