Multivariable Calculus

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Introduction

This course may be like no other course in mathematics you have ever taken. We'll discuss in class some of the key differences, and eventually this section will contain a complete description of how this course works. For now, it's just a skeleton.

I received the following email about 6 months after a student took the course:

> Hey Brother Woodruff,
>
> I was reading *Knowledge of Spiritual Things* by Elder Scott. I thought the following quote would be awesome to share with your students, especially those in Math 215 :)
>
> > Profound [spiritual] truth cannot simply be poured from one mind and heart to another. It takes faith and diligent effort. Precious truth comes a small piece at a time through faith, with great exertion, and at times wrenching struggles.

Elder Scott's words perfectly describe how we acquire mathematical truth, as well as spiritual truth.

Contents

Chapter 1

Vectors

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Define, draw, and explain what a vector is in 2 and 3 dimensions.
2. Add, subtract, and scalar multiply vectors. Be able to illustrate each operation geometrically.
3. Compute the dot product and use it to find angles, lengths, projections, and work.
4. Decompose a vector into parallel and orthogonal components.
5. Give equations of lines in both vector and parametric form.

You'll have a chance to teach your examples to your peers prior to the exam.

1.1 The Mars Rover - Curiosity

The Curiosity Rover left Florida in November 2011, and arrived on the surface of Mars in August 2012. Since then, NASA scientists have been using this rover to explore the surface of mars. The rover has a solar panel array that provides a limited amount of power during each Martian day, so activities are limited greatly by power consumption. Let's imagine that we are the scientists who control the movement of the rover. Here are some of our jobs.

1. Get the rover from point A to point B safely, calculating distance, speed, time needed, location at any point in time, etc.
2. Rotate scanners, solar arrays, cameras, etc., as the rover moves.
3. Determine the energy needed to make a trip, and relay the information on to other teams so they know how much power they have to perform other actions.

In this chapter, we'll tackle all the problems above and more. Let's get started.

1.2 Straight Line Motion

Once the Curiosity Rover landed, it started moving around the surface. For simplicity, let's put the landing site at the origin $(0,0)$ in a 2D grid looking down upon the surface of mars. A north-south-east-west grid has been imposed on the grid (which we may change later, but for now let's just assume it has been given). The numbers we'll use in most of our examples are for simplicity in calculation, and unless otherwise stated, all distances will be given in meters.

Problem 1.1 The Curiosity Rover is currently 20m east, and 10 m north, of the landing site (so at location (20,10)). The rover is on flat land and starts moving at a constant speed. Its position after 1 minute is $(15,12)$. After 2 minutes it's at $(10,14)$. After 3 minutes it's at $(5,16)$.

- Each minute, how far does the rover move?
- Where will the rover be after 4 minutes?
- Where will the rover be after t minutes?

Problem 1.2 Suppose for a short time the rover follows a path given by $(x,y) = (1t+3, -2t+4)$. This is the same as writing $(x,y) = (1,-2)t + (3,4)$. See 12.2: 1.

- Construct a plot that shows the location of the rover at time $t = 0,1,2$, and add some arrows as well as a line to illustrate the rover's path.
- What is the speed of the rover?
- When we write the path in the form $(x,y) = (1,-2)t + (3,4)$, what do the quantities $(1,-2)$ and $(3,4)$ have to do with the path?

You encountered an expression of the form $(x,y) = (a,b)t + (c,d)$. The quantity (x,y) represents a point, but the quantity (a,b) represents a change, rather than a point. Both are examples of what we call vectors. They represent a magnitude (for example a distance) in a direction. As one of our main goals in this course is to learn the language used in the sciences, let's formally make some definitions.

Definition 1.1. A vector is a magnitude in a certain direction. If P and Q are points, then the vector $\vec{PQ}$ is the directed line segment from P to Q. This definition holds in 1D, 2D, 3D, and beyond. If $V = (v_1, v_2, v_3)$ is a point in space, then to talk about the vector $\vec{v}$ from the origin O to V we'll use any of the following notations:

Most textbooks use a bold font to write vectors. When writing vectors by hand, it's common to use an arrow above a letter to represent that it's a vector.

$$\vec{v} = \mathbf{v} = \vec{OV} = \langle v_1, v_2, v_3 \rangle = (v_1, v_2, v_3) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$
$$= \underbrace{v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}}_{\text{common in engineering}} = \underbrace{v_1\hat{\mathbf{x}} + v_2\hat{\mathbf{y}} + v_3\hat{\mathbf{z}}}_{\text{common in physics}}.$$

We call v_1, v_2, and v_3 the x, y, and z components of the vector, respectively.

Note that (v_1, v_2, v_3) could refer to either the point V or the vector $\vec{v}$. The context of the problem we are working on will help us know if we are dealing with a point or a vector.

Definition 1.2. Let $\mathbb{R}$ represent the set real numbers. Real numbers are actually 1D vectors. Let $\mathbb{R}^2$ represent the set of vectors (x_1, x_2) in the plane. Let $\mathbb{R}^3$ represent the set of vectors (x_1, x_2, x_3) in space. There's no reason to stop at 3, so let $\mathbb{R}^n$ represent the set of vectors $(x_1, x_2, \ldots, x_n)$ in n dimensions.

In first semester calculus and before, most of our work dealt with problem in $\mathbb{R}$ and $\mathbb{R}^2$. Most of our work now will involve problems in $\mathbb{R}^2$ and $\mathbb{R}^3$. We've got to learn to visualize in $\mathbb{R}^3$.

To find the distance between the two points (x_1, y_1) and (x_2, y_2) in the plane, we create a triangle connecting the two points. The base of the triangle has length $\Delta x = (x_2 - x_1)$ and the vertical side has length $\Delta y = (y_2 - y_1)$. The Pythagorean theorem gives us the distance between the two points as $\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. What about two points in 3D?

Problem 1.3 The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in 3-dimensions is $\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. Construct an appropriate picture and show how to use the Pythagorean theorem repeatedly to prove this fact about distance in 3D.

Problem 1.4 The curiosity rover needs to climb a hill. It's currently sitting at a point $P = (2, 3, -4)$ and needs to get to the point $Q = (0, -1, 1)$ (we could add units and adjust numbers to make this completely realistic, but doing so would complicate the computations). See 12.1:41-58.

- What is the distance between these two points?
- Give an equation of the sphere passing though point Q whose center is at P. Hint: suppose (x, y, z) is another point Q_2 that is the same distance away from point P. What does the distance formula say?

Problem 1.5 For each of the following, construct a rough sketch of the set of points in space (3D) satisfying: See 12.1:1-40.

1. $2 \le z \le 5$
2. $x = 2, y = 3$
3. $x^2 + y^2 + z^2 = 25$

Now that we can compute distances in 3D, we can formally define the magnitude of a vector with a formula.

Definition 1.3. The **magnitude**, or **length**, or **norm** of a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. It is just the distance from the point (v_1, v_2, v_3) to the origin $(0, 0, 0)$.

A **unit vector** is a vector whose length is one unit. We commonly place a hat above unit vectors, as in $\hat{v}$ or $\hat{\mathbf{v}}$, The standard unit vectors are vectors of length one that point in the positive x, y, and z directions, namely

$$\mathbf{i} = \langle 1, 0, 0 \rangle = \hat{\mathbf{x}}, \quad \mathbf{j} = \langle 0, 1, 0 \rangle = \hat{\mathbf{y}}, \quad \mathbf{k} = \langle 0, 0, 1 \rangle = \hat{\mathbf{z}}.$$

Note that in 1D, the length of the vector $\langle -2 \rangle$ is simply $|-2| = \sqrt{(-2)^2} = 2$, the distance to 0. Our use of the absolute value symbols is appropriate, as it generalizes the concept of absolute value (distance to zero) to all dimensions.

Definition 1.4. Suppose $\vec{x} = \langle x_1, x_2, x_3 \rangle$ and $\vec{y} = \langle y_1, y_2, y_3 \rangle$ are two vectors in 3D, and c is a real number. We define vector addition and scalar multiplication as follows:

- Vector addition: $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ (add component-wise).
- Scalar multiplication: $c\vec{x} = (cx_1, cx_2, cx_3)$.

Problem 1.6 Consider the vectors $\vec{u} = (1, 2)$ and $\vec{v} = \langle 3, 1 \rangle$. Start by computing $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$. Then draw $\vec{u}$ and $\vec{v}$ with their tails placed at the origin. Finish by adding arrow to your drawing to show how $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are related to the original two vectors. Come ready to explain what connections you found.

See 12.2:23-24.

The previous problem focused on a geometric understanding of vector addition and vector subtraction. The next problem focuses on a geometric understanding of scalar multiplication.

Problem 1.7 Consider the vector $\vec{v} = (3, -1)$.

See 11.1: 3,4.

- Start by drawing the three vectors $\vec{v}$, $-\vec{v}$, and $3\vec{v}$.
- Suppose that the Curiosity Rover travels along the path given by $(x, y) = \vec{v}t = (3, -1)t$, where t represents time. Draw the rover's path, placing markers to show the location at time $t = 0, 1, 2$.
- How fast the rover is traveling? What property of the vector $\vec{v}$ did you use to answer this question?

Problem 1.8 Consider the two points $P = (1, 2, 3)$ and $Q = (2, -1, 0)$. Write the vector $\vec{PQ}$ in component form (a, b, c). Find the length of vector $\vec{PQ}$. Then find a unit vector in the same direction as $\vec{PQ}$. Finally, find a vector of length 7 units that points in the same direction as $\vec{PQ}$. [If you don't recall what "unit" vector means, please head back up to the definitions and reread them.]

See 12.2: 9,17,25,33 and surrounding.

We've encountered several expressions of the form $(x, y) = (a, b)t + (c, d)$. This is a function where the input is a number t and the output is a vector (x, y). For each input parameter t, we get a single vector output (x, y). Such a function we often call a **parametrization**, as we use a parameter t to describe something. Because the output is a vector, we also call this function a **vector-valued function**. Often, we'll use the variable $\vec{r}$ to represent the radial vector (x, y), or (x, y, z) in 3D, which points from the origin outwards. So we could rewrite the position of the rover as $\vec{r}(t) = (a, b)t + (c, d)$. We use $\vec{r}$ instead of r to remind us that the output is a vector.

Problem 1.9 The rover is no longer on flat ground, rather is sitting at point $P = (0, 2, 3)$. It starts to climb in the direction $\vec{v} = \langle 1, -1, 2 \rangle$.

See 12.5: 1-12.

- Write a vector equation $(x, y, z) = (?, ?, ?)$ for the line that passes through the point P and is parallel to $\vec{v}$.
- Modify your equation to give the position of the rover at any time t, provided you know the rover is moving 4 m per minute.
- Generalize your work to give an equation of the line that passes through the point $P = (x_1, y_1, z_1)$ and is parallel to the vector $\vec{v} = (v_1, v_2, v_3)$.

Problem 1.10 This problem has you connect the equations you have used for lines in 2D to parametrizations in vector form.

- Give a parameterization $(x, y) = (?, ?)t + (?, ?)$ of the line $y = mx + b$.
- Suppose a line has slope m and passes through the point (a, b). Give an equation of the line in the vector form $(x, y) = (?, ?)t + (?, ?)$.

Problem 1.11 Suppose the Curiosity Rover is at $P = (3, 1)$ and needs to get to $Q = (-1, 4)$. See 12.5: 13-20.

- Write a vector equation $\vec{r}(t) = (?, ?)$ for (i.e, give a parametrization of) the line that passes through P and Q, with $\vec{r}(0) = P$ and $\vec{r}(1) = Q$.
- Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is twice the speed of the first line.
- Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is one meter per minute.

1.3 Rotating Cameras - Angles, Dot Products

We now have the ability to accurately describe straight line motion in both 2D and 3D. When we tell the rover to start moving in a specific direction, we know how to track its position. Let's now tackle the issue of rotating the onboard camera. We'll start by assuming the rover is stopped, but eventually we'll want to rotate the camera as the rover moves. We'll need the law of cosines.

Theorem (The Law of Cosines)**.** *Consider a triangle with side lengths a, b, and c. Let θ be the angle between the sides of length a and b. Then the law of cosines states that*

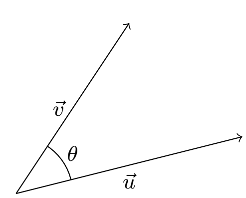
$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

If $\theta = 90^\circ$, then $\cos\theta = 0$ and this reduces to the Pythagorean theorem.

Problem 1.12 Assume that the rover is located at the origin. Currently the rover is pointed in the direction $\langle -1, 2\rangle$. The camera needs to be rotated to look in the direction of $\langle 3, 5\rangle$. Sketch an appropriate diagram and then use the law of cosines to find the angle between the vectors. See 12.3: 9-12.

In the next two problems, we'll develop a simplified version of the law of cosines that will make our work much simpler. We can then return to the above problem while the rover is moving.

Problem 1.13 Consider the two vectors $\vec{u}$ and $\vec{v}$ in the plane (so $\vec{u}, \vec{v} \in \mathbb{R}^2$) shown in margin to the right.



1. Add the vector $\vec{u} - \vec{v}$ to the picture to the right.
2. Use the law of cosines to explain why $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$.

Notice that in your work on the previous problem, the fact that $|\vec{u}-\vec{v}|^2 = |\vec{u}|^2+|\vec{v}|^2-2|\vec{u}||\vec{v}|\cos\theta$ did not require ever referring to the fact that the vectors were in $\mathbb{R}^2$. This fact is true for vectors in general.

Problem 1.14 Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$ (which we write as $\vec{u}, \vec{v} \in \mathbb{R}^3$).

See page 693 if you are struggling.

1. First use the result of the previous problem to explain why

$$|\vec{u}||\vec{v}|\cos\theta = \frac{|\vec{u}|^2+|\vec{v}|^2-|\vec{u}-\vec{v}|^2}{2}.$$

2. Now use the coordinates (u_1, u_2, u_3) and (v_1, v_2, v_3) to simplify the right hand side of the equation above. For example, you'll replace $|\vec{u}|^2$ with $\left(\sqrt{u_1^2+u_2^2+u_3^2}\right)^2 = u_1^2+u_2^2+u_3^2$. For the difference $|\vec{u}-\vec{v}|$, you'll need to subtract coordinates and then compute the magnitude, which gives something like $|\vec{u}-\vec{v}| = \sqrt{(u_1-v_1)^2+\cdots}$. When you are done simplifying you should end up with something quite simple.

Definition 1.5: The Dot Product. Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$. We define the dot product of these two vectors to be

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

A similar definition holds for vectors in $\mathbb{R}^n$, where $\vec{u}\cdot\vec{v} = u_1v_1+u_2v_2+\cdots+u_nv_n$. We just multiply corresponding components together and then add.

Theorem (The Law of Cosines - Dot Product Version)**.** *With the definition of the dot product, we can rewrite the law of cosines as*

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta.$$

Problem 1.15 Use our new rule $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$ to find the angle between each pair of vectors below. If the angle is messy, first write the answer in terms of arccos and then use a calculator to approximate the angle.

See 12.3: 9-12.

1. $1\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $-2\mathbf{i}+1\mathbf{j}+4\mathbf{k}$
2. $(1,2,3)$ and $(-2,1,0)$

In the previous problem, you should have found that one of the pairs of vectors had a dot product that was zero.

Definition 1.6. We say two vectors $\vec{u}$ and $\vec{v}$ are orthogonal when $\vec{u}\cdot\vec{v} = 0$.

Problem 1.16 Find two vectors orthogonal to $(1,2)$. Then find 4 vectors orthogonal to $(3,2,1)$.

