Introduction to Analysis

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Contents

Preface

The following problem set goes along with Steven R. Lay's *Analysis With an Introduction to Proof*.

Part I

Logic and Proof

Chapter 1

Logic through Proof Techniques

Our goal in this first chapter is to familiarize ourselves with the principles of logic. We'll then explore the basic proof techniques which follow immediately from these principles.

We have to start with some definitions.

Definition 1.1. A sentence that can be classified as either true or false is called a statement.

For a sentence to be a statement, it has to clearly be the case that the sentence is either true or false, even if we do not know the truth value. We now define some ways of connecting multiple statements.

Definition 1.2. Suppose P and Q are statements.

The text shows truth tables for these connectives on pages 3-5.

- The negation of P, written $\sim P$ is the statement whose truth values are exactly opposite P.
- The conjunction of P and Q, written $P \wedge Q$, is the statement P and Q. This statement is only true if both P and Q are both true.
- The disjunction of P and Q, written $P \vee Q$, is the statement P or Q. This statement is only false if both P and Q are both false.
- We call the statement "if P, then Q," an implication, and write $P \Rightarrow Q$. We call P the antecedent and Q the consequent. An implication is false only when P is true but Q is false.
- The statement P if and only if Q is the conjunction $P \Rightarrow Q$ and $Q \Rightarrow P$.

Problem 1.3 Construct a truth table for $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. Show that $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is true precisely when P and Q have the same truth value.

Problem 1.4 Construct a truth table containing $\sim (P \wedge Q)$, $\sim (P \vee Q)$, $\sim (P \Rightarrow Q)$, $(\sim P) \wedge (\sim Q)$, $(\sim P) \vee (\sim Q)$, $P \wedge (\sim Q)$. Which quantities are equal? Write your findings in an easy to remember form, such as "The negation of a conjuction is ..."

Problem 1.5 Complete 1.4acde.

Problem 1.6 Complete 1.7a, 1.8a, and then show using truth tables that $(p \Rightarrow q) \iff (\sim p \vee q)$.

Problem 1.7 Complete 1.10abfgh.

Problem 1.8 Complete 1.14e.

We need some addition notation that shows up commonly in logic. This notation is discussed in the text in section 2.

Definition 1.9.
- The symbol $\forall$ is read "for all" or "for every" or "for each." This symbol is called the universal quantifier.
- The symbol $\exists$ is read "there exists" or "for some" or "for at least one." This symbol is called the existential quantifier.
- The symbol $\ni$ is read "such that."
- The symbol $\in$ is read "is an element of." We'll see this symbol show up more when we start discussing sets. For now, variables are assumed to be real numbers unless additional information is provided.

There are two key points we need to focus on in learning to work with the symbols $\forall$ and $\exists$. First, the order in which you write them matters. Second, we need to learn to negate a statement involving one of these quantifiers.

Problem 1.10 Translate each of the following carefully into an English sentence. Then prove or disprove each statement.

- $\forall x \; \exists y \ni y + 1 > x$.
- $\exists y \ni \forall x, \; y + 1 > x$.

How does the order of the quantifiers change the meaning?

Problem 1.11 Complete 2.3a and 2.4ad.

Problem 1.12 Complete 2.5b and 2.6be.

Problem 1.13 Complete 2.12.

Problem 1.14 Sometimes it is easy to prove a statement false. All you have to do is create a counterexample. Complete 3.6acjk.

Problem 1.15 Construct a truth table for $p \Rightarrow q$, $q \Rightarrow p$, $(\sim p) \Rightarrow (\sim q)$, and $(\sim q) \Rightarrow (\sim p)$. Which of these implications, if any, are equivalent?

Definition 1.16. If p and q are statements, then given the implication $p \Rightarrow q$, we define the following.

- The converse is $q \Rightarrow p$.
- The inverse is $(\sim p) \Rightarrow (\sim q)$.
- The contrapositive is $(\sim q) \Rightarrow (\sim p)$.

In the previous problem, you showed that the implication and the contrapositive are logically equivalent, and that the converse and the inverse are logically equivalent. One method of proving that a statement is true is to instead prove that the contrapositive is true. If the contrapositive is true, then the original statement must be true as well.

Problem 1.17 Complete 3.3b, 3.4b, and 3.5b.

Problem 1.18 Complete 3.7aef.

Problem 1.19 Use truth tables to verify the tautologies (f) and (g) in Example 3.12 on page 23. Then write a sentence to represent what each tautology says, such as "To prove p is true, instead ..."

Problem 1.20 Complete 4.4.

Problem 1.21 Complete 4.5.

Problem 1.22 Complete 4.11.

Problem 1.23 Complete 4.13b and 4.14b.

Problem 1.24 Complete 4.20.

Problem 1.25 Complete 4.22.

Part II

Sets and Functions

Chapter 5

Basic Set Operations

We often speak about collections of objects, for example "the students in this class,", "the homework problems in sections 1-4", etc. We'll use the word "set" to mean a collection of objects. We'll often use capital letters to denote sets, and lower case letters to refer to the objects in the set. So if you see $a \in A$, that means that object a is a member of set A. For the set $A = \{1, 2, 3\}$, we have $1 \in A$ but $4 \notin A$. Elements of a set are most often listed between braces. Here are some sets you are familiar with.

- We'll use $\mathbb{N}$ to mean the set of natural numbers $\{1, 2, 3, \ldots\}$.
- We'll use $\mathbb{Z}$ to mean the set of integers.
- We'll use $\mathbb{Q}$ to mean the set of rational numbers.
- We'll use $\mathbb{R}$ to mean the set of real numbers.

To define a set A, the declaration "$a \in A$" must be a logical statement, however it is not required that we know the truth value of "$a \in A$."

Problem 5.1 Which of the following are sets? Explain.

1. All the current U.S. senators from Idaho.
2. All the U.S. senators from Idaho serving in 2060.
3. All tall BYU-I students this semester.
4. All the prime numbers between 24 and 28.

Definition 5.2. The empty set, written $\emptyset$, is a set with no members in the set. The answer to every question "is $a \in \emptyset$" is always "no."

Using LaTeXyou just write \emptyset.

We'll often use set builder notation to define a set. To talk about the real numbers between 0 and 1, we could use either of the following set-builder notations:

$$(0, 1) = \{x : x \in \mathbb{R} \text{ and } 0 < x < 1\} = \{x | x \in \mathbb{R} \text{ and } 0 < x < 1\}.$$

This is read "the set of all x such that x is a member of the real numbers and $0 < x < 1$." You can either use a colon or a vertical bar to replace the word "such that." When the elements of a set all belong to some larger set (like $x \in \mathbb{R}$ above), we abbreviate the notation to

$$(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\} = \{x \in \mathbb{R} | 0 < x < 1\}.$$

We'll use interval notation common to calculus to talk about intervals of real numbers. The formal definitions are on page 39.

Definition 5.3. Let B be a set. We say that A is a subset of B, written $A \subseteq B$, provided that every member of A is also a member of B. If $A \subseteq B$ but there is a member of B that is not in A, then we say that A is a proper subset of B (and we may write $A \subsetneq B$).

Some authors write $A \subseteq B$ to mean A is a subset of B, while some authors mean A is a proper subset of B.

The definition above tells us that to prove $A \subseteq B$, we must prove

if $a \in A$, then $a \in B$.

Problem 5.4 Prove that the empty set is subset of every set.

We now use this to define what it means for two sets to be equal.

Definition 5.5. If A and B are sets, we say $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Problem 5.6 Prove that the empty set is unique. That is, suppose that A and B are both empty sets, and then show that $A = B$. [Exercise 5.18.]

Problem 5.7 Complete problem 5.8. Make sure you can carefully explain why each statement is true or false.

We now define ways of obtaining new sets from old ones.

Definition 5.8. Let A and B be sets.
The union of A and B, written $A \cup B$, is the set

$$A \cup B = \{x | x \in A \text{ or } x \in B\}.$$

The intersection of A and B, written $A \cap B$, is the set

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

The complement of B in A, written $A \backslash B$, is the set

$$A \backslash B = \{x | x \in A \text{ and } x \notin B\}.$$

If $A \cap B = \emptyset$, then we say that A and B are disjoint.

I strongly suggest that you look at problem 5.3 in page 47 and check your answers with the back of the book before going on.

Problem 5.9 Complete problem 5.4.

The following theorem provides us with a way of simplifying complex set operations. These rules are very similar to our rules with logic (where the symbols $\sim, \wedge, \vee$ are replaced with $\backslash, \cap, \cup$ respectively).

Theorem 5.10. *Let A, B, and C be subsets of some universal set S. Then the following statements are true.*

1. $A \cup (U \backslash A) = U$
2. $A \cap (U \backslash A) = \emptyset$
3. $U \backslash (U \backslash A) = A$
4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6. $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$
7. $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$

The proofs of part (4) and (6) are spelled out on pages 43 and 44. I strongly suggest you read these before going on.

Problem 5.11 Prove part (5) of Theorem 5.10.

Problem 5.12 Prove part (7) of Theorem 5.10.

Problem 5.13 Prove part (1) of Theorem 5.10.

Problem 5.14 Use Theorem 5.10 to complete exercise 5.6 in the text (when you present in class, we'll only ask to see a few of these).

We will often need to talk about the union and intersection of a collection of sets (more than 2). Sometimes, we'll be working with finitely many sets. Sometimes we'll be working with infinitely many sets. The following definition allows us to discuss the union or intersection of an arbitrary collection of sets.

Definition 5.15. Let J be a non empty set (we'll call this the index set). If for each $j \in J$ there corresponds a set A_j, then we define

$$\mathcal{A} = \{A_j \mid j \in J\}$$

to be an indexed family (or indexed collection) of sets with index set J. The union of all the sets in $\mathcal{A}$ is

$$\bigcup_{j \in J} A_j = \{x \mid x \in A_j \text{ for some } j \in J\}.$$

The intersection of all the sets in $\mathcal{A}$ is

$$\bigcap_{j \in J} A_j = \{x \mid x \in A_j \text{ for every } j \in J\}.$$

Problem 5.16 Complete exercise 5.25a and 5.25b.

Problem 5.17 Complete exercise 5.26a.

Problem 5.18 Complete exercise 5.26c.

Chapter 6

Relations

Definition 6.1: Cartesian product $A \times B$**.** If A and B are sets, we define the Cartesian product $A \times B$ to be the set of ordered pairs (a, b) where $a \in A$ and $b \in B$. Notationally, we can write

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

If $A_1, A_2, A_3, \ldots, A_n$ are sets, then we define the Cartesian product of these sets to be the set of ordered n-tuples given by

$$\prod_{i=1}^{n} A = A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for each } i\}.$$

Problem 6.2 Let $A = \{g, h\}$ and $B = \{1, 2\}$. List all the elements of $A \times B$. List all the subsets of $A \times B$?

Definition 6.3: Relation. Suppose A and B are sets. We define a relation R between A and B to be a subset R of $A \times B$. We say that a is related to b, and write aRb, if $(a, b) \in R$. If $A = B$, then we will say that R is a relation on A.

Problem 6.4 Consider the sets $A = \{a\}$ and $B = \{1, 2, 3\}$. List all possible relations between A and B. This is problem 6.7.

Definition 6.5. Suppose R is a relation on a set S. We define the reflexive, symmetric, and transitive properties of a relation as follows.

- We say that R is reflexive iff for every $x \in S$ we have xRx.
- We say that R is symmetric iff xRy implies yRx.
- We say that R is transitive iff xRy and yRz implies xRz.

Problem 6.6 Complete each of the following sentences.

- We say that R is not reflexive iff ...
- We say that R is not symmetric iff ...
- We say that R is not transitive iff ...

Problem 6.7 Let X be a set. Let P be the set of subsets of X. Define a relation on P by XRY iff $A \subseteq B$ where $A, B \in P$ (so $A \subseteq X$ and $B \subseteq X$). Exercise 6.11 b.

1. Is R reflexive? Remember to always justify your answers.
2. Is R symmetric?
3. Is R transitive?

Problem 6.8 Let R be the relation on $\mathbb{R}$ defined by xRy iff $x-y$ is irrational. Is R reflexive? Is R symmetric? Is R transitive? Explain. Exercise 6.11 f.

Definition 6.9. We say that a relation is an equivalence relation iff the relation is reflexive, symmetric, and transitive.

If R is an equivalence relation on a set S, then for each $x \in S$, we define the equivalence class of x to be the set

$$E_x = \{y \in S \mid yRx\}.$$

Problem 6.10 Complete problem 6.14.

Problem 6.11 Complete problem 6.25. Along the way, please explain how this relates to simplifying fractions.

Problem 6.12 Let R be an equivalence relation. Prove that xRy if and only if the equivalence class of x equals the equivalence class of y, i.e. $E_x = E_y$. [Note: This is an if and only if proof. Inside of the if and only if proof, you have to show in one direction that two sets are the same. I'll have have someone prove the forward direction, and someone prove the reverse direction in class.]

Definition 6.13. A partition of a set S is a collection $\mathcal{P}$ of nonempty subsets of S such that

- For each $x \in S$, we have $x \in A$ for some $A \in \mathcal{P}$.
- For each $A, B \in \mathcal{P}$, we have $A = B$ or $A \cap B = \emptyset$.