The dot product provides a really easy way to determine when two vectors meet at a right angle. The dot product is precisely zero when this happens. The next problem has you justify this fact.

Problem 1.17 Show that if two nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal, then the angle between them is 90°. Then show that if the angle between them is 90°, then the vectors are orthogonal.

See page 694.

Note: There are two things to show above. First, assume that the vectors are orthogonal (so their dot product is zero) and use this to compute the angle. Then second, assume that the angle between them is 90° and use this to compute the dot product.

We invented a new operation. Let's see what properties the dot product has.

Problem 1.18 Mark each statement true or false. Then make up an example to illustrate why you gave your answer. I have done the first as an example. You can assume that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ and that $c \in \mathbb{R}$.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

   Solution: This is true. If $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$, then we know $\vec{u} \cdot \vec{v} = (a, b) \cdot (c, d) = ac + bd$ and $\vec{v} \cdot \vec{u} = (a, b) \cdot (c, d) = ca + db$. Since $ab = ba$ and $cd = dc$, we see that $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ is true.

2. $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$.
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$.
4. $\vec{u} + (\vec{v} \cdot \vec{w}) = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{w})$.
5. $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$.
6. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

The last property above is extremely important, namely it connects the length of a vector to the dot product. Any time we are working with either lengths or angles, there is a dot product hiding in the background.

Problem 1.19 Suppose that the rover is moving in the straight line path given by $(x, y) = (a, b)t + (c, d)$, where the landing site is located at position $(0, 0)$. One of the science teams wants you to capture footage of an anomaly located at position (m, n), ideally they'd like several shots from multiple angles as the rover moves. Your job is to make sure the camera is properly rotated, throughout the drive, so that the camera is constantly pointing at the anomaly. Give a formula, $\theta(t)$, for the angle between the direction the rover is headed, and the direct line of sight to the anomaly. In addition, give a formula for the distance $d(t)$ from the rover to the anomaly.

1.4 Energy, Work, Vector Projections

How much energy does it take to get the rover from point A to point B? In this last section, we'll tackle this question.

To start, we first need to discuss the concept of work - a transfer of energy. When our bodies process food into mechanical energy, work is done (energy is transferred from chemical to mechanical). We'll mostly be examining the work that occurs as a force acts on a object through a displacement. As Curiosity moves along the surface of Mars, there are several forces acting on the rover. Two of these forces are gravity and surface friction. As the rover rises, negative work is done by gravity which means we must supply positive work to enable the

There are more forces than just gravity and surface friction acting on the rover. Air resistance is probably negligible on the surface of mars. However, we cannot ignore the internal friction from the moving parts of the rover. We'll just focus on the two external forces, gravity and surface friction, in this chapter.

rover to go uphill. Surface friction opposes any motion, and so negative work is done throughout all movement which means we must supply energy (work) to counteract the negative work from surface friction. Both of these forces, gravity and friction, act on the rover throughout its entire path. We now have precisely the tools needed to understand the work (energy transfer) that occurs because of these forces.

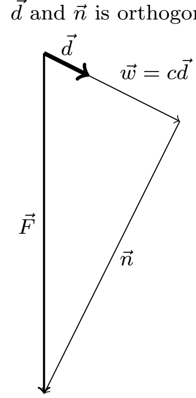
Experiments show that if a force $\vec{F}$ acts on an object through a displacement $\vec{d}$, then the work done by $\vec{F}$ through the displacement $\vec{d}$ is $W = |\vec{F}||\vec{d}|$, the product of the magnitudes of the force and the displacement. This basic definition has a few assumptions.

- The force $\vec{F}$ must act in the same direction as the displacement. If the force and displacement oppose each other, then the work done is simply $W = -|\vec{F}||\vec{d}|$.
- The force $\vec{F}$ must be constant throughout the entire displacement.
- The displacement must be in a straight line.

We'll eventually remove all these assumptions and be able to compute work done by non constant forces as objects move along curved paths. For now, we'll keep the paths straight and the forces constant. However, we will use what we've learned about vectors to remove the first assumption about forces and displacements acting in the same, or opposite, directions.

Suppose a heavy box needs to be lowered down a ramp. The box exerts a downward force of say 200 Newtons (gravity), which we could write in vector notation as $\vec{F} = \langle 0, -200\rangle$. If the ramp was placed so that the box needed to be moved right 6 m, and down 3 m, then we'd need to get from the origin $(0, 0)$ to the point $(6, -3)$. This displacement can be written as $\vec{d} = \langle 6, -3\rangle$. The force $\vec{F}$ acts straight down, rather than parallel to the displacement. Let's find out how much of the force $\vec{F}$ acts in the direction of the displacement. We are going to break the force $\vec{F}$ into two components, one component in the direction of $\vec{d}$, and another component orthogonal to $\vec{d}$. We'll use the component that is orthogonal to $\vec{d}$ to analyze surface friction afterwards.

In the diagram below, we have $\vec{F} = \vec{w} + \vec{n}$ where $\vec{w}$ is parallel to $\vec{d}$ and $\vec{n}$ is orthogonal to $\vec{d}$.



Problem 1.20 Read the preceding paragraph. Rather than working with the specific numbers given in that paragraph, please use $\vec{F}$ and $\vec{d}$ to represent any vector, so that when we are done with this problem we'll have a symbolic solution.

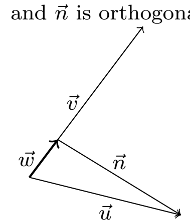
We want to write $\vec{F}$ as the sum of two vectors $\vec{F} = \vec{w} + \vec{n}$, where $\vec{w}$ is parallel to $\vec{d}$ and $\vec{n}$ is orthogonal to $\vec{d}$. Since $\vec{w}$ is parallel to $\vec{d}$, we can write $\vec{w} = c\vec{d}$ for some unknown scalar c. This means that $\vec{F} = c\vec{d} + \vec{n}$. Use the fact that $\vec{n}$ is orthogonal to $\vec{d}$ to show that $c = \dfrac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}$.

[Hint: Dot each side of $\vec{F} = c\vec{d} + \vec{n}$ with $\vec{d}$ and distribute. You'll need to use the fact that $\vec{n}$ and $\vec{d}$ are orthogonal to remove $\vec{n} \cdot \vec{d}$ from the problem. This should turn the vectors into numbers, so you can use division and solve for c directly. Don't spent more than 10 minutes on this problem.]

Problem 1.21 Consider the vectors $\vec{u}$ and $\vec{v}$ in the diagram to the right. We can write $\vec{u}$ as the sum of a vector that is parallel to $\vec{v}$ (called $\vec{w}$ below) and a vector that is orthogonal to $\vec{v}$ (called $\vec{n}$ below). This gives us $\vec{u} = \vec{w} + \vec{n}$.

1. Let θ be the angle between $\vec{u}$ and $\vec{v}$. Use right triangle trigonometry to explain why the length of $\vec{w}$ is given by $|\vec{w}| = |\vec{u}| \cos\theta$.

In the diagram below, we have $\vec{u} = \vec{w} + \vec{n}$ where $\vec{w}$ is parallel to $\vec{v}$ and $\vec{n}$ is orthogonal to $\vec{v}$.



Notice the right angle where vectors $\vec{n}$ and $\vec{w}$ meet.

2. Now that we know the length of $\vec{w}$, explain why $\vec{w} = (|\vec{u}|\cos\theta)\dfrac{\vec{v}}{|\vec{v}|}$. See problem 1.8 if you need help.

3. We have a formula that connects the dot product to the cosine of the angle between two vectors. Show the steps that transform the equation above into the equation

$$\vec{w} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|}.$$

Can you explain why this also means

$$\vec{w} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}?$$

The previous two problems give us the definition of a projection.

Definition 1.7. The projection of $\vec{F}$ onto $\vec{d}$, written $\text{proj}_{\vec{d}}\vec{F}$, is defined as

$$\text{proj}_{\vec{d}}\vec{F} = \underbrace{\left(\frac{\vec{F}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\right)\vec{d}}_{\text{quick computation method}} = \underbrace{\left(\frac{\vec{F}\cdot\vec{d}}{|\vec{d}|}\right)\frac{\vec{d}}{|\vec{d}|}}_{\substack{\text{geometric method}\\ \text{magnitude times direction}}}.$$

Definition 1.8: Vector Decomposition into parallel and orthogonal components. We can write $\vec{F}$ as the sum of a vector parallel to $\vec{d}$ plus a vector orthogonal to $\vec{d}$, written

$$\vec{F} = \vec{F}_{\|d} + \vec{F}_{\perp d},$$

These two vectors we call the vector component of $\vec{F}$ that is parallel to $\vec{d}$, and the vector component of $\vec{F}$ that is orthogonal to $\vec{d}$. Note that the projection of $\vec{F}$ onto $\vec{d}$ is precisely the vector component of $\vec{F}$ that is parallel to $\vec{d}$. Also notice that the orthogonal component is simply the difference $\vec{F}_{\perp d} = \vec{F} - \vec{F}_{\|d}$.

Let's practice using these new definitions before we return to studying work.

Problem 1.22 Let $\vec{F} = (-1, 2)$ and $\vec{d} = (3, 4)$. Start by computing $\vec{F}_{\|d} = \text{proj}_{\vec{d}}\vec{F}$ and $\vec{F}_{\perp d}$. Then construct a picture that shows the relationship between $\vec{F}$, $\vec{d}$, $\text{proj}_{\vec{d}}\vec{F}$, and $\vec{F}_{\perp d}$.

See 12.3:1-8 (part d).

Once you have finished the computations and related picture above, change the force to $\vec{F} = (-2, 0)$. but keep $\vec{d} = (3, 4)$. Then construct a similar picture, showing the relationship between $\vec{F}$, $\vec{d}$, $\text{proj}_{\vec{d}}\vec{F}$, and $\vec{F}_{\perp d}$. Feel free to construct this picture with, or without, doing any computations.

We're now ready to return the box on a ramp, described prior to problem 1.20. Gravity exerts a force of $\vec{F} = \langle 0, -200\rangle$ N. Gravity will do work on the box through a displacement of $\langle 6, -3\rangle$ m. This work transfers the potential energy of the box into kinetic energy (remember that work is a transfer of energy). If the surface were frictionless and the only force acting on the box were gravity, then 100% of the work done by gravity would become kinetic energy of the box. Let's first find the work done gravity, and then the work done by friction.

Problem 1.23 We will find the amount of work done by the force $\vec{F} = \langle 0, -200\rangle$ through the displacement $\vec{d} = \langle 6, -3\rangle$ by doing the following:

See 12.3: 24, 41-44.

1. Find the projection of $\vec{F}$ onto $\vec{d}$. This tells us how much force acts in the direction of the displacement. Find the magnitude of this projection.
2. Since work equals $W = |\vec{F}||\vec{d}|$, provided $\vec{F}$ acts in the direction of $\vec{d}$, multiply $|\vec{F}_{\|\vec{d}}|$ by $|\vec{d}|$ to obtain the work, and simplify your answer.
3. Now simply compute the dot product $\vec{F}\cdot\vec{d}$.

   (Hint: Did you get the same answer as the second part, but with a lot less computations? You have just shown that the dot product gives work, $W = \vec{F}\cdot\vec{d}$, when $\vec{F}$ and $\vec{d}$ are not in the same direction.)

The dot product gives us the work done by $\vec{F}$ through a displacement $\vec{d}$ when $\vec{F}$ and $\vec{d}$ are not in the same direction. Remember that the dot product is a number, which means it may be hard to visualize. Connecting the dot product to work done by one vector in the direction of another can often lead to a good geometric description of the dot product.

Problem Answer each of the following, assuming that none of the vectors are the zero vector.

1. Suppose $\vec{u}\cdot\vec{v} = 0$. What do you know about the two vectors?
2. Suppose $\vec{u}\cdot\vec{v} > 0$. What do you know about the two vectors?
3. Suppose $\vec{u}\cdot\vec{v} < 0$. What do you know about the two vectors?

See [1] for a solution.

Now let's consider the work done by surface friction. Surface friction is often classified into two types, kinetic friction and static friction. The only difference is whether the object is moving (kinetic) at the point of contact, or is not moving (static) at the point of contact. A box sliding down a ramp results in kinetic friction. However the tire treads of a rover moving across Mars result in static friction (the treads don't actually move while on the surface - hopefully no skidding). In either case, we can often model surface friction $\vec{F}_f$ as a vector whose magnitude is proportional to the force between the surfaces (often called the normal force $\vec{N}$). Symbolically, we write this as $|\vec{F}_f| = \mu|\vec{N}|$ where μ is the proportionality constant (some use μ_k for kinetic friction, and μ_s for static friction). For a box sliding down a ramp, or a rover traveling across the surface of mars, the normal force between the two surfaces is just the orthogonal component of the force from gravity, so $\vec{N} = \vec{F}_{\perp\vec{d}}$. Note that the direction of friction is always opposite the motion, so in our case $-\vec{d}$.

Problem 1.24 Suppose that the Curiosity rover needs to climb a hill, moving through a displacement $\vec{d} = (a, b)$ (so a meters horizontally and b meters vertically). Gravity on Mars creates a force of $\vec{F} = (0, -mg)$, where m is the mass of the rover and g is the gravitational constant associated with mars (about $3.711 m/s^2$ - a little more than a third of earth's).

[1] When the dot product is zero, we know that the two vectors meet at a 90° angle. Thinking about this in terms of work, this means that the force has no portion in the direction of the displacement, hence there is no work done. If the dot product is positive, then the force has a portion acting in the direction of the displacement. This means that the angle between the two vectors is acute. Similarly if the dot product is negative then the angle must be obtuse (greater than 90°.)

1. Compute the component of $\vec{F}$ that is orthogonal to $\vec{d}$ (i.e. find $\vec{N}$). Show how to simplify your result to obtain
$$\vec{N} = \vec{F}_{\perp d} = \frac{mga}{a^2+b^2}(b, -a).$$

2. We know that the magnitude of frictional force is proportional to $|\vec{N}|$. Use this fact to obtain the formula
$$|\vec{F}_f| = \mu \frac{mga}{|\vec{d}|}.$$

3. We know that the frictional force $\vec{F}_f$ points opposite the displacement $\vec{d}$. We also have a formula now for the magnitude of this frictional force. Use these two pieces of information to give a formula for $\vec{F}_f$.

   (Hint: We already know how to give a vector of a length 3 that points in the direction $(1, 2)$, it's just $3\frac{(1,2)}{\sqrt{5}}$. We repeat this idea here as we know the magnitude from the previous part, and the direction is given.)

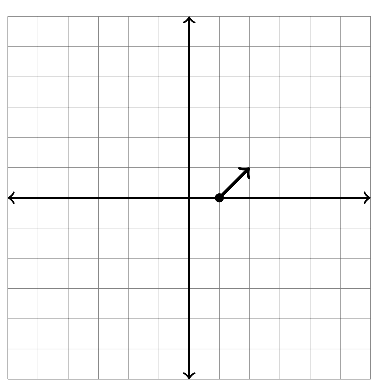
The previous problem was an introduction to frictional forces. These forces can get quite complicated, but analyzing them always requires being able to analyze the vector component of a force that is orthogonal to the point of contact. You can study this topic more in future courses as it pertains to your major.

Gravity is our first example of a vector field. Other important vector fields arise when we study magnetism, electricity, fluid flow, and more. To analyze how a river flows, we can construct a plot of the river and at each point in the river we draw a vector that represents the velocity at that point. This creates a collection of many vectors drawn all at once, where the base of each velocity vector is placed at the point where the velocity occurs. For gravity, a similar picture can be drawn, though all the vectors will point down with the same magnitude. The next problem has us construct a plot of a vector field.