A member of $\mathcal{P}$ is called a piece (or block) of the partition.

Problem 6.14 Suppose R is an equivalence relation on a set S. Prove that the set of equivalence classes of R, written $\{E_x \mid x \in S\}$ is a partition of S.

Problem 6.15 Suppose $\mathcal{P}$ is a partition of a set S. Let R be a relation on S defined by xRy if and only if x and y are in the same piece of the partition $\mathcal{P}$. Prove that R is an equivalence relation.

The previous two problem show that equivalence relations and partitions are really the same idea. If you have an equivalence relation, it creates a partition. If you have a partition, it creates an equivalence relation.

Problem 6.16 Complete problem 6.20. This problem introduces the idea of modular arithmetic. You'll study this a lot more in abstract algebra.

Chapter 7

Functions

Definition 7.1. A function between A and B is a nonempty relation f between A and B such that if $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$. The domain of f, written dom f, is the set of all first elements of members of f. The range of f, written range$f$ is the set of all second elements of members of f. We can write these two sets symbolically as

- dom $f = \{a \in A \mid (a, b) \in f \text{ for some } b \in B\}$ and
- range $f = \{b \in B \mid (a, b) \in f \text{ for some } a \in A\}$.

When A is the domain of f, we will write $f : A \to B$ to remind us that is a function from A into B. The set B is called the codomain of f. If $(x, y) \in f$, then we often say that f maps x to y, and will write $x \mapsto y$ or $f(x) = y$ (rather than xfy or $(x, y) \in f$).

Problem 7.2 Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Which of the following are functions between A and B? Which are functions from A into B? If it is a function, state the domain and range.

1. $\{(a, 1), (b, 3), (c, 4), (a, 3)\}$
2. $\{(a, 1), (b, 2), (c, 3)\}$
3. $\{(a, 4), (b, 2), (c, 2)\}$
4. $\{(a, 1), (b, 3), (a, 4)\}$
5. $\{(b, 1), (c, 3)\}$

Definition 7.3. Let $f :: A \to B$ be a function.

- We say that f is injective (one-to-one) iff for every $x, y \in A$, we have $f(x) = f(y)$ implies $x = y$.
- We say that f is surjective (onto) iff for every $y \in B$, there exists $x \in A$ such that $f(x) = y$. Equivalently, we way that f is surjective if the range of f is the codomain B.
- If f is both injective and surjective, then we say f is bijective.

Problem 7.4 Complete problem 7.9. This has you analyze 6 different attempts at proving a function is injective. With each one, either state that it is correct, or identify the logical flaw in their proof (did they try to prove the converse, the inverse, or something else?). As a hint, three of the proofs give are completely correct. Be prepared to explain the flaw in the three incorrect proofs.

Problem 7.5 Complete 7.8 on page 74. This problem asks you to use the definitions of surjective and injective to examine two functions with different domains. The functions return the area of a circle.

Definition 7.6. Suppose that $f : A \to B$. Suppose $C \subseteq A$ and $D \subseteq B$. We define $f(C)$, called the image of C, to be the set of images of elements from C, i.e.

$$f(C) = \{f(x) \mid x \in C\} = \{y \in B \mid f(x) = y \text{ for some } x \in C\}.$$

We define $f^{-1}(D)$, called the pre-image of D or inverse image of D, to be the set of elements in A whose image is in D, i.e.

$$f^{-1}(D) = \{x \in A \mid f(x) \in D\}.$$

If $D = \{d\}$ is a one element set, we often write $f^{-1}(d)$ instead of $f^{-1}(\{d\})$.

Problem 7.7 Complete problem 7.13. Be prepare to explain the visual way of representing the function as shown in the problem.

Problem 7.8 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x$.

1. Find $f(\mathbb{R})$ and $f([0, \pi/2))$.
2. Find $f^{-1}(0)$ and $f^{-1}([0, \pi/2))$.
3. Find a set $A \subseteq \text{dom} f$ so that $f : A \to \mathbb{R}$ is injective. Make A as large as possible.
4. Find a set $B \subseteq \text{codom} f$ so that $f : A \to B$ is surjective (use A from above).

Theorem 7.9. *Suppose that $f : A \to B$. Suppose C, C_1, and C_2 are subsets of A and D, D_1, D_2 are subsets of B. Then the following are true.*

1. $C \subseteq f^{-1}(f(C))$
2. $f[f^{-1}(D)] \subseteq D$
3. $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$
4. $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$
5. $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$
6. $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$
7. $f^{-1}(B \backslash D) = f^{-1}(B) \backslash f^{-1}(D) = A \backslash f^{-1}(D)$

Problem 7.10 Complete problem 7.16. This asks you to produce examples of functions and subsets so that equality is not obtained in the first three conclusions above.

You can use the functions in problems 7.7 and 7.8 to complete this problem.

Problem 7.11 Prove that $f[f^{-1}(D)] \subseteq D$ (part 2 of theorem 7.9).

Problem 7.12 Prove that $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$ (part 5 of theorem 7.9).

Theorem 7.13. *Under the assumptions of theorem 7.9, the following are true.*

1. *If f is injective, then $C = f^{-1}(f(C))$.*
2. *If f is surjective, then $f[f^{-1}(D)] = D$.*
3. *If f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.*

Problem 7.14 Prove that if f is injective, then $C = f^{-1}(f(C))$. This is part 1 of the previous theorem.

Problem 7.15 Complete problem 7.21, parts a and b. Make conjectures for c and d, but don't worry about proving them here. You may be asked to prove them on an exam.

Definition 7.16. Suppose that $f : A \to B$ and $g : B \to C$. The composition of f and g, written $g \circ f$, is the function defined by $(g \circ f)(x) = g(f(x))$. In set notation we write

$$g \circ f = \{(a, c) \in A \times C \mid \exists b \in B \text{ s.t. } (a, b) \in f \text{ and } (b, c) \in g\}.$$

Problem 7.17 Prove that if both $f : A \to B$ and $g : B \to C$ are injective, then the composition $g \circ f : A \to C$ is also injective. (It is also true that if both f and g are surjective, then the composition is surjective. This is proved on page 68.)

Problem 7.18 Complete problem 7.26 and 7.27.

Definition 7.19. Let $f : A \to B$ be bijective. The inverse function of f is defined to be the function $f^{-1} : B \to A$ defined by $f^{-1}(y) = x$ if and only if $f(x) = y$. This is the same as writing in set notation

$$f^{-1} = \{(y, x) \in B \times A \mid (x, y) \in f\}.$$

The definition above actually includes a theorem. Given any relation f between A and B, we can easily create the inverse relation $f^{-1} = \{(y, x) \in B \times A \mid (x, y) \in f\}$. The theorem included in the definition above is that the inverse relation is always a function if f is bijective.