Problem 1.25: Vector Fields Consider the function $\vec{F}(x, y) = \langle x - 2y, x + y\rangle$. This is a function where the input is a point (x, y) in the plane, and the output is the vector $\langle x - 2y, x + y\rangle$. For example, if we input the point $(1, 0)$, then the output is $\langle 1 - 2(0), 1 + 0\rangle = \langle 1, 1\rangle$. To construct a vector field plot, we draw the vector $\langle 1, 1\rangle$ with its base located at the input $(1, 0)$. In the picture below, based at $(1, 0)$ we draw a vector that points right 1 and up 1.

1. Complete the table below and add the other 7 vectors to the graph.

(x, y)	$\langle x - 2y, x + y\rangle$
$(1, 0)$	$\langle 1, 1\rangle$
$(1, 1)$	
$(1, -1)$	
$(0, 1)$	
$(0, -1)$	
$(-1, 0)$	
$(-1, 1)$	
$(-1, -1)$	



2. Repeat the above for the vector field $\vec{F}(x, y) = (-2y, 3x)$, constructing a vector field plot consisting of 8 vectors.

Problem 1.26 Suppose a rover is currently moving and has a velocity vector $\vec{v} = (3,4)$. A force acts on the rover causing an acceleration of $\vec{a} = (-1,5)$. The rover is currently at the location $(2,-3)$. Start by drawing a picture that shows the rover's location along with the velocity and acceleration vectors drawn with the base at the rover's location.

1. Find the vector component of the acceleration that is parallel to the velocity (so find $\vec{a}_{\|\vec{v}}$).
2. Find the vector component of the acceleration that is orthogonal to the velocity (so find $\vec{a}_{\perp\vec{v}}$).
3. Will this acceleration cause the rover to speed up or slow down? Explain.
4. Will this acceleration cause the rover to turn left or right? Explain.

Problem 1.27 A probe above Mars is currently moving and has a velocity vector $\vec{v} = (-2,1,2)$. The onboard thrusters apply a force that causes an acceleration of $\vec{a} = (0,2,-3)$.

1. Find the vector component of the acceleration that is parallel to the velocity (so find $\vec{a}_{\|\vec{v}}$).
2. Find the vector component of the acceleration that is orthogonal to the velocity (so find $\vec{a}_{\perp\vec{v}}$).
3. Will this acceleration cause the satellite to speed up or slow down? Explain.
4. How would you interpret $\vec{a}_{\perp\vec{v}}$?

This final problem has you practice using the new words we developed.

Problem 1.28 Let $P = (2,5)$ and $Q = (3,-4)$.

1. Give a vector equation of the line through P and Q.
2. Compute both $\vec{P}_{\|\vec{Q}}$, and $\vec{P}_{\perp\vec{Q}}$. Construct a picture that shows these two vectors and their relationship to $\vec{P}$ and $\vec{Q}$.
3. Compute the work done by $\vec{P}$ through a displacement $\vec{Q}$.
4. What is the angle between $\vec{P}$ and $\vec{Q}$.

1.5 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 2

Curved Motion

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Be able to graph and give equations of parabolas, ellipses, and hyperbolas.
2. Use a change-of-coordinates involving translation and stretching to give an equation of and graph a curve.
3. Model motion in the plane using parametric equations.
4. Find derivatives and tangent lines for parametric equations. Explain how to find velocity, speed, and acceleration from a parametrization.
5. Use integrals to find the length of a parametric curve, the work done by a non constant force along a curve, and related quantities.

You'll have a chance to teach your examples to your peers prior to the exam.

In the previous chapter, our rover always moved in a straight path. Any satellite orbiting Mars will clearly not move in straight path. We need the ability to move in paths that are not straight. In addition, we may need to change our coordinate system to use a different origin, or adjust the scale we use to measure things. In this chapter, we'll add the abilities to move in nonlinear paths, as well as change one coordinate system into another. Let's get started.

2.1 A New View - Changing Coordinates

In this first section, we'll tackle changing the coordinate system. Given a graph of a function $y = f(x)$, how do we modify the equation $y = f(x)$ to obtain a new function that has been shifted? You might recall several rules that allow you to translate functions left and right, up and down, or even rescale (stretch) the functions vertically and horizontally. For example, if we start with the parabola $y = x^2$, then the equation $y = (x-2)^2 + 3$, or equivalently $y - 3 = (x-2)^2$, is the same parabola except we have shifted it right 2 and up 3.

In this section, we'll revisit the concepts of translating and stretching functions. All of these ideas are part of a bigger picture which we'll refer to as changing coordinates. In the example above we had two curves, namely $y = x^2$ and the translated $y - 3 = (x-2)^2$. To simplify our work, let's use the variables u and v for the starting equation and x and y for the translated equation. Notice then that we have $v = u^2$ and $y - 3 = (x-2)^2$. If we just let $v = y - 3$ and $u = x - 2$, or equivalently $x = u + 2$ and $y = v + 3$, then we have equations

In practice, we generally don't use new variables but might instead write the change-of-coordinates as $x_n = x_o + 2$ and $y_n = y_o + 3$ where n stands for "new" and o stands for "old". After making the change, we just drop subscripts.

that allow us to change between uv and xy coordinates. We call each pair of equations a change-of-coordinates. We'll often write our changes of coordinates by solving for x and y, as the equations $x = u + 2$ and $y = v + 3$ clearly show us that the x-values should be the old u-values shifted 2 units right and the y-values should be the old v-values shifted 3 units up.

Problem 2.1 Consider the circle $u^2 + v^2 = 1$ and the change-of-coordinates given by $x = 2u + 1$ and $y = 3v + 4$. If you didn't read the paragraphs above this problem, please do so before you start working on this problem.

1. Draw the curve $u^2 + v^2 = 1$ in the uv plane.

2. The change of coordinates given above allows us to construct a graph of this curve in the xy plane. Once simple way to do this is make a u, v, x, y table. We know the circle above passes through the points $(\pm 1, 0)$ and $(0, \pm 1)$, so we can use the change-of-coordinates equations $x = 2u + 1$ and $y = 3v + 4$ to find the corresponding points in the xy plane, as seen on the right. Use this table to construct a graph of the curve in the xy plane.

(u, v)	(x, y)
$(1, 0)$	$(3, 4)$
$(-1, 0)$	$(-1, 4)$
$(0, 1)$	$(?, ?)$
$(0, -1)$	$(?, ?)$

3. Solve the change-of-coordinate equations for u and v and use substitution to give an equation of the curve using x and y coordinates.

4. Use the same change-of-coordinates with the curve $v = u^2$ to graph the curve in both the uv and xy plane. Then state an equation of the curve in the x and y coordinates. You may find the table to the right helpful.

(u, v)	(x, y)
$(-2, 4)$	$(-3, 16)$
$(-1, 1)$	$(?, ?)$
$(0, 0)$	$(?, ?)$
$(1, 1)$	$(3, 7)$
$(2, 4)$	$(?, ?)$

5. How would you describe the connection between the graphs you made in the uv plane and their corresponding graph in the xy plane?

In the previous problem you were given a curve using uv coordinates, and then asked to use a change-of-coordinates to construct a graph in the xy plane. The next problem has you do this in reverse, namely gives you curve in the xy plane and asks you to state the change of coordinates that would reduce the curve to a simple object in the uv plane.

Problem 2.2 Start by graphing the parabola $y = 3(x - 1)^2 + 2$.

1. Give a change-of-coordinates of the form $x = ?u + ?$, $y = ?v + ?$ that will transform the curve $v = u^2$ in the uv plane to the parabola $y = 3(x-1)^2+2$.

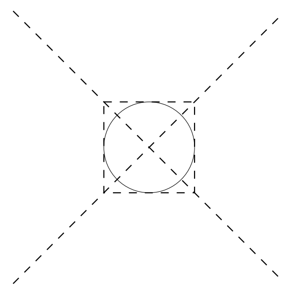
2. Which of $y = 3(x - 1)^2 + 2$ or $\dfrac{y - 2}{3} = (x - 1)^2$ makes it easier to see the change of coordinates?

3. Construct a graph of the parabola $\dfrac{y + 1}{2} = \left(\dfrac{x - 3}{4}\right)^2$. Optionally, state the change-of-coordinates you used.

Problem 2.3 Consider the curve $x^2 - y^2 = 1$, which we call a hyperbola.

1. Show that $y = \pm x\sqrt{1 - \frac{1}{x^2}}$, and then use this fact to explain why y approaches the lines $y = \pm x$ as x gets large. We call these two lines the asymptotes of the hyperbola, and any good graph of a hyperbola should include them.

2. We'll now construct a graph of the hyperbola. One simple way to draw the asymptotes is to start by constructing a rectangular box with corners at $(1, \pm 1)$ and $(-1, \pm 1)$. Connecting opposing corners of this box gives the asymptotes $y = \pm x$. The circle $x^2 + y^2 = 1$ should fit nicely inside your box (see the picture on the right). Now use software to view a graph of the hyperbola $x^2 - y^2 = 1$ and add it to your picture, making sure the hyperbola follows the asymptotes as $|x|$ gets large. When you construct your graph on your paper, make sure your sketch includes the box, lines, and circle, as well as the hyperbola.



3. Now construct a graph of $\frac{(x-1)^2}{4} - \frac{(y-4)^2}{9} = 1$, including an appropriate box and asymptotes. If you want to find the box easily, start by drawing the ellipse $\frac{(x-1)^2}{4} + \frac{(y-4)^2}{9} = 1$, and then add the box, the asymptotes, and finally the hyperbola.

Problem 2.4 Consider the parabola $v = u^2$ and the hyperbola $u^2 - v^2 = 1$. With each problem below, please make a u, v, x, y table before constructing your graph.

1. Using the change of coordinates $x = v$, $y = u$, draw the corresponding parabola and hyperbola in the xy-plane.

2. Using the change of coordinates $x = 2v + 1$, $y = 3u + 4$, draw the corresponding parabola in the xy-plane.

3. Draw both the ellipse $\frac{(y-4)^2}{9} + \frac{(x-1)^2}{4} = 1$ and hyperbola $\frac{(y-4)^2}{9} - \frac{(x-1)^2}{4} = 1$ in the xy-plane.

Problem 2.5 Consider the change of coordinates $x = au + h$, $y = bv + k$.

1. Use this change of coordinates to rewrite the parabola $v = u^2$, the ellipse $u^2 + v^2 = 1$, and the hyperbola $u^2 - v^2 = 1$ using xy coordinates.

2. In your own words, how do each of the values of a, b, h, and k, change the graph of the curve in the uv plane when you draw the graph in the xy plane. Include pictures to accompany your words.

Problem 2.6 Graph each of the ellipses below by hand. Be prepared to explain how you obtained the graph. See 11.6: 17-24.

1. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

2. $16x^2 + 25y^2 = 400$ [Hint: divide by 400.]

3. $\frac{(x-1)^2}{5} + \frac{(y-2)^2}{9} = 1$.

Problem 2.7 Graph each of the hyperbolas below by hand. Make sure your graph shows the hyperbola's asymptotes. See 11.6: 27-34.

1. $\frac{x^2}{25} - \frac{y^2}{9} = 1$ and $\frac{y^2}{9} - \frac{x^2}{25} = 1$
2. $25y^2 - 16x^2 = 400$ [Hint: divide by 400.]
3. $\frac{(x-1)^2}{5} - \frac{(y-2)^2}{9} = 1.$

Problem 2.8 Consider the hyperbola $\frac{(x-1)^2}{5} - \frac{(y-2)^2}{9} = 1$ from the previous problem. Use Mathematica and the ContourPlot[] command to produce a nice plot with reasonable bounds. Then add to your plot the asymptotes, using a different color. Your final plot should include both the hyperbola and the asymptotes in the same plot.

See 11.6: 27-34.

If you are struggling with getting the graphs to show up in the same plot, try using the Show[] command to combine several plots. Look up Show[] in the help menu, and you'll see several examples of how to combine several plots into one. Then you can make one plot for each curve, pick the color you want for that plot, and finish by combining all the plots with Show[].

If you need help changing the color, open the help menu for ContourPlot[]. Scroll to the bottom of the examples and expand the "Options" section. There are several options that have Color in the name, and Contour in the name. You want to change the style of the Contour, so expand the "ContourStyle" option. From there, look for an example that you like.

2.2 Parametric Equations

In the previous chapter, we learned how to describe the position (x, y) of a rover with respect to a parameter t. All of the equations we used in the previous chapter were lines. We now extend this work to study curved path, allowing our rover to follow any path at all.

In middle school, we learned to write an equation of a line as $y = mx + b$. In vector notation, we can now write this as the vector equation $(x, y) = (1, m)t + (0, b)$. Equivalently we can write the two equations

$$x = 1t + 0, y = mt + b,$$

which we call parametric equations for the line. We were able to quickly develop equations of lines in space, by just adding a third equation for z. Parametric equations provide us with a way of specifying the location (x, y, z) of an object by giving an equation for each coordinate.

Definition 2.1. If each of f and g are continuous functions, then the curve in the plane defined by $x = f(t), y = g(t)$ is called a parametric curve, and the equations $x = f(t), y = g(t)$ are called parametric equations for the curve. You can generalize this definition to 3D and beyond by just adding more variables.

Problem 2.9 By plotting points, construct graphs of the three parametric curves given below (just make a t, x, y table, and then plot the (x, y) coordinates). Place an arrow on your graph to show the direction of motion.

See 11.1: 1-18. This is the same for all the problems below.

1. $x = \cos t, y = \sin t$, for $0 \le t \le 2\pi$.
2. $x = \sin t, y = \cos t$, for $0 \le t \le 2\pi$.

3. $x = \cos t, y = \sin t, z = t$, for $0 \le t \le 4\pi$. You'll need an x, y, z, t table. Plot your points (x, y, z) in 3D.

4. Now use Mathematica to plot these curves. Use the ParametricPlot[] command for the first two, and ParametricPlot3D[] for the last.

Problem 2.10 Plot the path traced out by the parametric curve $x = 1 + 2\cos t, y = 3 + 5\sin t$. Then use the trig identity $\cos^2 t + \sin^2 t = 1$ to give a Cartesian equation of the curve (an equation that only involves x and y). What kind of motion would you model with this kind of parametrization?

Problem 2.11 Find parametric equations for a line that passes through the points $(0, 1, 2)$ and $(3, -2, 4)$. What kind of motion would you model with this kind of parametrization?

What we did in the previous chapter should help here.

Problem 2.12 Plot the path traced out by the parametric curve $\vec{r}(t) = (t^2 + 1, 2t - 3)$. Give a Cartesian equation of the curve (eliminate the parameter t).What kind of motion would you model with this kind of parametrization?

Problem 2.13 Consider the parametric curve given by $x = \tan t, y = \sec t$. Plot the curve for $-\pi/2 < t < \pi/2$. Give a Cartesian equation of the curve. (A trig identity will help - what identity involves both tangent and secant?) [Hint: this problem will probably be easier to draw if you first find the Cartesian equation, and then plot the curve.] What kind of motion would you model with this kind of parametrization?

2.2.1 Derivatives and Tangent lines

We now tackle calculus on parametric curves. The derivative of a vector valued function is defined using the same definition as first semester calculus.