Problem 7.20 Suppose $f : A \to B$ is a function from A into B. Consider the inverse relation $f^{-1} = \{(y, x) \in B \times A \mid (x, y) \in f\}$. Prove that this relation is a function from B into A if and only if f is both surjective and injective. In other words, you are proving that a function $f : A \to B$ has in inverse function if and only if f is bijective. [Note the "if and only if," which means there are two directions.]

Theorem 7.21. *Let $f : A \to B$ and $g : B \to C$ be bijective. Prove that the composition $g \circ f$ is bijective, and that the inverse is $(g \circ f)^{-1}) = f^{-1} \circ g^{-1}$.*

Problem 7.22 Prove the previous theorem.

Problem 7.23 Complete problem 7.32.

Chapter 8

Cardinality

How do we determine if two sets have the same number of elements? If both sets are finite, the answer is simple as you can count the number of elements. However, what if both sets have infinitely many elements? The sets $\mathbb{Z}$ and $\mathbb{N}$ both have infinitely many elements. How do we compare the size of these sets? You might want to say there are two times as many plus one integers than there are natural numbers. This is because you can see an injection $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = n$, and this injection misses all the negative integers and zero. To help you build some intuition, here's a classical fun way to look at this issue.

Problem 8.1: The Hilbert Hotel Sally owns a rather strange hotel. This hotel has infinitely many rooms which are numbered $1, 2, 3, 4, \ldots$. Business has been good, and Sally manages to fill the entire hotel. Frank, Sally's brother, shows up at the main lobby and asks to get a room. The front clerk says, "Sorry, we're full." Luckily Sally overhears this comment and interrupts by saying, "Frank, I think we can find a room for you. We may have to ask some people to switch around rooms, we can fit you in." Sally comes to you and asks you to find a room for Frank.

1. Prepare new room assignments for every single occupant in Sally's hotel, so that after each occupant has moved to their new room, there will be an empty room available for Frank. In other words, if someone is currently in room k, which room do you want them to move to. You're allowed to move as many people in the hotel as you want, but you can't make anyone share rooms.
2. If George, who is Sally's cousin, had shown up at the same time as Frank, how could you have prepared the room assignments to make space for two empty rooms?
3. If n people show up to Sally's full hotel, how can you make room for n extra people?

Definition 8.2. Suppose A and B are two sets. The cardinality of A is written $|A|$.

- If set A has finitely many elements, say n elements, then we say the cardinality of A is n and write $|A| = n$.
- If there is an injection $f : A \to B$, then we say the cardinality of A is no bigger than B, and we write $|A| \le |B|$.

- If there is a surjection $g : A \to B$, then we say the cardinality of A is no smaller than B, and we write $|A| \geq |B|$.
- If there is a bijection $h : A \to B$, then we say the cardinality of A equals the cardinality of B, and we write $|A| = |B|$.

Problem 8.3 What does problem 8.1 tell us about the cardinality of sets $\mathbb{N}$ and $\mathbb{N} \cup \{0\}$? What about the cardinality of sets $\mathbb{N}$ and $\mathbb{N} \cup \{0, -1, -2, -3, -4\}$? Which has a greater cardinality, or are they the same?

Problem 8.4 Business has been really good for Sally, so her brother Frank decides to open another hotel across town, next to the river. Both Sally and Frank manage to fill their hotels completely every night (really good business). However, a week of excessive rain causes the river to flood, and Frank has to find lodging for all of his hotel guests (quite a feat). Frank calls Sally (whose hotel is full) and he tells her about the problem. Sally immediately says, "Oh, I've got room for all your guests. Bring them over." Since you were the one who figured out how to make room for Frank, George, and m friends, Sally figures you can find a way to fit in an extra infinite number of guests. She asks you to prepare the room assignments for all of Franks guests. Not wanting to disappoint her, you accept the challenge.

1. Prepare new room assignments for every single occupant of both hotels. If someone is currently in room k in your hotel, which room should they move to? If someone was in room j of Frank's hotel, which room will you give them in your hotel? You can't make anyone share rooms. Sally would also like you to make sure the hotel is still full when you're done (if there are vacancies, it might look bad, we have to keep up appearances).
2. George, remember Sally's cousin, decides to open a hotel as well (he puts it next to a lake).A month later, severe flooding causes both Frank and George to vacate their hotel. All three of Sally, Frank, and George have completely full hotels, but Sally tells the boys to bring all their guests over and she'll fit them all into the hotel. She then tells you to make room assignments.

Problem 8.5 One night, Sally decides to take her hotel plan and sell the concept to potential investors. The idea is quite popular, and she fills each room in her hotel with someone who wants to build a hotel just like hers. For sake of ease, we'll name the new entrepreneurs $p_1, p_2, p_3, \ldots$. Every entrepreneur p_k goes out and opens a new hotel H_k that is just like Sally's. After a few months, every single hotel H_k is filled to capacity. The business men p_k decide to have a party in recognition of Sally, and as part of the party they decide that everyone should bring all their occupants to Sally's hotel on the same night (so there will be an infinite number of hotels, each bringing infinitely many guests). Sally decides to clear her hotel of all guests for the day, so her hotel is empty to start with. She asks you to prepare room assignments for all the inhabitants of all the hotels. Sally insists that you can do this. Your job is clear. When each guest arrives, they tell you which hotel H_k they came from, and then they tell you which room number j they were occupying at hotel H_k. How will you make the room assignments so that you can fill every single room in Sally's hotel with exactly one occupant? Which room will you assign guest (j, k) to (j was their old room assignment in hotel H_k)?

Problem 8.6 One day, an alien space ship lands next to Sally's hotel. They had heard about her hotel and it's amazing feats from an intergalactic newspaper. The alien ship is quite full, and Sally is having a hard time remembering names. She discovers that you can name all the aliens if you name them a_x for each $x \in \mathbb{R}$. The aliens want to all stay in Sally's hotel, so Sally asks you to prepare room assignments for them all. You tell her it can't be done. She insists that it can be done, but you are firm in your convictions and tell her that it can't be done (you're not discriminating against aliens, rather you are just stating a fact). Sally doesn't believe you, so she tries to prepare the room assignments herself. She brings a list to you that tells you which alien should be in room 1, which alien to put in room two, etc. Before she takes it to the ship, she ask you to look over the list to make sure she hasn't missed anyone.

Your job. Show Sally why her room assignment plan is missing an alien. When you have completed this problem, you will have shown that the cardinality of the real numbers is larger than the cardinality of the natural numbers.