Definition 2.2. If $\vec{r}(t)$ is a vector equation of a curve (or in parametric form just $x = f(t), y = g(t)$), then we define the derivative to be

$$\frac{d\vec{r}}{dt} = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

Because vector addition is done component-wise, this is the same as just taking the derivative of each component separately, so $\dfrac{d\vec{r}}{dt} = \left(\dfrac{dx}{dt}, \dfrac{dy}{dt}\right)$.

In first semester calculus, we used derivatives to find velocity and acceleration. Let's verify that this is true for parametric curves as well.

Problem 2.14 Suppose the Curiosity rover travels in a circular path given by the parametric curve $\vec{r}(t) = (3\cos t, 3\sin t)$.

See 13.1:5-8 and 13.1:19-20

1. Graph the curve $\vec{r}$, and compute $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$.

2. On your graph, draw the vectors $\frac{d\vec{r}}{dt}\left(\frac{\pi}{4}\right)$ and $\frac{d^2\vec{r}}{dt^2}\left(\frac{\pi}{4}\right)$ with their tail placed on the curve at $\vec{r}\left(\frac{\pi}{4}\right)$. Are these vectors the velocity and acceleration?

3. Give a vector equation of the tangent line to this curve at $t = \frac{\pi}{4}$. (You know a point and a direction vector.)

Definition 2.3. If an object moves along a path $\vec{r}(t)$, we can find the velocity and acceleration by just computing the first and second derivatives. The velocity is $\frac{d\vec{r}}{dt}$, and the acceleration is $\frac{d^2\vec{r}}{dt^2}$. Speed is a scalar, not a vector. The speed of an object is just the length of the velocity vector.

Problem 2.15 Consider the curve $\vec{r}(t) = (2t + 3, 4(2t - 1)^2)$.

1. Construct a graph of $\vec{r}$ for $0 \leq t \leq 2$.
2. If this curve represents the path of a rover (meters for distance, minutes for time), find the velocity of the rover at any time t, and then specifically at $t = 1$. What is the rover's speed at $t = 1$?
3. Give a vector equation of the tangent line to $\vec{r}$ at $t = 1$. Include this on your graph.
4. Explain how to obtain the slope of the tangent line, and then write an equation of the tangent line using point-slope form. [Hint: How can you turn the direction vector, which involves (dx/dt) and (dy/dt), into the number given by the slope (dy/dx)?]

The next problem has you decompose an acceleration vector into the components that are parallel and orthogonal to the velocity vector. It's a partial review of what we did in the previous chapter. For those of you taking dynamics in the future, this decomposition is central to that course.

Problem 2.16 Suppose an object travels along the path given by $\vec{r}(t) = (3t, -2t^2)$. The velocity is $\vec{v}(t) = (3, -4t)$ and the acceleration is $\vec{a}(t) = (0, -4)$. At time $t = 1$, these vectors are $\vec{v}(1) = (3, -4)$ and $\vec{a}(1) = (0, -4)$.

1. Why do we know that the acceleration and velocity vectors are not in the same direction?
2. What is the vector component of the acceleration vector that is parallel to the velocity vector? In other words, what is $\text{proj}_{\vec{v}}\vec{a}$. We'll call this vector $\vec{a}_{\parallel\vec{v}}$.
3. What is the vector component of the acceleration vector that is orthogonal to the velocity vector? We'll call this vector $\vec{a}_{\perp\vec{v}}$.
4. Draw a picture that shows the relationship among $\vec{v}$, $\vec{a}$, $\vec{a}_{\parallel\vec{v}}$, and $\vec{a}_{\perp\vec{v}}$.

2.2.2 Integration, Arc Length, Work, and More

We've been focusing on describing non linear motion by using parametric curves. In the previous section, we used derivatives to obtain the velocity and acceleration vectors. We'll finish this chapter by analyzing how to obtain the work done by a non constant force, along an arbitrary path.

Let's think about the Curiosity rover again. In the first chapter, we saw that the work done by surface friction through a displacement $(5, 2)$ is the same as through a displacement $(5, 0)$. A rise or fall in height does not affect the work done by surface friction. The only thing that mattered was the distance we traveled horizontally on the surface. If we moved right 5 meters, and then back 5 meters, the total distance traveled is 10 meters (though the displacement is 0 meters). The 10 meters traveled is the part we need to determine work done by surface friction.

Imagine now that the rover starts moving along the surface of mars following the path parametrized by $\vec{r}(t) = (x(t), y(t), z(t))$. The third coordinate, $z(t)$, won't affect the work done by surface friction. The thing we need to determine is the distance traveled by the rover through the path $\vec{r}(t) = (x(t), y(t))$. If the path is a straight line, then we just use the Pythagorean theorem. If the path is not a straight line, then we have a tougher problem. The key is to break this tough problem up into lots of smaller problems, each simple, and then use integration to find the total distance. The following exercise, whose solution is provided in the footnotes, reminds us how to find the area of a region by breaking the region up into lots of little regions. Try doing this exercise before moving on.

Exercise Consider a function $y = f(x)$ for $a \leq x \leq b$ and assume that $f(x) \geq 0$. Imagine cutting the x-axis up into many little bits, where we use dx to represent the length of each little bit. See [1] for a solution.

1. If we pick one of the tiny bits of length dx whose left endpoint is located at x, what does the quantity $dA = f(x)dx$ give us? Construct a picture to illustrate this.
2. Why is the total area given by $A = \int_a^b f(x)dx$.

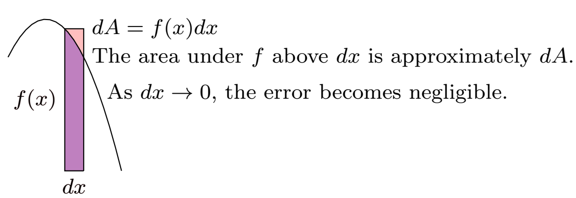
We are ready to tackle the problem of finding the length of a path. This length we call arc length. If our rover moves at a constant speed, then the distance traveled is simply

$$\text{distance} = \text{speed} \times \text{time}.$$

This requires that the speed be constant. What if the speed is not constant? Over a really small time interval dt, the speed is almost constant, so we can still use the idea above.

Problem 2.17: Derivation of the arc length formula Suppose a rover (or other object) moves along the path given by $\vec{r}(t) = (x(t), y(t))$ for $a \leq t \leq b$. We know that the velocity is $\dfrac{d\vec{r}}{dt}$, and so the speed is just the magnitude of this vector.

[1]The quantity $dA = f(x)dx$ is the area of a rectangle whose base is dx wide and whose height is $f(x)$. If dx is really small, then the function f is almost constant, so $f(x)$ and $f(x + dx)$ are really close. The little bit of area dA is extremely close the actual area under f that lies above the x axis between x and $x + dx$, off by the small amount of the rectangle that lies above the curve as shown below. This extra area becomes negligible as $dx \to 0$.



To find the total area under the curve, all we have to do is add up the little bits of area. In terms of Riemann sums, we would write $\sum dA$. The integral symbol just means that we're letting $dx \to 0$, and so the total area is found using $A = \int dA$. To obtain the total area, we just add up the little bits of area. When we replace dA with $f(x)dx$, we put the bounds $x = a$ to $x = b$ on the integral to obtain $A = \int dA = \int_a^b f(x)dx$.

1. Show that we can write the rover's speed at any time t as $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

2. If the rover moves at speed $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ for a little time length dt, what's the little distance ds that the rover traveled?

3. Explain why the length of the path given by $\vec{r}(t)$ for $a \leq t \leq b$ is

$$s = \int ds = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

This is the arc length formula. Ask me in class for an alternate way to derive this formula.

Problem: Alternate derivation of arc length formula Suppose an object moves along the path given by $\vec{r}(t) = (x(t), y(t))$ for $a \leq t \leq b$. Imagine slicing the path up into hundreds of tiny slices. Let ds represent the length of each tiny slice.

1. Draw an appropriate diagram showing an arbitrary curve, a tiny chunk of the curve of length ds, and a triangle so that the Pythagorean theorem gives the approximation $ds = \sqrt{(dx)^2 + (dy)^2}$.

2. Use algebra to show that $\sqrt{(dx)^2 + (dy)^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$.

3. Explain why the length of the path given by $\vec{r}(t)$ for $a \leq t \leq b$ is

$$s = \int ds = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Now that we have a formula for computing arc length, let's practice using it. First, we'll actually evaluate an integral. Next, we'll walk through setting up a block of code to do the same thing in Mathematica. Then, we'll set up several more integrals to find the arc length of several curves. You'll find that arc length problems can become quite messy and sometimes impossible to compute exactly because of the square root term in the integrand.

Problem 2.18 Compute each integral below. Be prepared to show how you use substution to complete the integral.

1. $\int e^{4x} dx$
2. $\int \sin x e^{\cos x} dx$
3. $\int \frac{x^2}{\sqrt{x^3+1}} dx$

Problem 2.19 Find the length of the curve $\vec{r}(t) = \left(t^3, \frac{3t^2}{2}\right)$ for $t \in [1, 3]$. The notation $t \in [1, 3]$ means $1 \leq t \leq 3$. Be prepared to show us your integration steps in class (you'll need a substitution).

See 11.2: 25-30

Problem 2.20 Now let's use the parameterization from the previous problem to write a block of code in Mathematica to compute the arc length of a parameterized curve. We'll use the previous problem as a test problem.

1. First, define a vector function in Mathematica to represent the parameterized curve $\vec{r}(t) = \left(t^3, \frac{3t^2}{2}\right)$. In addition, define some variables to hold the upper and lower limits for the parameter t (i.e., $a = 1$ and $b = 3$).

2. Add a line to your block of code that uses ParametricPlot[] to create a graph of the function. This verifies that the function is defined correctly.

3. Using the vector function and limits you defined, add another line to your block of code to set up and evaluate an integral that will compute the path length of the curve. Use the derivative function in the integrand where necessary. *Hint: you may have to use a square root and a dot product to find the magnitude of a vector function.*

4. Copy the block of code that you created, then change the interval of integration to $2 \leq t \leq 5$.

5. Finally, copy your block of code one more time and use it to compute the length of the curve given by $x = \cos t + t \sin t, y = \sin t - t \cos t$, for $0 \leq t \leq 4\pi$. This curve, called the *involute* of the circle, is the path you would trace if you were skating around a barrel of radius 1 while holding taut a string that was initially wound around the barrel.

For more, visit http://mathworld.wolfram.com/Involute.html to see an animation of an involute of a circle, as well as more details.

Problem 2.21 For each curve below, set up an integral formula which would give the length, and sketch the curve. Do not worry about integrating them.

1. The parabola $\vec{r}(t) = (t, t^2)$ for $t \in [0, 3]$.

2. The ellipse $\vec{r}(t) = (4 \cos t, 5 \sin t)$ for $t \in [0, 2\pi]$.

3. The hyperbola $\vec{r}(t) = (\tan t, \sec t)$ for $t \in [-\pi/4, \pi/4]$.

The reason I don't want you to actually compute the integrals is that they will get ugly really fast. Try doing one in Wolfram Alpha and see what the computer gives. To actually compute the integrals above and find the lengths, we would use a numerical technique to approximate the integral (something akin to adding up the areas of lots and lots of rectangles).

We now have the ability to compute the work done by friction along any path $\vec{r}(t) = (x(t), y(t))$ our rover takes. Can we generalize what we've done to find the work done by any force, acting on any object, along any path? Recall that work is a transfer of energy. Consider the following examples:

- A tornado picks up a couch and applies forces to the couch as it swirls around the center. Work is the transfer of the energy from the tornado to the couch, giving the couch its kinetic energy.

- When an object falls, gravity does work on the object. The work done by gravity converts potential energy to kinetic energy.

- If we consider the flow of water down a river, it is gravity that gives the water its kinetic energy. We can place a hydroelectric dam next to a river to capture a lot of this kinetic energy. Work transfers the kinetic energy of the river to rotational energy of the turbine, which eventually ends up as electrical energy available in our homes.

When we study work, we are really studying how energy is transferred. This is one of the key components of modern life. Recall that the work done by a vector field $\vec{F}$ through a displacement $\vec{d}$ is the dot product $\vec{F} \cdot \vec{d}$.

Review An object moves from $A = (6, 0)$ to $B = (0, 3)$. Along the way, it encounters the constant force $\vec{F} = (2, 5)$. How much work is done by $\vec{F}$ as the object moves from A to B? See [2].

Problem 2.22 An object moves from $A = (6, 0)$ to $B = (0, 4)$. A parametrization of the object's path is $\vec{r}(t) = (-3, 2)t + (6, 0)$ for $0 \le t \le 2$.

1. For $0 \le t \le 1$, the force encountered is $\vec{F} = (2, 5)$. For $1 \le t \le 2$, the force encountered is $(2, 7)$. How much work is done in the first second? How much work is done in the last second? How much total work is done?
2. If we encounter a constant force $\vec{F}$ over a little displacement $d\vec{r}$, explain why the little work done is $dW = \vec{F} \cdot d\vec{r} = \vec{F} \cdot \frac{d\vec{r}}{dt} dt$.
3. Suppose that the force constantly changes as we move along the curve. At t, we'll assume we encountered the force $\vec{F}(t) = (2, 5 + 2t)$, which we could think of as the wind blowing stronger and stronger to the north. Explain why the total work done by this force along the path is
$$W = \int \vec{F} \cdot d\vec{r} = \int_0^2 (2, 5 + 2t) \cdot (-3, 2) dt.$$
Then compute this integral and show you get 16.
4. (Optional) If you are familiar with the units of energy, complete the following. What are the units of $\vec{F}$, $d\vec{r}$, and dW.

You can visualize what's happening in this problem as follows. Attach a clothesline between the points (maybe representing two trees in your backyard). Put a cub scout space derby ship on the clothesline. Then the wind starts to blow. As the ship moves along the clothesline, the wind changes direction.

If a force F acts through a displacement d, then the most basic definition of work is $W = Fd$, the product of the force and the displacement. This basic definition has a few assumptions.

- The force F must act in the same direction as the displacement.
- The force F must be constant throughout the displacement.
- The displacement must be in a straight line.

The dot product let's us remove the first assumption as work is $W = \vec{F} \cdot \vec{r}$, where $\vec{F}$ is a force acting through a displacement $\vec{r}$. The previous problem showed that we can remove the assumption that $\vec{F}$ is constant to obtain

$$W = \int \vec{F} \cdot d\vec{r} = \int_a^b F \cdot \frac{d\vec{r}}{dt} dt,$$

provided we have a parametrization of $\vec{r}$ with $a \le t \le b$. The next problem gets rid of the assumption that $\vec{r}$ is a straight line.

Problem 2.23 Suppose that we move along the circle C parametrized by $\vec{r}(t) = (3 \cos t, 3 \sin t)$. As we move along C, we encounter a rotational force $\vec{F}(x, y) = (-2y, 2x)$.

Watch a YouTube video about work.

1. Draw C. Then at several points on the curve, draw the vector field $\vec{F}(x, y)$. For example, at the point $(3, 0)$ you should have the vector $\vec{F}(3, 0) = (-2(0), 2(3)) = (0, 6)$, a vector sticking straight up 6 units. Are we moving with the vector field, or against the vector field?

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[2] The displacement is $B - A = (-6, 3)$. The work is $\vec{F} \cdot \vec{d} = (2, 5) \cdot (-6, 3) = -12 + 15 = 3$.

2. Explain why we can state that a little bit of work done over a small displacement is $dW = \vec{F} \cdot d\vec{r}$. Why does it not matter that $\vec{r}$ does not move in a straight line?

3. Since a little work done by $\vec{F}$ along a small bit of C is $dW = \vec{F} \cdot d\vec{r}$, we know that the total work done is $\int dW = \int \vec{F} \cdot d\vec{r}$. This gives us

$$W = \int_C (-2y, 2x) \cdot d\vec{r} = \int_0^{2\pi} (-2(3\sin t), 2(3\cos t)) \cdot (-3\sin t, 3\cos t) dt.$$

Complete the integral, showing that the work done by $\vec{F}$ along C is 36π.

We put the C under the integral $\int_C$ to remind us that we are integrating along the curve C. This means we need to get a parametrization of the curve C, and give bounds before we can integrate with respect to t.

Definition 2.4. The work done by a vector field $\vec{F}$, along a curve C with parametrization $\vec{r}(t)$ for $a \leq t \leq b$, is

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt.$$

If we let $\vec{F} = (M, N)$ and we let $\vec{r}(t) = (x, y)$, so that $d\vec{r} = (dx, dy)$, then we can write work in the differential form

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C (M, N) \cdot (dx, dy) = \int_C Mdx + Ndy.$$

Example 2.5. Consider the curve $y = 3x^2 - 5x$ for $-2 \leq x \leq 1$. Give a parametrization of this curve. See [3].

Problem 2.24 Consider the parabolic curve $y = 4 - x^2$ for $-1 \leq x \leq 2$, and the vector field $\vec{F}(x, y) = (2x + y, -x)$.

Please use this Sage link to check your work.

1. Give a parametrization $\vec{r}(t)$ of the parabolic curve that starts at $(-1, 3)$ and ends at $(2, 0)$. See the example problem above if you need a hint.

2. Compute $d\vec{r}$ and state dx and dy. What are M and N in terms of t?

3. Compute the work done by $\vec{F}$ on an object that moves along the parabola from $(-1, 3)$ to $(2, 0)$ (i.e. compute $\int_C Mdx + Ndy$).

Click the link to check your answer with Sage.

4. How much work is done by $\vec{F}$ to move an object along the parabola from $(2, 0)$ to $(-1, 3)$. In other words, if you traverse along a path backwards, how much work is done?

Problem 2.25 Again consider the vector field $\vec{F}(x, y) = (2x + y, -x)$. In the previous problem we considered how much work was done by $\vec{F}$ as an object moved along the the parabolic curve $y = 4 - x^2$ for $-1 \leq x \leq 2$. We now want to know how much work is done to move an object along a straight line from $(-1, 3)$ to $(2, 0)$.

1. Give a parametrization $\vec{r}(t)$ of the straight line curve that starts at $(-1, 3)$ and ends at $(2, 0)$. Make sure you give bounds for t.

2. Compute $d\vec{r}$ and state dx and dy. What are M and N in terms of t?

3. Compute the work done by $\vec{F}$ to move an object along the straight line path from $(-1, 3)$ to $(2, 0)$. Check your answer with Sage.

When you enter your curve in Sage, remember to type the times symbol in "(3*t-1, ...)". Otherwise, you'll get an error.

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[3] Whenever you have a function of the form $y = f(x)$, you can always use $x = t$ and $y = f(t)$ to parametrize the curve. So we can use $\vec{r}(t) = (t, 3t^2 - 5t)$ for $-2 \leq t \leq 1$ as a parametrization.

4. Optional (we'll discuss this in class if you don't have it). How much work does it take to go along the closed path that starts at $(2,0)$, follows the parabola $y = 4 - x^2$ to $(-1,3)$, and then returns to $(2,0)$ along a straight line. Show that this total work is $W = -9$.

Let's finish this chapter with some examples that use integration along a parametrization to give us more than just arc length and work. We'll first find the mass of a rod, whose density is not constant.

Density is generally a mass per unit volume. However, when talking about a wire, it's simpler to let density be the mass per unit length. We can make objects out of composite material, and the density is different at different places in the object. For example, we might have a straight wire where one end is copper and the other end is gold. In the middle, the wire slowly transitions from being all copper to all gold. Such composite materials are engineered all the time (though probably not our example wire). The density at point (x, y, z) is given by the quantity $\delta(x, y, z)$. In future mechanical engineering courses, you would learn how to determine the density δ (mass per unit length) at each point on such a composite wire.

Problem 2.26: Mass Suppose a wire C has the parameterization $\vec{r}(t)$ for $t \in [a, b]$. Suppose the wire's density (mass per unit length) at a point (x, y, z) on the wire is given by the function $\delta(x, y, z)$. Since density is a mass per length, multiplying density by a small length ds gives us the mass of a small portion of the curve. We represent this symbolically using $dm = \delta(\vec{r}(t_0))ds$.

Watch a YouTube video.

1. Explain why the mass m of the wire is given by the formulas below (explain why each equal sign is true):

$$m = \int_C dm = \int_C \delta ds = \int_a^b \delta(\vec{r}(t)) \left|\frac{d\vec{r}}{dt}\right| dt.$$

2. Now suppose a wire lies along the straight segment from $(0,2,0)$ to $(1,1,3)$. A parametrization of this line is $\vec{r}(t) = (t, -t+2, 3t)$ for $t \in [0,1]$. The wire's density (mass per unit length) at a point (x, y, z) is $\delta(x, y, z) = x + y + z$.

   (a) Is the wire heavier at $(0,2,0)$ or at $(1,1,3)$?

   (b) What is the total mass of the wire? [Replace x, y, z, and ds with what they equal in terms of t and then integrate.]

This last problem comes from physics and asks you to find the total charge on a wire if you know the charge per length.

Problem 2.27 A wire lies along the curve $\vec{r}(t) = (7\cos t, 7\sin t)$ for $0 \le t \le \pi$. The wire contains charged particles where the charge per unit length at location (x, y) is given by $q(x, y) = y$. In this problem we'll compute the total charge on the wire.

If the wire were a conductor, then the charged particles (electrons) would not stay put, but rather flow freely along the wire until the repulsive forces are minimized. This wire is an insulator.

1. Draw the curve. Then at several points on the curve write the value of $q(x, y)$ at that point. (Optional: Should the total charge be positive or negative?)

2. Why is the little charge dQ over a little distance ds approximately given by $dQ = q(x, y)ds$?

3. The total charge is the sum of the charges over all the little pieces on the rod. This gives us the total charge as

$$Q_{\text{total}} = \int_C dQ = \int_C q(x,y)ds = \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt.$$

   Replace x and y with what they are in terms of t and then finish by computing the integral above.

2.3 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 3

New Coordinates

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Use a change-of-coordinates to convert between rectangular and other coordinate systems. In particular, convert both points and equations between rectangular and polar coordinates.
2. Graph curves from other coordinate systems (such as polar functions $r = f(\theta)$) in the xy plane.
3. Find the differentials dx and dy of a change-of-coordinates. Then compute tangent vectors, slope $\frac{dy}{dx}$, equations of tangent lines, and arc length.
4. Use the area of a parallelogram to express the relationship between the area of a region in two different coordinate systems.
5. Compute double integrals to find the area of regions in the xy plane.
6. Shade regions in the plane bounded by $\alpha \leq \theta \leq \beta$ and $r_1(\theta) \leq r \leq r_2(\theta)$, and use double integrals to compute their area.

You'll have a chance to teach your examples to your peers prior to the exam.

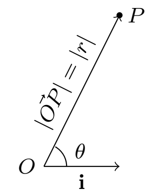
3.1 Polar Coordinates

Up to now, we most often give the location of a point (or coordinates of a vector) by stating the (x, y) coordinates. These are called the Cartesian (or rectangular) coordinates. Some problems are much easier to work with if we know how far a point is from the origin, together with the angle between the x-axis and a ray from the origin to the point.

Problem 3.1 There are two parts to this problem. See 11.3:5-10.

1. Consider the point P with Cartesian (rectangular) coordinates $(2, 1)$. Find the distance r from P to the origin. Consider the ray $\vec{OP}$ from the origin through P. Find an angle between $\vec{OP}$ and the x-axis.
2. Given a generic point $P = (x, y)$ in the plane, write a formula to find the distance r from P to the origin (in terms of x and y) as well as a formula to find the angle θ between the vector $(1, 0)$ (the positive x-axis) and the vector from the origin to P. [Hint: A picture of a triangle will help here.]

Definition 3.1. Let P be be a point in the plane with Cartesian coordinates (x, y). Let $O = (0, 0)$ be the origin. We say that (r, θ) is a polar coordinates of P if (1) we have $|\vec{OP}| = |r|$, and (2) the angle between $\mathbf{i} = (1, 0)$ and $\vec{OP}$ is θ, or coterminal with θ.



Problem 3.2 The following points are given using polar coordinates. Plot the points in the Cartesian plane, and give the Cartesian (rectangular) coordinates of each point. The points are

See 11.3:5-10.

$$(r, \theta) = (1, \pi), \left(6, \frac{\pi}{4}\right), \left(-3, \frac{\pi}{4}\right), \left(3, \frac{5\pi}{4}\right), \text{ and } \left(-2, -\frac{\pi}{6}\right).$$

Finish by explaining why a general formula for x and y if we know a point has polar coordinates (r, θ) is $x = r\cos\theta$ and $y = r\sin\theta$.

See page 647.

The equations above, namely

$$x = r\cos\theta, \quad y = r\sin\theta$$

are a typical example of what we call a change-of-coordinates. We've seen that these equations allow us transfer points back and forth between Cartesian coordinates and polar coordinates. We can also use this change-of-coordinates to transfer equations back and forth between coordinate system. The next two problems have you do this.

Problem 3.3 Each of the following equations is written in the Cartesian (rectangular) coordinate system. Convert each to an equation in polar coordinates, and then solve for r so that the equation is in the form $r = f(\theta)$. You'll want to use the change-of-coordinates to replace any x and y you see so that it is in terms of r and θ.

See 11.3: 53-66.

1. $x^2 + y^2 = 7$
2. $2x + 3y = 5$
3. $x^2 = y$

Problem 3.4 Each of the following equations is written using polar coordinates. Convert each to an equation in using Cartesian coordinates (sometimes called rectangular coordinates). You'll want to use the change-of-coordinates to replace any r and θ you see so that it is in terms of x and y.

See 11.3: 27-52. I strongly suggest that you do many of these as practice.

1. $r = 9\cos\theta$
2. $r = \dfrac{4}{2\cos\theta + 3\sin\theta}$
3. $\theta = 3\pi/4$

We've been writing the change-of-coordinates by listing the two equations $x = r\cos\theta$, $y = r\sin\theta$. We can also write this in vector notation as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}.$$

This is a vector equation in which you input polar coordinates (r, θ) and get out Cartesian coordinates (x, y). So you input one thing to get out one thing, which means that we have a function. We could also write $\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta)$, where we've used the letter T as the name for the function because the function is a $t$ransformation between coordinate systems. The arrow above $\vec{T}$ reminds us that the output is more than one dimensional. To emphasize that the domain and range are both two dimensional systems, we write $\vec{T} : \mathbb{R}^2 \to \mathbb{R}^2$.

3.2 Differentials

The derivative of a function gives us the slope of a tangent line to that function. We can use this tangent line to estimate how much the output (y values) will change if we change the input (x-value). If we rewrite the notation $\frac{dy}{dx} = f'$ in the form $dy = f'dx$, then we can read this as "A small change in y (called dy) equals the derivative (f') times a small change in x (called dx)."

Definition 3.2. We call dx the differential of x. If f is a function of x, then the differential of f is $df = f'(x)dx$. Since we often write $y = f(x)$, we'll interchangeably use dy and df to represent the differential of f.

We will often refer to the differential notation $dy = f'dx$ as "a change in the output y equals the derivative times a change in the input x."

Problem 3.5 Let $f(x) = x^2 \ln(3x+2)$ and $g(t) = e^{2t}\tan(t^2)$. Compute the derivatives $\frac{df}{dx}$ and $\frac{dg}{dt}$, and then state the differentials df and dg. If you skipped reading the definition of a differential, you'll find it is given directly above this problem.

See 3.10:19-38.

Exercise The manufacturer of a spherical storage tank needs to create a tank with a radius of 5 m. Recall that the volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$. No manufacturing process is perfect, so the resulting sphere will have a radius of 5 m, plus or minus some small amount dr. The actual radius will be $5 + dr$. Find the differential dV. Then use differentials to estimate the change in the volume of the sphere if the actual radius is 5.02 m instead of the planned 5 m. See [1] for a solution.

See 3.11:45-62.

Problem 3.6 A forest ranger needs to estimate the height of a tree. The ranger stands 50 feet from the base of tree and measures the angle of elevation to the top of the tree to be about 60°.

1. If this angle of 60° is correct, then what is the height of the tree?

2. If the ranger's angle measurement could be off by as much as 5°, then how off could his estimate of the height be? Use differentials to give an answer.

If your answer here is quite large (much larger than the height of the tree), then look back at your work and see if using radians instead of degrees makes a difference. Why does it? Feel free to ask in class.

How do we use the ideas above when dealing with more than 2 variables? Let's use a change-of-coordinates, of the form $x = au + bv, y = cu + dv$, to first answer this questions. We will see that lines map to lines in our work below, which is the reason why any change-of-coordinates of this form we call a linear change-of-coordinates.

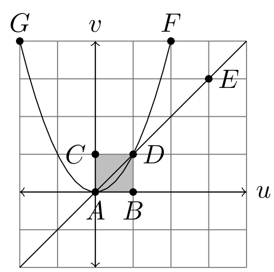
[1] Since $V(r) = \frac{4}{3}\pi r^3$, we know $dV = 4\pi r^2 dr$. The problem gave us $r = 5$ m and $dr = 5.02 - 5 = 0.02$ m. This means $dV = 4\pi(5)^2(.02) = 2\pi$ cubic meters. An error of 0.02 meters on the radius could cause a total error of approximately 6.28 cubic meters in volume.

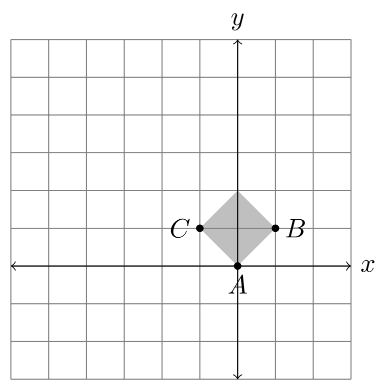
Problem 3.7 Consider the change-of-coordinates $x = u-v$, $y = u+v$, which we could also write as the coordinate transformation $\vec{T}(u,v) = (u-v, u+v)$.

1. In the table below, you're given several (u,v) points. Find the corresponding (x,y) pair.

Name	(u,v)	(x,y)
A	$(0,0)$	$(0,0)$
B	$(1,0)$	$(1-0, 1+0) = (1,1)$
C	$(0,1)$	$(0-1, 0+1) = (-1,1)$
D	$(1,1)$	
E	$(3,3)$	
F	$(2,4)$	
G	$(-2,4)$	

2. There are two graphs below. One is a plot in the uv plane of the points from the table, along with the parabola $v = u^2$, the line $v = u$, and the shaded box whose corners are the first four points. Complete a similar plot in the xy plane by adding the remaining points, and then connect the points in your xy plot to show how the parabola, line, and shaded box (done for you) transform because of this change-of-coordinates. How would you describe what this change-of-coordinates is doing?





Problem 3.8 Consider the change-of-coordinates from the problem above, namely $x = u-v$, $y = u+v$, or equivalently $\vec{T}(u,v) = (u-v, u+v)$.