Part III

The Real Numbers

Chapter 10

Induction

In this section [i], we will explore a technique for proving statements of the form

For every $n \in \mathbb{N}$, the statement $P(n)$ is true.

Notice that this is a statement about natural numbers and not some other set. We will be learning how to prove that infinitely many statements are true.

Consider the following two claims.

1. For all $n \in \mathbb{N}$, we have $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.
2. For all $n \in \mathbb{N}$, the number $n^2 + n + 41$ is prime.

Let's take a look at potential proofs.

"Proof" of Claim (1). If $n = 1$, then $1 = \frac{1(1+1)}{2}$. If $n = 2$, then $1 + 2 = 3 = \frac{2(2+1)}{2}$. If $n = 3$, then $1 + 2 + 3 = 6 = \frac{3(3+1)}{2}$, and so on. □

"Proof" of Claim (2). If $n = 1$, then $n^2 + n + 41 = 43$, which is prime. If $n = 2$, then $n^2 + n + 41 = 47$, which is prime. If $n = 3$, then $n^2 + n + 41 = 53$, which is prime, and so on. □

Are these actual proofs? The answer is NO! In fact, the second claim isn't even true. If $n = 41$, then $n^2 + n + 41 = 41^2 + 41 + 41 = 41(41 + 1 + 1)$, which is not prime since it has 41 as a factor.

It turns out that the first claim is true, but what we wrote cannot be a proof since the same type of reasoning when applied to the second claim seems to prove something that isn't actually true.

We need a rigorous way of capturing "and so on" and a way to verify whether it really is "and so on."

Axiom 10.1: Axiom of Induction. Let $S \subseteq \mathbb{N}$ such that both

1. $1 \in S$, and
2. if $k \in S$, then $k + 1 \in S$.

Then $S = \mathbb{N}$.

[i] Much of the material in this section came from Dana Ernst's materials located at https://github.com/dcernst/IBL-IntroToProof

Remark 10.2. An axiom is a basic mathematical assumption. We are assuming that the Axiom of Induction is true, which I'm hoping that you can agree is a pretty reasonable assumption. I like to think of the first hypothesis of the Axiom of Induction as saying that we have a first rung of a ladder. The second hypothesis says that if we have some random rung, we can always get to the next rung. Taken together, this says that we can get from the first rung to the second, from the second to the third, and so on. Again, we are assuming that the "and so on" works as expected here.

Here's another way to see why this axiom is a reasonable assumption. If $S \neq \mathbb{N}$, then we could look at the nonempty set $\mathbb{N} \setminus S$. This set ought to have a first element (more on this later), which we could call n. We know that $n \neq 1$, so if we subtract one from n we must be in the set S. This means that $n-1 \in S$. However, since $n-1 \in S$, we can add one to $n-1$ and still be in the set. This would mean that $n \in S$, a contradiction.

There is problem with assuming that $\mathbb{N} \setminus S$ has a first element. There is no way to prove this. The "Well Ordering Principle" states that a nonempty subset of the natural numbers has a smallest element. However, the well ordering principle is an axiom as well. If we assume the axiom of induction is true, we could prove the well ordering principle is true. Similarly, if we assume the well ordering principle is true, we can prove the axiom of induction is true. But we have to just accept one of them as an axiom without proof.

Theorem 10.3 (Principle of Mathematical Induction)**.** *Let $P_1, P_2, P_3, \ldots$ be a sequence of statements, one for each natural number. Assume the following.*

1. *The first statement P_1 is true.*
2. *If P_k is true, then P_{k+1} is true.*

Then P_n is true for all $n \in \mathbb{N}$.

Remark 10.4. The Principal of Mathematical Induction (PMI) provides us with a process for proving statements of the form: "For all $n \in \mathbb{N}$, the statement P_n is true," where P_n is some statement involving $n \in \mathbb{N}$. Hypothesis (1) above is called the **base step** while (2) is called the **inductive step**. Here is what a proof by induction looks like (remarks are in parentheses):

Proof. We proceed by induction.

(i) Base step: (Verify that P_1 is true. This often amounts to plugging $n = 1$ into two sides of some claimed equation and verifying that both sides are actually equal. Don't assume that they are equal!)

(ii) Inductive step: (Your goal is to prove that "For all $k \in \mathbb{N}$, if P_k is true, then P_{k+1} is true.") Let $k \in \mathbb{N}$ and assume that P_k is true. (Now, do some stuff to show that P_{k+1} is true.) Therefore, P_{k+1} is true.

Thus, by the the principle of mathematical induction, P_n is true for all $n \in \mathbb{N}$. □

Problem 10.5 For all $n \in \mathbb{N}$, we have $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. (*Note:* $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$, by definition.)

This is example 10.3, on page 101. Try to complete the proof without looking. If you need to use the book, you can.

Problem 10.6 For all $n \in \mathbb{N}$, we have $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Problem 10.7 For all $n \in \mathbb{N}$, we have $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

Problem 10.8 Notice that

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \\ 1+3+5+7+9 &= 25. \end{aligned}$$

Make a conjecture about the sum of odd numbers by giving a general formula for what you see above. Then prove your formula is valid using mathematical induction.

Problem 10.9 For all $n \in \mathbb{N}$, the number $5^{2n} - 1$ is a multiple of 8.

Problem 10.10 In first semester calculus, you proved (or rather your teacher probably proved) that $(f+g)' = f' + g'$, provided that both f and g are differentiable. However, you then started using this fact to take derivatives of the sum of any number of things (so $(x^3 + 2x + \cos x)' = (x^3)' + (2x)' + (\cos x)' = 3x^2 + 2 - \sin x$). There was no theorem which justified taking derivatives of 3 things, or 4, or more. This sounds like a perfect problem for induction.

Prove that for all $n \in \mathbb{N}$, we have $(f_1 + f_2 + \cdots + f_n)' = (f_1)' + (f_2)' + \cdots + (f_n)'$, provided that f_i is differentiable for each $i \in \{1, 2, \ldots, n\}$.

Sometimes a statement of the form $P(n)$ may be true for $n = 1, n = 2, n = 3$, and so on but false at say $n = 7$. However, after $n = 7$, the statement is true for all n after that "bad case." We can modify the principle of mathematical induction to take care of instances like this. If we know that $P(8)$ is true, and we know that if $P(k)$ is true, then $P(k+1)$ is true, then we know that $P(n)$ is true for all $n \geq 8$.

Problem 10.11 For which natural numbers is the statement $n^2 < 2^n$ true? Prove your result with induction.

Chapter 11

Axioms for the Real Numbers

The properties, listed on page 108-109 as A1-A5, M1-M5, DL, and O1-O4, provide the axioms that we will assume are true for the real numbers. As they are axioms, we will never prove they are true, rather we accept them as the rules of the game. Any set which satisfies these axioms is called an ordered field. The completeness axiom (which we'll get to in a bit) is the only remaining axiom that is needed to completely characterize the real numbers. Any other property of the real numbers can be proved from these axioms. These axioms form a minimal collection of axioms, in that if you removed any one of these axioms, then there would be more sets that satisfy these axioms than just the real numbers. The claim I just made we will not prove in this class.

Using the axioms on page 108-109, we could spend an entire semester proving all the other facts about real numbers that you have come to accept as true. We will not take time to do this, rather we will just prove a few facts.

Problem 11.1 Prove that $0 \cdot x = 0$. Then prove that $x \cdot 0 = 0$. Use only the axioms on page 108-109.

Problem 11.2 Prove that $xy = 0$ iff $x = 0$ or $y = 0$. Again, only use the axioms (though you can use what you just proved above).

You could spend an entire semester proving all the facts about the real numbers that we commonly use. See problem 11.3 for a small list of facts that you could prove. Here are two facts which will help us be prepared for sequences and limits.

Problem 11.3 Prove that if $x \geq 0$ and $x \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$. Only use the axioms. (This is problem 11.4.)

Problem 11.4 Prove that if $|x - y| < \epsilon$ for all $\epsilon > 0$, then $x = y$. (This is 11.6 c. You may want to use the previous problem.)

Problem 11.5 Prove that for all $n \in \mathbb{N}$, if $x_1 x_2 x_3 \cdots x_n = 0$ then there exists $i \in \{1, 2, 3, \ldots, n\}$ such that $x_i = 0$. Use induction and problem 11.2.

Chapter 12

The Completeness Axiom

Before introducing the completeness axiom, we must make a few definitions. Our goal is to understand the difference between the words upper bound, lower bound, maximum, minimum, supremum, and infimum.

Definition 12.1. Let S be a set of real numbers. Let $m \in \mathbb{R}$.

- We say m is an upper bound for S if $m \geq x$ for all $x \in S$. We say S is bounded above if it has an upper bound.
- We say m is a lower bound for S if $m \leq x$ for all $x \in S$. We say S is bounded below if it has a lower bound.
- We say S is bounded if it is both bounded above and below.
- We say m is a maximum of S if $m \in S$ and m is an upper bound.
- We say m is a minimum of S if $m \in S$ and m is a lower bound.

Problem 12.2 For each part below, construct an example as requested.

1. Give a set S_1 that is bounded below but not bounded above. State 4 different lower bounds for the set.
2. Give a set S_2 that is bounded. Make S have a maximum, but no minimum. State an upper and lower bound for S.
3. Give an example of a function $f : \mathbb{N} \to \mathbb{R}$ so that the range $f(\mathbb{N})$ is bounded. Give both upper and lower bounds for $f(\mathbb{N})$.
4. Give an example of an injective function $f : \mathbb{N} \to \mathbb{R}$ so that the range $\mathbb{N}$ is bounded. Give both upper and lower bounds for $f(\mathbb{N})$.

Definition 12.3: Infimum. When a set is bounded below, there are infinitely many lower bounds. The infimum of such a set is the greatest lower bound, written $\inf S$. So if S is a set, then we can write $m = \inf S$ if and only if

1. m is a lower bound, so $m \leq x$ for all $x \in S$, and
2. m is the greatest lower bound, so if m' is another lower bound then $m \geq m'$.

[As a side note, the text restates part 2 above as "m is the greatest lower bound, so if $m' > m$, then there exists $s' \in S$ with $s' < m'$." They are equivalent.]

Definition 12.4: Supremum. When a set is bounded above, there are infinitely many upper bounds. The supremum of such a set is the least upper bound, written $\sup S$. So if S is a set, then we can write $m = \sup S$ if and only if

1. m is an upper bound (so $m \geq x$ for all $x \in S$), and
2. m is the least upper bound, so if m' is another upper bound then $m \leq m'$.

Problem 12.5 Consider the interval $[0, 4)$. Show the set is bounded. Then state the minimum, maximum, infimum, and supremum of the set, if possible. Justify each claim you make, by showing the number you have chosen satisfies the corresponding definition, or by showing why the number does not exist.

Problem 12.6 Find the supremum and infimum of the set $\left\{\dfrac{n}{n+1} : n \in \mathbb{N}\right\}$. Justify your claims by using the definition.

Axiom 12.7: The Completeness Axiom. Every nonempty subset of the real numbers that is bounded below has a greatest lower bound. Equivalently, every nonempty subset of the real numbers that is bounded above has a least upper bound.

The completeness axiom guarantees that whenever you have a bounded set, you can write $\inf S$ and $\sup S$ without worry. The maximum and minimum of a set may not exist, but the supremum and infimum always will. This is not true of every set.

Problem 12.8 Let $S = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$. Show that this set is bounded by rational numbers (give upper and lower bounds using only rational numbers). Show that the supremum of S is not rational. [Hint: A proof by contradiction may be the best route.]

The proof is basically in the text. You're proving why $\sqrt{2}$ is not rational. You can assume $\sqrt{2}$ is the supremum without proof.

The previous problem showed that if you start with a bounded set of rational numbers, the supremum does not have to be rational. The rational numbers are incomplete because thy do not contain all supremums and infimums. There is something missing. By including the completeness axiom, the real numbers becomes precisely the set needed to guarantee that you can find the supremum (and infimum) of a set, without having to leave your set. This makes the real numbers "complete."

The following property seems obvious, but it's proof relies on the completeness axiom. Please prove it now.

Problem 12.9: The Archimedean Property Prove that the set of natural numbers is unbounded above in $\mathbb{R}$. [Hint: Suppose it were not. What does this imply?]

There are many ways to rewrite the Archimedian property. The following problem has you show that 3 things are equivalent to this property. Remember, to show that statements are equivalent, there is really an "iff" between each. If you want to prove a statement such as $p \iff q \iff r \iff s$, then you can just show $p \Rightarrow q \Rightarrow r \Rightarrow s \Rightarrow p$ instead.

Problem 12.10 Prove that the following are all equivalent.

1. The set of natural numbers is unbounded above in $\mathbb{R}$.

2. For each $z \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > z$.
3. For each real number $x > 0$ and $y \in \mathbb{R}$, there exists and $n \in \mathbb{N}$ such that $nx > y$.
4. For each real number $x > 0$, there exists and $n \in \mathbb{N}$ with $0 < 1/n < x$.

We already know that the rational numbers are not complete. However, there are a lot a rational numbers. The real numbers complete the rational numbers. We'll now show that between any two real numbers, there is always a rational number. Similarly, we'll show that between any two rational numbers, there is always an irrational.