1. If we assume x, y, u, v are all functions of t, we can compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$. For example, $\frac{dx}{dt} = \frac{du}{dt} - \frac{dv}{dt}$, which gives the differential $dx = du - dv$. Obtain a similar formula for the differentials dy. Finish by writing your answer as the vector equation

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 1 \\ ? \end{pmatrix} du + \begin{pmatrix} -1 \\ ? \end{pmatrix} dv.$$

2. Examine your vector equation above. The two missing parts above represent 2 vectors. Compare this problem to the previous, and look for these two vectors in your drawing. Explain where these two vectors appear.

3. Describe what the vector equation above means geometrically (what does it physically say about the relationship among dx, dy, du, and dv)? For example, if $du = .1$ and $dv = .5$, then how can we use vector addition and scalar multiplication to find dx and dy based off this information.

The next problem returns to polar coordinates, and connects errors in measuring r and θ to errors in the distances x and y.

Problem 3.9 We have two equations $x = r\cos\theta$ and $y = r\sin\theta$. Suppose that a point is moving through space and x, y, r, θ all depend on time t.

1. Explain why $\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}$. Obtain a similar equation for $\frac{dy}{dt}$. Hint: Use implicit differentiation.
2. We can obtain the differential dx and dy in terms of r, θ, dr, and $d\theta$ if we multiply through by dt. This gives $dx = \cos\theta dr - r\sin\theta d\theta$ and $dy =$?. Write you answer as the vector equation
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta \\ ? \end{pmatrix} dr + \begin{pmatrix} -r\sin\theta \\ ? \end{pmatrix} d\theta.$$
3. Suppose the point is currently at $r = 2$, $\theta = \pi/2$ (so $(x, y) = (0, 2)$). Use this information to obtain dx and dy in terms of dr and $d\theta$, as in
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos(\pi/2) \\ ? \end{pmatrix} dr + \begin{pmatrix} -2\sin(\pi/2) \\ ? \end{pmatrix} d\theta = \begin{pmatrix} 0 \\ ? \end{pmatrix} dr + \begin{pmatrix} -2 \\ ? \end{pmatrix} d\theta.$$
4. Examine your vector equation above. How would you describe what the vector equation above means geometrically (what does it physically say about the relationship among dx, dy, dr, and $d\theta$)? It's OK if your answer is not perfect, rather for this part of the problem just do your best to visualize geometrically what the vector equation above means.

3.3 Graphing Transformed Equations

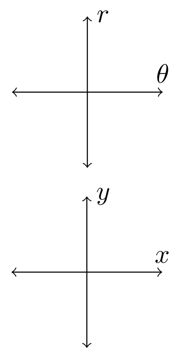
You've spent a lot of time in your past graphing equations of the form $y = f(x)$. Let's now graph equations of the form $r = f(\theta)$ in the xy plane.

Exercise What use does a function like $r = f(\theta)$ have? See [2].

Problem 3.10 In the θr plane, graph the curve $r = \sin\theta$ for $\theta \in [0, 2\pi]$ (make a table where you pick several values for θ and then compute r). Then graph the curve $r = \sin\theta$ for $\theta \in [0, 2\pi]$ in the xy plane (add to your table the corresponding x and y values). The graphs should look very different. If one looks like a sine wave, and the other looks like a circle, you're on the right track. Here's the start of a table to help you, as well as the axes you'll need to put your graphs on.

[2]The function $r = f(\theta)$ tells us a distance from the origin, based off an angle. So if something's distance from the origin depends on an angle from the x-axis, this is exactly what we want. Planetary orbits are one example, along with satellites traveling around the earth, or probes around other planets. Along the same lines, we can use this type of function to describe the motion of electrons around an atom. It's also great for describing spirals, and any object with a nice symmetric pattern centered at some point.

θ	r	$x = r\cos\theta$	$y = r\sin\theta$
0	$\sin(0) = 0$	0	0
$\frac{\pi}{6}$	$\sin\frac{\pi}{6} = \frac{1}{2}$	$\frac{1}{2}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{4}$	$\frac{1}{2}\sin\frac{\pi}{6} = \frac{1}{4}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}\cos\frac{\pi}{4} = \frac{1}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$			
$\frac{\pi}{2}$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$



In general, to construct a graph of a polar curve in the xy plane, we create an r, θ table. We choose values for θ that will make it easy to compute any trig functions involved. If you need to, add x and y to your table before plotting the location of the polar point in the xy plane. Then connect the points in a smooth manner, making sure that your radius grows or shrinks appropriately as your angle increases. Ask me in class to show you some animations of this, or you can see these animations before class if you open up the Mathematica Technology Introduction in I-Learn.

Problem 3.11 Graph the polar curve $r = 2 + 2\cos\theta$ in the xy plane. See 11.4: 1-20.

Problem 3.12 Graph the polar curve $r = 2\sin 3\theta$ in the xy plane. [Hint: You'll want to chose values for theta so that 3θ hits all multiples of ninety degrees, the places where r attains it maximums and minimums.]

Problem 3.13: Mathematica Problem In this problem we'll use Mathematica to plot the polar curve $r = a\cos(n\theta)$ for various values of a and n.

1. Use the command PolarPlot[] to plot the curve $r = 3\cos 2\theta$ for $0 \leq \theta \leq 2\pi$.
2. Use the command ParametricPlot[] to plot the curve $r = 3\cos 2\theta$ for the same bounds. We know that $x = r\cos\theta$ and $y = r\sin\theta$, so you just need to plot $\vec{r}(t) = \langle (3\cos 2\theta)\cos\theta, (3\cos 2\theta)\sin\theta \rangle$.
3. Use your code above to graph $r = 3\cos(n\theta)$ for $0 \leq \theta \leq 2\pi$ for each integer n from 2 to 8. What patterns do you see? Make a conjecture and then plug in higher values for n to see if you are correct.
4. With software you can quickly change parts of a function to see how they affect behavior. In the function $r = a\cos(n\theta)$, how does the graph change if instead of having $a = 3$ you pick a to be another number? What happens if you pick n to be something other than an integer? What happens if you change cos to sin?

3.4 Slopes and Arc Length

Problem 3.14 We saw in some previous problems that for polar coordinates we can express the differentials dx and dy using the vector equation

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} dr + \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix} d\theta.$$

1. Use the vector equation above to compute $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of r and $\frac{dr}{d\theta}$, if we assume that r is a function of θ.

Hint: Just multiply everything out and divide by $d\theta$.

2. Explain why the slope of a tangent line in the xy plane to the curve $r = f(\theta)$ is
$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

For parametric curves $\vec{r}(t) = (x(t), y(t))$, the slope of the curve is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

A polar curve of the form $r = f(\theta)$ is just the parametric curve $(x, y) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. The previous problem showed us that we can find the slope by computing

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Problem 3.15 Consider the polar curve $r = 3 + 2\cos\theta$. Start by graphing this curve in the xy plane.

See 11.2: 1-14.

1. Remember that $x = r\cos\theta$ and $y = r\sin\theta$. Compute $dx/d\theta$ and $dy/d\theta$.
2. Find the slope dy/dx of the curve at $\theta = \pi/2$.
3. Give both a vector equation of the tangent line, and a Cartesian equation of the tangent line at $\theta = \pi/2$.

The above process works with any change-of-coordinates.

Problem 3.16 Consider the parabola $v = u^2$ and the change-of-coordinates $x = 2u + v$, $y = u - 2v$.

1. Construct a graph of the parabola in the xy plane.
2. Compute both dx/du and dy/du. Then find the slope dy/dx of the parabola at $u = 1$.
3. Give both a vector equation of the tangent line, and a Cartesian equation of the tangent line at $u = 1$.

We showed in the curves section that you can find the arc length for a parametric curve by using the formula

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

If we replace t with θ, this becomes a formula for the arc length of a curve given in polar coordinates.

Problem 3.17 Set up (do not evaluate) an integral formula to compute each of the following (draw the curve to be sure your bounds are correct - getting the right bounds is perhaps the toughest part of this problem.):

See 11.5: 21-28.

1. The length of one petal of the rose $r = 3\cos 2\theta$.
2. The length of the entire rose $r = 2\sin 3\theta$.

3.5 Area from Double Integrals

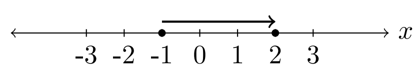
We've now seen one example of how we can use a change-of-coordinates to compute an integral, namely to find arc length. You've actually been using a change-of-coordinates since first semester calculus, every time you performed a substitution to complete an integral. The next problem has you revisit this, and notice something crucial about differentials.

Problem 3.18 Consider the integral $\displaystyle\int_{-1}^{2} e^{-3x}\,dx$.

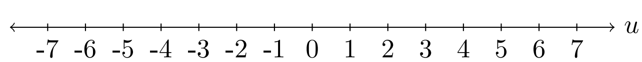
1. To complete this integral we use the substitution $u = -3x$. Solve for x and compute the differential dx.
2. Now perform the substitution, filling in the missing parts of
$$\int_{x=-1}^{x=2} e^{-3x}\,dx = \int_{u=?}^{u=?} e^{u} ?du.$$
To find the u bounds, just ask, "If $x = -1$, then $u = ?$" Don't spend any time completing the integral, rather just focus on completing the substitution above.

Note: When a definite integral ends with du, the bounds should be in terms of u. Many of you have always ignored this step, and instead would first compute $\int e^u ?du$ without bounds, replace u with $-3x$, and then finish. We need the approach on the left in high dimensions.

3. The x values range from -1 to 2. This is a directed interval whose width is 3 units, pointing from left to right along the x-axis (shown below).



Our substitution $u = -3x$ transforms this directed interval into a different interval along the u-axis. Draw the transformed interval below.



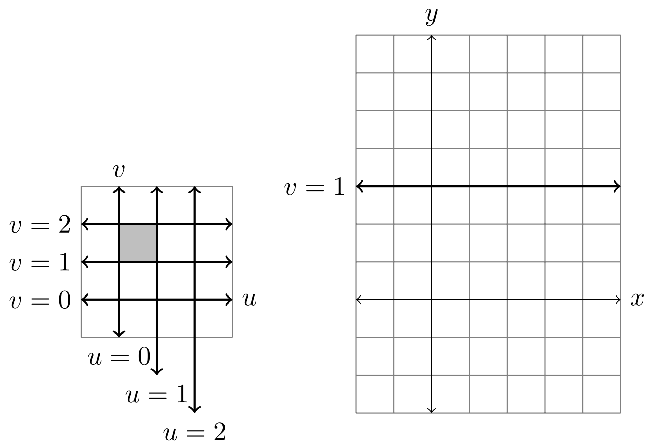
4. How long is the new interval along the u-axis? What does your differential equation $dx = -\frac{1}{3}du$ have to do with this problem? What does the negative sign do?

We've now seen that the differential equation $dx = \frac{dx}{du}du$ tells us how to relate lengths along the u-axis to lengths along the x-axis. The next two problems have you focus on how a two dimensional change-of-coordinates helps us connect areas in the uv plane to areas in the xy plane.

Problem 3.19 Consider the change-of-coordinates $x = 2u$, $y = 3v$.

1. The lines $u = 0, u = 1, u = 2$ and $v = 0, v = 1, v = 2$ correspond to lines in the xy plane. Draw these lines in the xy-plane ($v = 1$ is done for you).

[Hint: One option is to find the xy coordinates of the (u, v) points $(0, 0)$, $(0, 1)$, $(0, 2)$ and connect the dots to make a line. Then repeat with the (u, v) coordinates $(1, 0)$, $(1, 1)$, $(1, 2)$ and draw another line. Eventually you'll have a grid.]



2. The box in the uv plane with $0 \leq u \leq 1$ and $1 \leq v \leq 2$ corresponds to a box in the xy plane. Shade this box in the xy plane and find its area.

3. Compute the differentials dx and dy. State them using the vector form

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} du + \begin{pmatrix} ? \\ ? \end{pmatrix} dv$$

4. What do the two vectors above have to do with your picture?

5. Draw the box given by $-1 \leq u \leq 1$ and $-1 \leq v \leq 1$ in both the uv-plane and xy-plane. State the area A_{uv} of this box in the uv-plane, and then state the area A_{xy} of the corresponding rectangle in the xy-plane.

6. Consider the circle $u^2 + v^2 = 1$, whose area inside is $A_{uv} = \pi$. Guess the area A_{xy} inside the corresponding ellipse in the xy plane. Explain.

Before we continue, we need a quick way to compute the area of a parallelogram. Our work with vectors gives us all the tools we need to tackle this problems, so the proof of the next theorem is 100 percent within our reach. Proving the following theorem is the next problem.

Theorem 3.3. *The area of a parallelogram whose edges are parallel to the two vectors* (a, b) *and* (c, d) *is given by* $|ad - bc|$.

Problem 3.20 Suppose a parallelogram has edges that are parallel to the vectors $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$. Prove that the area of this parallelogram is given by $|ad - bc|$. If you want some help, here are some steps you can follow:

1. Draw the parallelogram. Add to your picture the projection of $\vec{u}$ onto $\vec{v}$, so $\vec{u}_{\|\vec{v}}$. Then include the component of $\vec{u}$ that is orthogonal to $\vec{v}$, so $\vec{u}_{\perp\vec{v}}$.

2. Explain why the area is $A = |\vec{v}||\vec{u}_{\perp\vec{v}}|$. The base length is $|\vec{v}|$ and the height is the magnitude of $\vec{u}_{\perp\vec{v}}$.

3. Compute $\vec{u}_{\perp\vec{v}}$ (there are several ways).

4. Compute both lengths, multiply them together, and then simplify down until you get $\sqrt{(ad - bc)^2}$, which equals $|ad - bc|$. This might be long.

Whether you are able to complete the proof above or not, let's practice using the result in another problem.

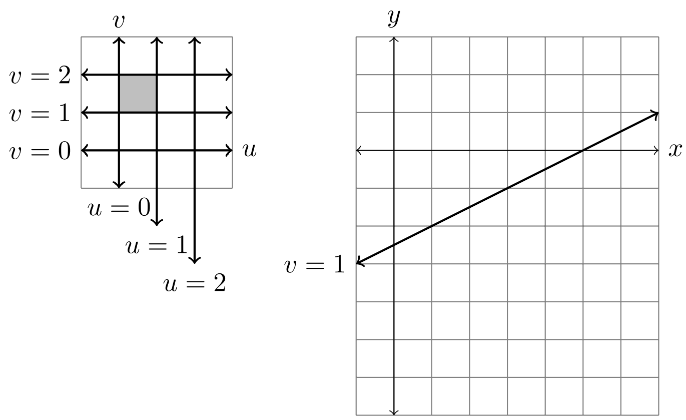
Problem 3.21 For each of the three problems below, you'll want to first find the area of an appropriate parallelogram (using $|ad - bc|$).

1. A parallelogram has vertices $(0, 0)$, $(-2, 5)$, $(3, 4)$, and $(1, 9)$. Find its area.
2. Find the area of the triangle with vertices $(0, 0)$, $(-2, 5)$, and $(3, 4)$.
3. Find the area of the triangle with vertices $(-3, 1)$, $(-2, 5)$, and $(3, 4)$.

We're now use areas of parallelograms to analyze a change-of-coordinates.

Problem 3.22 Consider the change-of-coordinates $x = 2u + v$, $y = u - 2v$.

1. The lines $u = 0, u = 1, u = 2$ and $v = 0, v = 1, v = 2$ correspond to lines in the xy plane. Draw these lines in the xy-plane (the line $v = 1$ is drawn for you). [Hint: One option is to find the xy coordinates of the (u, v) points $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, etc., and then just connect the dots to make a rotated grid.]