Problem 12.11 Suppose $x > 0$ is a real number. Show that there exists an integer m such that $m - 1 \leq x < m$. Then prove that the number m you have found is unique. [Hint: To use the completeness axiom you need a nonempty set that is bounded below. Create a set so that m is the infimum.]

Problem 12.12 Let $x, y \in \mathbb{R}$ such that $0 < x < y$. Show that there exists a rational number r with $x < r < y$. This shows that between any two positive real numbers, there exists a rational number.

Problem 12.13 Let $r, s \in \mathbb{Q}$ such that $0 < r < s$. Show that there exists an irrational x with $r < x < s$. This shows that between any two positive rational numbers, there exists an irrational number.

The previous three problems all assumed we were working with positive numbers. You could easily repeat any of the proofs if you assumed the numbers were negative, and the inequalities were all reversed. Any time you apply the completeness axiom, you would just use a supremum instead of an infimum.

Chapter 13

Topology of the reals

We are just about ready to start limits. Before doing so, we'll introduce some vocabulary that will make our discussion of limits much simpler. Our real goal is to talk about what it means for two points to be close. The absolute value function $|x - y|$ was created precisely to talk about the distance between two numbers x and y. If we want to talk about all points that are close to x, we would first need to pick a maximum distance ϵ away, and then we could consider all points y that are within ϵ of x by considering the set $\{y : |x - y| < \epsilon\}$. Since the points y are close to x, we could say that the points "live in x's neighborhood," just as people who live close to each other are called neighbors. We're ready for some definitions.

Definition 13.1. Let $x \in \mathbb{R}$. Let $\epsilon > 0$ be given. The ϵ-neighborhood of x is the set

$$N(x; \epsilon) = \{y \in \mathbb{R} : |x - y| < \epsilon\}.$$

This is the set of all real numbers that are within ϵ of x. The number ϵ is called the radius of this neighborhood. We will often just call this a neighborhood, where ϵ is assumed to be known.

Definition 13.2. Let $x \in \mathbb{R}$. Let $\epsilon > 0$ be given. The deleted ϵ-neighborhood of x is the set

$$N^*(x; \epsilon) = \{y \in \mathbb{R} : 0 < |x - y| < \epsilon\}.$$

Problem 13.3 Consider the interval $S = (0, 7)$.

1. Find real numbers x and ϵ so that $N(x; \epsilon) = S$.
2. Find real numbers x and ϵ so that $N(x; \epsilon)$ is a proper subset of S.
3. For every $x \in S$, find a real number ϵ so that $N(x; \epsilon) \subseteq S$. Your choice of ϵ will depend on x.

Definition 13.4. Let $S \subseteq \mathbb{R}$.

- We say that x is an interior point of S iff there exists some neighborhood N of x with $N \subseteq S$.
- The interior of S is the set, int S, of all interior points of S.
- We say that S is open iff every $x \in S$ is an interior point.
- We say that S is closed if the complement $\mathbb{R} \setminus S$ is open.

Problem 13.5 Why does 13.3 part 3 show that $(0,7)$ is an open set? Show that any interval of the form (a,b) with $a<b$ is open.

Problem 13.6 Let U_1 and U_2 be open sets. Show that the union $U_1 \cup U_2$ is open. Then show how to slightly modify your proof to show that the union of any collection of open sets is an open set.

Problem 13.7 Show that any interval of the form $[a,b]$ for $a<b$ is closed.

Problem 13.8 Find a subset of $\mathbb{R}$ that is both open and closed. Show that your set is both open and closed.

Problem 13.9 Let U_1 and U_2 be open sets. Show that the intersection $U_1 \cap U_2$ is open. Then prove that the intersection of finitely many open sets is an open set.

Problem 13.10 Give an example of a collection of open sets whose intersection is not open. Then give an example of a collection of closed sets whose union is not closed.

Problem 13.11 In the problems above, we showed that union of any collection of open sets is again open. We also showed that the intersection of any finite collection of open sets is again closed. Write two related statements concerning intersections and unions of closed sets, and prove that one of these statements is true. (In other words, how many closed sets can you union together and still get a closed set? Similarly, how many closed sets can you intersect and still get a closed set. One should be infinite, and one should be finite.]

Definition 13.12. Let $S \subseteq \mathbb{R}$.

- We say that $x \in \mathbb{R}$ is an accumulation point of S iff every deleted neighborhood of x contains an element of S. Symbolically, we say x in accumulation point of S iff

$$\text{for every } \epsilon > 0, \text{ we have } N^*(x;\epsilon) \cap S \neq \emptyset.$$

- Let S' denote the set of all accumulation points of S.
- If $x \in S$ but $x \notin S'$, then we say x is an isolated point of S.
- The closure of S is the union $\text{cl } S = S \cup S'$.
- The boundary of S is the complement (cl S) $\setminus$ (int S).

Problem 13.13 Consider the set $S=(0,1]$.

1. Show that 0 is an accumulation point of S.
2. Show that 1 is an accumulation point of S.
3. Show that for each $x \in S$, we know x is an accumulation point of S.

4. Show that if $x \notin [0,1]$, then x is not an accumulation point of S.
5. Prove that $S' = [0,1]$. (How do the previous 4 parts show this?)
6. What is the closure of S? What is the interior of S? What is the boundary of S.

Problem 13.14 Consider the set $S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$.

1. Show that 0 is an accumulation point of S.
2. Show that 1 is an isolated point of S.
3. Show that if $x \neq 0$, then x is not an accumulation point. In other words, prove that $S' = \{0\}$.
4. What is the closure of S? What is the interior of S? What is the boundary of S.

Problem 13.15 Let S be the set of rational numbers, so $S = \mathbb{Q}$. Find the closure and interior of S.

Definition 13.16. Let S be a subset of the real numbers. We say that S is compact if and only if S is closed and bounded.

When we are dealing with subsets of the real numbers, the Heine-Borel theorem (14.5 on page 139 in your book) shows that the definition given above is equivalent to the more general topological definition of compact. There is a more general definition of compact, but we'll leave that definition to a course in topology. For our purposes, we'll just say a set is compact iff it is closed and bounded.

Part IV

Sequences and Functions

Chapter 16

Sequences

In second semester calculus, you learned how to work with sequences and series. In this section, we'll take some time to work with the formal definition of a sequence. I've decided to change the structure of the document by listing all the definitions first, and then putting the problems afterwards.

16.1 Definitions

Definition 16.1. A sequence is a function whose domain is the set $\mathbb{N}$. So we could write a sequence as $f : \mathbb{N} \to \mathbb{R}$. However, typically we denote a sequence by (s_n), where $s_n = f(n)$ for each $n \in \mathbb{N}$. If the pattern of values in (s_n) is clear, we may express a sequence by writing the ordered lists $(s_1, s_2, s_3, \ldots)$. The number s_n is called the nth term of the sequence.