2. The box in the uv plane with $0 \le u \le 1$ and $1 \le v \le 2$ should correspond to a parallelogram in the xy plane. Shade this parallelogram in your picture above.
3. Compute the differentials dx and dy. State them using the vector form
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} du + \begin{pmatrix} ? \\ ? \end{pmatrix} dv$$
What do the two vectors above have to do with your picture?
4. Show that the area of the parallelogram formed using these two vectors is 5. What does this area have to do with this problem? How would you describe the change in area between the graph in the uv plane, and the graph in the xy plane?

The next problem has us analyze the integrals $\int_C dx$ and $\int_C dy$, and from them develop a way to compute area using double integrals.

Problem 3.23 Consider the portion of the ellipse parametrized by $\vec{r}(t) = (3\cos t, 4\sin t)$ for $0 \le t \le \pi/2$ (so $x = 3\cos t$ and $y = 4\sin t$).

1. Draw the curve, paying attention to the given bounds.

2. The integral $\int_C dx$ literally says "Sum up little changes in x." Adding up little changes in x gives the total change in x. For this problem note that $dx = -3\sin t dt$. Now compute $\int_C dx = \int_{t=0}^{t=\pi/2} dx = \int_{t=0}^{t=\pi/2} -3\sin t dt$ and verify that it gives the physical change in x from $t = 0$ to $t = \pi/2$.
3. Compute both dy and then $\int_{t=0}^{t=\pi/2} dy$. Explain how you could obtain this answer without doing any integration.
4. Give the values of $\int_{t=\pi}^{t=2\pi} dx$ and $\int_{t=\pi}^{t=2\pi} dy$ without any integration.

Problem 3.24 Consider the region R between the functions $y = x^2$ and $y = -x$ for $0 \leq x \leq 3$. Draw both functions and shade the region R. Your goal in this problem is to explain why the iterated integral $\int_{x=0}^{x=3} \left(\int_{y=-x}^{y=x^2} dy \right) dx$ gives the area of the region R.

1. Compute the integral $\int_{y=-x}^{y=x^2} dy$ for arbitrary x. Then explain what physical quantity this integral measures. See the margin if you need help.
2. Recall that dx is a small width. When we multiply the previous integral by this width dx, we will obtain the area of a small region. Construct a picture that includes the original region R together with the small region whose area is given by $dA = \left(\int_{y=-x}^{y=x^2} dy \right) dx$.
3. Explain why $\int_{x=0}^{x=3} \left(\int_{y=-x}^{y=x^2} dy \right) dx$ gives the area of the region R.

Pick a value of x, such as $x = 2$. The inner integral $\int_{y=-x}^{y=x^2} dy$ adds up little changes in y for that specific x value. Compute this integral when $x = 2$ (so $\int_{y=-2}^{y=4} dy$) to verify that you get a total change in y of 6 units, the vertical distance between the two points $(2, -2)$ and $(2, 4)$. Draw a vertical line segment inside your region that connects these two points. Then repeat this for various other values of x, adding appropriate segments.

Problem 3.25 Consider the double integral

$$\int_{y=-1}^{y=2} \left(\int_{x=y^2}^{x=y+2} dx \right) dy.$$

1. The bounds in the integral above describe a region in xy plane where $-1 \leq y \leq 2$ and $y^2 \leq x \leq y + 2$. Sketch this region.
2. Consider the inner integral $\int_{x=y^2}^{x=y+2} dx$. This integral adds up changes in x, so gives a total x distance? Add to your sketch several line segments whose widths are given by this integral.
3. When we multiply a width $\int_{x=y^2}^{x=y+2} dx$ by a small height dy, we get a little bit of area dA. Pick a value y between -1 and 2, and then at that height draw a small rectangle whose area is given by $dA = \left(\int_{x=y^2}^{x=y+2} dx \right) dy$.
4. Adding up little bits of area gives total area, so the double integral at the start of this problem gives an area. Compute the integral.

Draw the four curves $-1 = y$, $y = 2$, $y^2 = x$, and $x = y + 2$, and then shade the appropriate region that satisfies all 4 inequalities.

Problem 3.26 The double integral $\int_{x=a}^{x=b}\left(\int_{y=g(x)}^{y=f(x)} dy\right) dx$ computes the area of a region in the xy plane that you should be quite familiar with. Compute the inner integral $\int_{y=g(x)}^{y=f(x)} dy$ to obtain the single variable formula you should be more familiar with. Provide a sketch of the region, using some specific functions to illustrate this abstract idea.

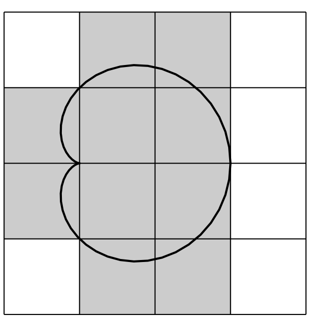
Since area is a two dimensional quantity, a double integral provides a natural way to compute the area. The above problems have shown that the area A of a region R can be found by adding up little bits of area using any of

$$A = \int_{x=a}^{x=b}\left(\int_{y=g(x)}^{y=f(x)} dy\right) dx = \int_{y=c}^{y=d}\left(\int_{x=a(y)}^{x=b(y)} dx\right) dy.$$

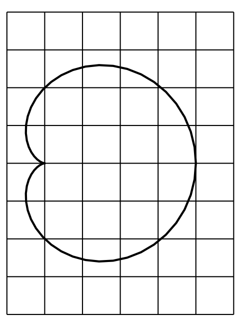
We call these iterated integrals, as we iteratively give the bounds for each variable. Notice that in each of the integrals above, we took a slice of the region, thickened it up to get a thin rectangle whose area was dA, and then found the area by adding up these thin rectangles.

Another way to compute the area of a region R is overlay the region with a rectangular grid, where dx and dy are the distances between the vertical and horizontal lines of the grid. To find the area of the region, we first determine which of the rectangles contains a portion of the region R, and then add up the areas of of all such rectangles. This will overestimate the area, but we then use limits to shrink both dx and dy to zero to obtain the area.

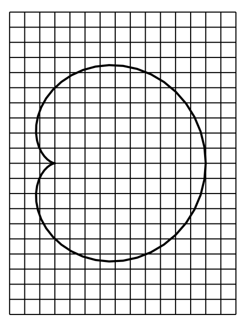
Problem 3.27 Consider the polar curve $r = 1 + \cos\theta$. We will use the approach described above this problem to estimate the area of region R that is inside this polar curve. The bounds for each graph below are $-1 \leq x \leq 2$ and $-2 \leq y \leq 2$. To present this problem in class, please print this page so you can appropriately shade things as asked below.



$dx = dy = 1$



$dx = dy = .5$



$dx = dy = .2$

1. For the first picture above, there are 10 rectangles (shaded) that contain a portion of the region R. Each of these rectangles has area $dA = dxdy = (1)(1) = 1$, which means an overestimate for the area of R is $A \approx 10\,dA = 10(1) = 10$.

2. Now use the middle picture above (where $dx = dy = .5$) to shade and then count the number of rectangles that contain a portion of R. What is the area dA of each little rectangle. Finish by giving an estimate for A.

3. Now use the last picture with $dx = dy = .2$ to estimate the area of R.

4. How can we obtain the exact value for the area of R?

We've been considering double integrals of the form

$$\int_{x=a}^{x=b}\left(\int_{y=g(x)}^{y=f(x)} dy\right) dx \quad \text{and} \quad \int_{y=c}^{y=d}\left(\int_{x=a(y)}^{x=b(y)} dy\right) dx.$$

These integrals give us the area of a region R in the (x, y) plane. Setting up the bounds for these integrals requires being able to describe the bounds of the region using inequalities of the form $a \leq x \leq b$ and $g(x) \leq y \leq f(x)$, or of the form $c \leq y \leq d$ and $a(y) \leq y \leq b(y)$. This can become a problem if the region is not easily described using rectangular coordinates.

Problem 3.28 Shade the region in the xy plane described by each set of inequalities.

1. $0 \leq \theta \leq \pi/4$ and $0 \leq r \leq 4$
2. $0 \leq x \leq 3$ and $0 \leq y \leq \sqrt{9-x^2}$
3. $-\pi/6 \leq \theta \leq \pi/6$ and $0 \leq r \leq 2\cos 3\theta$
4. $0 \leq \theta \leq 2\pi$ and $2 \leq r \leq 5 + 2\cos\theta$

Our goal now is to learn how to use double integrals to compute area if the region is easily described using polar coordinates instead of rectangular coordinates. Basically, we need to perform a substitution from (x, y) to (r, θ) coordinates. Earlier we saw that for the change-of-coordinates $x = 2u + v$, $y = u - 2v$, we can write the differentials dx and dy in the vector form

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} du + \begin{pmatrix} 1 \\ -2 \end{pmatrix} dv.$$

The area of the parallelogram formed from the two vectors above, namely 5, gave us the scale factor that connected areas in the xy plane to areas in the uv plane. A rectangle with width du and height dv in the uv plane would have an area 5 times larger when transformed to the xy plane. We can write this as $dA_{xy} = 5dudv$. The next problem has use repeat this process with polar coordinates.

Problem 3.29 Consider the change-of-coordinates $x = r\cos\theta$, $y = r\sin\theta$.

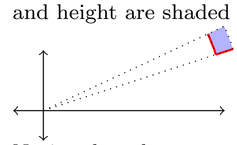
1. The lines $r = 1, r = 2, r = 3$ and $\theta = 0, \theta = \frac{\pi}{6}, \theta = \frac{\pi}{3}$ correspond to circles and lines in the xy plane. Draw these circles and lines in the xy-plane.
2. The box in the $r\theta$ plane with $2 \leq r \leq 3$ and $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$ corresponds to a region in the xy plane. Shade this region in the xy plane.
3. Compute the differentials dx and dy and give them in vector form
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} dr + \begin{pmatrix} ? \\ ? \end{pmatrix} d\theta.$$
4. We know that the area of a parallelogram with edges given by (a, b) and (c, d) is $|ad - bc|$. Apply this formula to show that the area of a parallelogram whose edges are given by the two vectors above is $|r|$.

Did you obtain $|r|$ in the last step above? This means that a little rectangle in the $r\theta$ plane will have its area increased by a scale factor of r when transforming the region to the xy plane. We can express this as $dA_{xy} = rdrd\theta$, or just $dA = rdrd\theta$. The next problem gives us a geometric proof of the same fact.

Problem 3.30 Let (r, θ) be an arbitrary point. Our goal is to develop a formula for the area of the region R in the xy plane where the radius ranges from r to $r + dr$ and the angle ranges from θ to $\theta + d\theta$, shown in the diagram to the right. Copy a similar diagram on to your paper and then do the following.

1. Add the labels r, θ, dr, $d\theta$, $r + dr$, and $\theta + d\theta$ to appropriate places in your diagram.
2. The shaded region is approximately a rectangle. Explain why the area of this rectangle is $dA = rdrd\theta$ by first finding the width and height.

A small polar rectangle, when transformed into the xy plane, looks like a rectangle whose width and height are shaded red below.



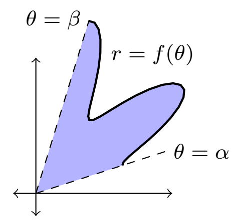
Notice that the rectangle's area will increase as r increases.

The area of a region R in the xy plane can be found using the double integral $A = \int\int_R dA$. If it's easy to describe the bounds using rectangular coordinates, then we can use either $A = \int_a^b \left(\int_{g(x)}^{f(x)} dy\right) dx$ or $A = \int_c^d \left(\int_{a(y)}^{b(y)} dx\right) dy$. The next problem gives us a formula for computing areas of regions that are easy to describe using polar coordinates.

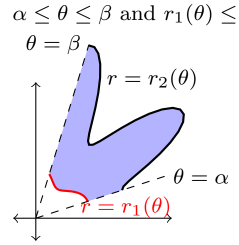
Problem 3.31 Consider the region R in the xy plane bounded by $\alpha \leq \theta \leq \beta$ and $0 \leq r \leq f(\theta)$.

1. The area of a region R in the xy plane is the double integral $A = \int\int_R dA$. Explain why the area of the region in the xy plane is
$$A = \int_\alpha^\beta \int_0^{f(\theta)} rdrd\theta.$$
2. Now consider the region R bounded by $\alpha \leq \theta \leq \beta$ and $r_1(\theta) \leq r \leq r_2(\theta)$, as shown in the diagram to the right. Set up a double integral that would give the area of this region R.

Here's a typical region with $\alpha \leq \theta \leq \beta$ and $0 \leq r \leq f(\theta)$.



Here's a typical region with $\alpha \leq \theta \leq \beta$ and $r_1(\theta) \leq r \leq r_2(\theta)$.



Let's use what we have just developed to examine several polar integrals.

Problem 3.32 Complete both parts below.

1. Draw the region in the xy plane whose area is given by the polar integral $\displaystyle\int_0^{3\pi/2} \int_1^{4+3\cos\theta} rdrd\theta.$
2. Set up a double integral to find the area in the xy plane that is inside one petal of the curve $r = 3\sin 2\theta$.

Problem 3.33 Find the area of the region enclosed by the positive x-axis and the spiral $r = 4\,\theta/3$ for $0 \leq \theta \leq 2\pi$. The region looks like a snail shell.

Problem 3.34 Find the area enclosed by one leaf of the rose $r = 5\cos 3\theta$ (a sketch may help you define limits for θ). Compute the integral by hand.

You may need the power reduction formula $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$.

For the remainder of the semester, any time an integral involves a power reduction formula, you may use software to finish the integral.

Problem 3.35 For each region R described below, start by drawing the region. Then set up a formula involving an iterated integral to find the area of R.

1. R is inside the cardioid $r = 1 + \cos\theta$ but outside the circle $r = 1$.
2. R is inside both the circles $r = \cos\theta$ and $r = \sin\theta$.
3. R is inside the circle $r = 5\cos\theta$ but to the right of the line $r = 3\sec\theta$.

3.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Exam 1 Review

At the end of each chapter, the following words appeared.

> Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam.

I've summarized the objectives from each chapter below. For our in class review, please come to class with examples to help illustrate each idea below. You'll get a chance to teach another member of class the examples you prepared. If you keep the examples simple, you'll have time to review each key idea.

Vectors

1. Define, draw, and explain what a vector is in 2 and 3 dimensions.
2. Add, subtract, and scalar multiply vectors. Be able to illustrate each operation geometrically.
3. Compute the dot product and use it to find angles, lengths, projections, and work.
4. Decompose a vector into parallel and orthogonal components.
5. Give equations of lines in both vector and parametric form.