Occasionally we may want to change the domain of a sequence to include a few more, or a few less, integers than $\mathbb{N}$. For example, if we wanted to start the sequence at 0 or 4, then we would write $(s_n)_{n=0}^{\infty}$ or $(s_n)_{n=4}^{\infty}$. If no mention is made of the domain, we assume the sequence has the domain $\mathbb{N}$.

Definition 16.2. Suppose x and y are two real numbers. We define $d(x, y)$ to be the distance between x and y. [Note: we could just write $|x - y|$ instead of $d(x, y)$. We'll use $d(x, y)$ to emphasize that we are after distances.]

Definition 16.3: Converge and Diverge. Let (s_n) be sequence. We say that the sequence (s_n) converges to the real number A if and only if

> for every $\epsilon > 0$, there exists a real number N such that for every $n \in \mathbb{N}$ with $n > N$ we have $d(s_n, A) < \epsilon$.

If a sequence (s_n) converges to A, then we call A *the* limit of the sequence and write $\lim_{n\to\infty} s_n = A$, or we may just write $\lim s_n = A$ or $s_n \to A$, A sequence diverges if and only if the sequence does not converge to a real number.

Definition 16.4: Diverge to $\pm\infty$. Let (s_n) be sequence. We say that the sequence (s_n) diverges to $+\infty$, and write $s_n \to \infty$, if and only if

> for every $M \in \mathbb{R}$, there exists a real number N such that for every $n \in \mathbb{N}$ with $n > N$ we have $s_n > M$.

The sequence (s_n) diverges to $-\infty$, and we write $s_n \to -\infty$, if and only if

> for every $M \in \mathbb{R}$, there exists a real number N such that for every $n \in \mathbb{N}$ with $n > N$ we have $s_n < M$.

Definition 16.5: Monotone. Let (s_n) be sequence.

- We say that (s_n) is increasing (or more appropriately nondecreasing) if and only if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$. If $s_n < s_{n+1}$ for all $n \in \mathbb{N}$, then we say the sequence is strictly increasing.
- We say that (s_n) is decreasing (or more appropriately nonincreasing) if and only if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$. If $s_n > s_{n+1}$ for all $n \in \mathbb{N}$, then we say the sequence is strictly decreasing.
- If s_n is increasing or decreasing, then we say s_n is monotone.

Definition 16.6. Let (s_n) be a sequence. We say that (s_n) is a Cauchy sequence if and only if for every $\epsilon > 0$, there exists a real number N such that for every $n, m \in \mathbb{N}$ with $n, m > N$, we have $d(s_n, s_m) < \epsilon$.

16.2 Problems and Theorems

For each problem below, provide a solution. If a theorem shows up on the list below, provide a proof. Remember that it is OK to skip a problem if you are struggling with proving it.

Problem 16.7: The triangle inequality Prove that if x, y, and z are real numbers, then $d(x, y) \leq d(x, z) + d(z, y)$. If x and y are known, for which z does equality hold? [Note: this is often written $|x - y| \leq |x - z| + |z - y|$.]

Problem 16.8 Show that if a sequence has a limit, then the limit is unique. This justifies saying, "*The* limit of a sequence."

Problem 16.9 Let (s_n) be the sequence defined by $s_n = \dfrac{1}{n}$ for each $n \in \mathbb{N}$. Prove that (s_n) converges to 0. If $\epsilon > 0$, what is the smallest N you can choose to satisfy the definition of converges?

Problem 16.10 Let (s_n) be the sequence defined by $s_n = \dfrac{2n+1}{3n+4}$ for each $n \in \mathbb{N}$. Prove that $s_n \to 2/3$. If $\epsilon > 0$, what is the smallest N you can choose to satisfy the definition of converges?

Problem 16.11 Let (s_n) be the sequence defined by $s_n = \dfrac{n^2}{n^2+1}$ for each $n \in \mathbb{N}$. Prove that $s_n \to 1$. If $\epsilon > 0$, what is the smallest N you can choose to satisfy the definition of converges?

Problem 16.12 Suppose $(s_n) \to A$ and $(t_n) \to B$. Prove that $(s_n + t_n)$ converges to $A + B$.

Problem 16.13 Suppose $(s_n) \to A$ and $(t_n) \to B$. Prove that $(s_n t_n)$ converges to AB.

Problem 16.14 Suppose $(s_n) \to A$ and $(t_n) \to B \neq 0$, with $t_n \neq 0$ for all natural numbers n. Prove that (s_n/t_n) converges to A/B.

Problem 16.15 Let (s_n) be a sequence, and A a real number. Prove that (s_n) converges to A if and only if the sequence $(s_n - A)$ converges to 0. This will make proving that sequences converge a lot simpler.

Problem 16.16 Consider the sequence $(s_n) = \left(\dfrac{4n^3 - 2n^2 - 7n}{5n^3 - 3n^2 + 2n - 1}\right)$. In this problem, your job is to prove that $s_n \to 4/5$. You may assume that for all natural numbers, we have $5n^3 - 3n^2 + 2n - 1 > 0$.

1. Show that $(s_n - 4/5) = \left(\dfrac{2n^2 - 43n + 4}{5\left(5n^3 - 3n^2 + 2n - 1\right)}\right)$.
2. Find a k and N so that $|2n^2 - 43n + 4| \leq kn^2$ for all natural numbers $n > N$.
3. Find a k and N so that $|5\left(5n^3 - 3n^2 + 2n - 1\right)| \geq kn^3$ for all natural numbers $n > N$.
4. Prove that (s_n) converges to $4/5$. [Hint: use the previous problem.]

Problem 16.17 Consider the sequence $(s_n) = (1 + (-1)^n)$.

1. Show that $s_n \not\to 2$.
2. Show that $s_n \not\to 0$.
3. Show that s_n diverges.

Problem 16.18 Prove that if a sequence converges, then the sequence is bounded.

Problem 16.19 Prove that if a sequence is monotonic and bounded, then the sequence must converge.

Problem 16.20 Prove that the sequence (n^2) diverges to $+\infty$. Then prove that the sequence $(-n^3 + 2n)$ diverges to $-\infty$.

Problem 16.21 Prove that the sequence $(1/2^n)$ is a Cauchy sequence.

Problem 16.22 Prove that if a sequence is convergent, then the sequence is Cauchy.

Problem 16.23 Prove that if a sequence is Cauchy, then the sequence is convergent.

Chapter 17

Functions

17.1 Definitions

17.2 Problems