Curved Motion

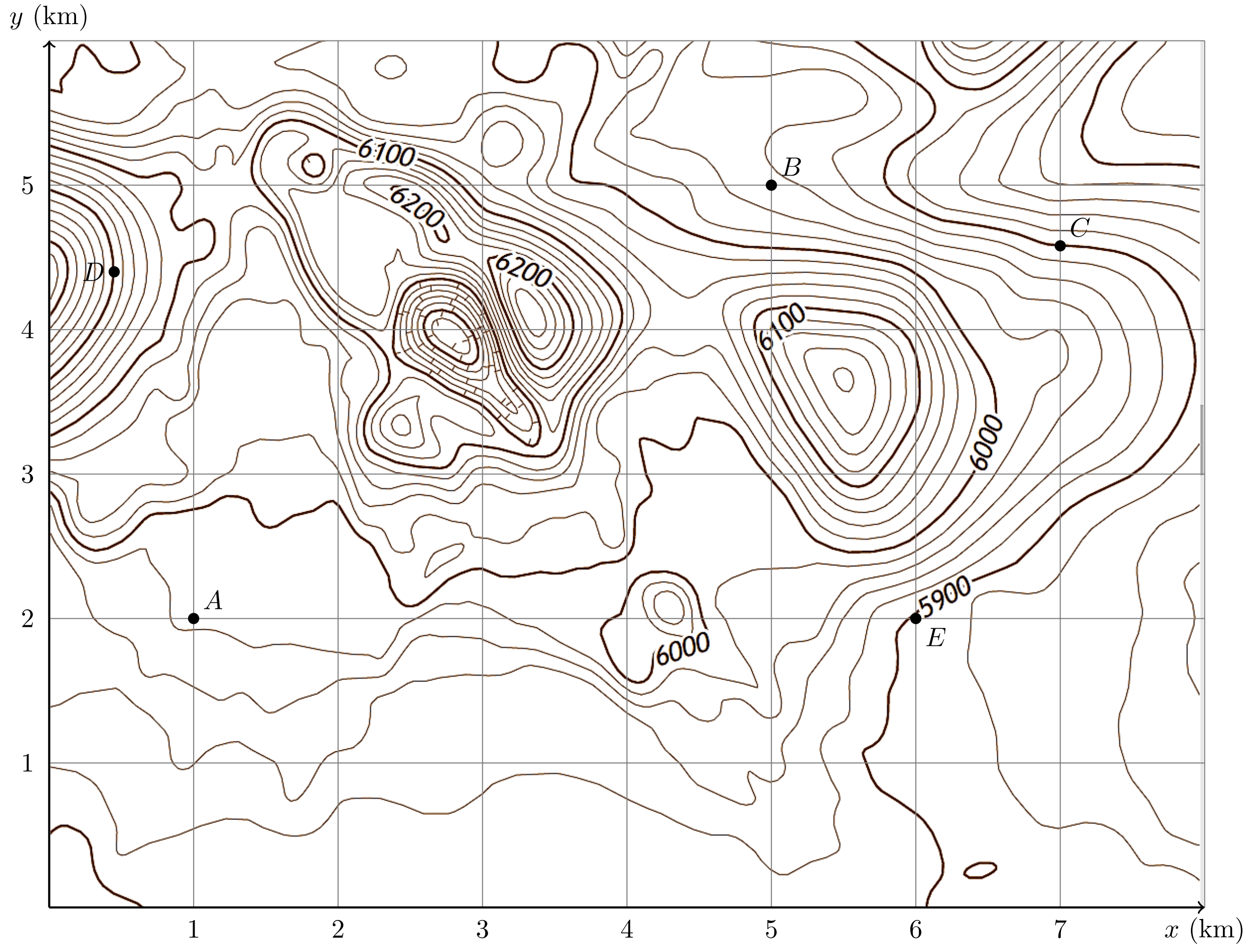
1. Be able to graph and give equations of parabolas, ellipses, and hyperbolas.
2. Use a change-of-coordinates involving translation and stretching to give an equation of and graph a curve.
3. Model motion in the plane using parametric equations.
4. Find derivatives and tangent lines for parametric equations. Explain how to find velocity, speed, and acceleration from a parametrization.
5. Use integrals to find the length of a parametric curve, the work done by a non constant force along a curve, and related quantities.

New Coordinates

1. Use a change-of-coordinates to convert between rectangular and other coordinate systems. In particular, convert both points and equations between rectangular and polar coordinates.
2. Graph curves from other coordinate systems (such as polar functions $r = f(\theta)$) in the xy plane.
3. Find the differentials dx and dy of a change-of-coordinates. Then compute tangent vectors, slope $\frac{dy}{dx}$, equations of tangent lines, and arc length.
4. Use the area of a parallelogram to express the relationship between the area of a region in two different coordinate systems.
5. Compute double integrals to find the area of regions in the xy plane.
6. Shade regions in the plane bounded by $\alpha \leq \theta \leq \beta$ and $r_1(\theta) \leq r \leq r_2(\theta)$, and use double integrals to compute their area.

Mars Rover - Contour Plots, Surface Plots, and Gradient Fields - Local Optimization

Let's pretend we're the current operations team for the Mars rover Curiosity. We'll suppose the map below shows the general vicinity where Curiosity currently is.



The contours on the map represent a function $z = f(x, y)$, where z is the elevation in meters, with contours drawn at 20 m increments. We've added a 1 km grid over the map to aid in navigation. The rover is currently at A. We need to get it to B. Discuss the answers to each question below with your group.

1. What is the elevation 2000 m east and 3000 m north, so at $(2, 3)$? In other words give $f(2, 3)$.
2. Note that $A = (1, 2)$. Estimate $f(A)$. Then estimate each of $f(B)$, $f(C)$, $f(D)$, and $f(E)$.
3. On the map, are there any hill tops (local maximums)? Where are they?
4. Mark the spot on the map with the highest elevation. [Hint: it is not on a hill top.]
5. How do you locate local minimums (low points)? Mark them.
6. On the map, where are the steepest inclines? Where are the mostly flat bits of land?
7. Curiosity is currently at A, and needs to get to B. Why is a straight path from A to B not a good idea?
8. What's the steepest slope encountered by Curiosity on this straight line path.

9. Plan a route to get from point A to point B that passes through one other point (call it F), with straight lines connecting A and B to F, avoiding steep rises and falls. Draw this route on your map.

10. What's the steepest slope that Curiosity will encounter while following your chosen route? Mark the spot on your route where this steepest slope occurs. Then estimate the slope (estimate the rise Δz over a run Δs).

11. There are three other points on the map (C, D, and E). At each of these points, let's estimate the slope of the hill in two different directions, namely moving east or north.

    (a) At point C, notice that moving right $\Delta x \approx 100$ m (so 0.1 km) doesn't really change the elevation much at all (maybe there is a slight drop, but very small). This suggests that $\Delta z \approx 0$ and so the slope when moving east is $\dfrac{\Delta z}{\Delta x} \approx \dfrac{0}{100} = 0$.

    (b) At point C, moving north about $\Delta y = 100$ m gets us to the next contour, which is a drop of $\Delta z \approx -20$ m. This gives a slope of $\dfrac{\Delta z}{\Delta y} \approx \dfrac{-20}{100} = -.2$, a 20% downhill grade.

    (c) Estimate $\dfrac{\Delta z}{\Delta x}$ at point D.

    (d) Estimate $\dfrac{\Delta z}{\Delta y}$ at point D.

    (e) Estimate $\dfrac{\Delta z}{\Delta x}$ at point E.

    (f) Estimate $\dfrac{\Delta z}{\Delta y}$ at point E.

In the rover problem, we were handed a function $z = f(x, y)$ as a contour plot, without a rule. For the rest of class time, lets instead start with a function $z = f(x, y)$ handed to us in rule form.

- Given $z = x^2 + 3xy$, compute dz and write it in the form $dz = (?_1)dx + (?_2)dy$. Note that if $dy = 0$, then $?_1$ is the slope $\frac{dz}{dx} \approx \frac{\Delta z}{\Delta x}$ in the x direction. A similar computation shows $?_2 = \frac{dz}{dy} \approx \frac{\Delta z}{\Delta y}$. Rather than using the notation $\frac{dz}{dx}$, we instead use $\frac{\partial z}{\partial x}$ to remind us that this slope only has meaning if the other variable is not changing.
- Use the dot product to rewrite your previous answer in the form $dz = (?, ?) \cdot (dx, dy)$.

The vector $(?, ?)$ above we call the gradient of f and write $\vec{\nabla} f$. The gradient of f is a vector field, which means we can construct a vector field plot in 2D. Software is ideal for doing this, so we'll use Mathematica to help us for the remainder of class. Our goal over the next few weeks is to learn how to use both contour plots and gradients to make decisions. Please download the following Mathematica notebook.

- ContourVsGradient.nb at https://www.dropbox.com/s/pwvge1p820s4u3b/ContourVsGradient.nb?dl=1

We'll go through the first portion of the notebook as a class, and then use what we learned to explore several different examples and make some conjectures.

As you work through the examples in Mathematica, once you feel like you know what the gradients vectors represent physically, add them to your map of mars at the points A, B, C, D, and E.

Chapter 4

Optimization

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. For a function of the form $f(x, y)$ or $f(x, y, z)$, construct (by hand and with software) contour plots, surface plots, and gradient field plots.

2. Compute differentials, partial derivatives, and gradients.

3. Compute slopes (directional derivatives), tolerances (differentials), and equations of tangent planes.

4. Obtain and use the chain rule to analyze a function f along a parametrized path $\vec{r}(t)$. In particular, calculate slopes and locate maximums and minimums of f along $\vec{r}$.

5. Use Lagrange multipliers to locate and compute extreme values of a function f subject to a constraint $g = c$.

6. Apply the second derivative test, using eigenvalues, to locate local maximum and local minimum values of a function f over a region R.

You'll have a chance to teach your examples to your peers prior to the exam.

In this chapter, we'll also be utilizing technology to help us construct contour plots in 2D and 3D, surface plots in 3D, and vector fields plots in 2D and 3D. Use the following Mathematica notebook to help you throughout the chapter.

- ContourSurfaceGradient.nb

The commands in Mathematica we'll be using are

- ContourPlot[ ] - Constructs a 2D contour plot of $f(x, y)$ which shows several level curves of f.

- Plot3D[ ] - Constructs a 3D surface plot of $f(x, y)$.

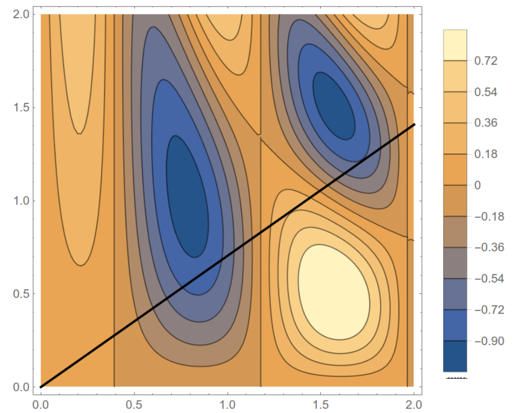
- VectorPlot[ ] - Constructs a 2D vector field plot of $\vec{\nabla} f(x, y)$.

- ContourPlot3D[ ] - Constructs a 3D contour plot of $f(x, y, z)$ which shows several level surfaces of f.

- VectorPlot3D[ ] - Constructs a 3D vector field plot of $\vec{\nabla} f(x, y, z)$.

4.1 Problems

Problem 4.1 The contour plot below shows some terrain near Curiosity. The rover will follow the linear path shown.

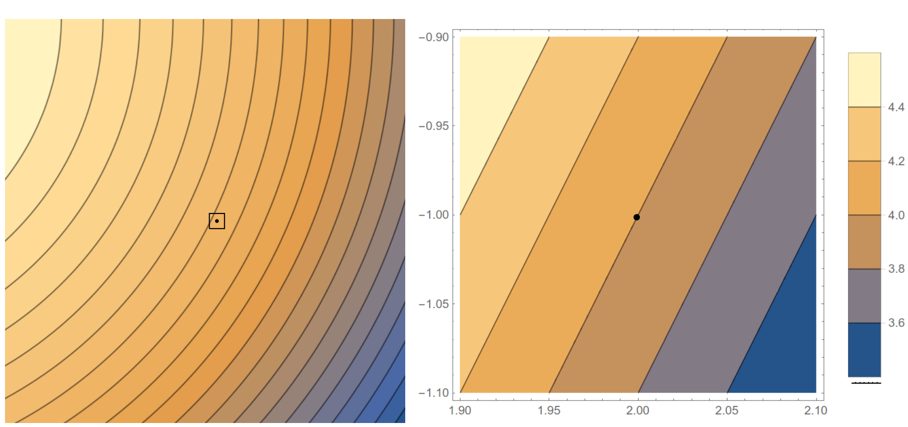


1. Mark every local maximum with $\times$ and every minimum with $+$. Explain.
2. The rover's path does not cross any of the points you just marked. However, as the rover moves, it does reach its own local maximums and minimums. Circle each of these points on the path. Explain.
3. At points where the rover hits a local maximum or local minimum value on its path, what's the relationship between the contours and the path?
4. As curiosity moves along the path, estimate the most extreme slope encountered. Use a ruler to approximate distances and compare them to the scale. You may assume that x, y, and z all have the same units, so units can be ignored. Explain how you got your answer.

We answered all the questions above by visual inspection. The goal of this unit is to gain the tools needed to answer all the questions above without needing a human to do a visual inspection. Once we're done, we'll know the tools needed to program a rover to answer all these questions itself.

Problem 4.2 Curiosity is currently on a hill, and its position is at the center of the map on the left below. Zooming in on the rover's position yields the map on the right below (the color legend applies to the graph on the right).

Why would we zoom in to estimate slopes?



The contours in the graph to the right each represent a change in height of 0.2 units. The bounds for the graph are $1.9 \leq x \leq 2.1$ and $-1.1 \leq y \leq -0.9$. For simplicity of computations, let's assume the x, y and z axes use the same units. The rover is currently located at the point $(2, -1)$, shown as a dot.

The rover can head in many directions. In this problem we'll estimate the slope in several directions. For example, if the rover follows the vector $(0, 1)$, heads north, then it has to move a distance (run) of 0.1 units to hit the next contour, resulting in a change in height of $\Delta z = +0.2$ units. This means the slope in the $(0, 1)$ direction is

$$\frac{\text{rise}}{\text{run}} = \frac{\Delta z}{\text{distance moved in } xy \text{ plane}} = \frac{+0.2}{0.1} = 2.$$

1. Estimate the slope if the rover heads east, following $(1, 0)$.
2. If the rover head south, following $(0, -1)$, estimate the slope.
3. If the rover follows the direction $(1, 1)$ (so northeast), what distance must the rover travel to hit the next contour? Use this to estimate the slope in the $(1, 1)$ direction.
4. Estimate the slope in the $(1, 2)$ direction. Then repeat with $(-1, 2)$.
5. What direction yields the greatest uphill slope? greatest downhill slope?
6. Why did we zoom in on the surface, instead of estimating the slope from the graph on the left?

Rather than start with a contour plot and use it to visually estimate slopes, let's start with a function of the form $z = f(x, y)$ and use it to compute slopes.

Problem 4.3 Suppose the elevation z of terrain near the rover is given by the formula $z = f(x, y) = x^2 + 3xy$, and the rover is currently at $P = (2, -1)$.

1. Compute the differential dz and write it in the form $dz = (?)dx + (?)dy$.
2. Evaluate dz at the rover's location $P = (2, -1)$.

   Did you get $dz = (1)dx + (6)dy$?
3. If the rover follows the direction (dx, dy), explain why the slope is $\frac{dz}{\sqrt{(dx)^2+(dy)^2}}$.
4. Estimate the slope if the rover heads east, following $(dx, dy) = (1, 0)$.
5. Estimate the slope if the rover heads north, following $(dx, dy) = (0, 1)$.
6. Estimate the slope in the $(1, 1)$ direction. Then repeat with $(1, 2)$.
7. What direction yields the greatest uphill slope? greatest downhill slope?

The slope questions above are similar to estimating tolerances, though with tolerances there is no need to divide by $\sqrt{(dx)^2 + (dy)^2}$. The next few problems have you analyze this idea, as well as introduce partial derivatives (parts of the differential) and the gradient vector.

Problem 4.4 The volume of a right circular cylinder is $V(r,h) = \pi r^2 h$. Imagine that each of V, r, and h depends on t (we might be collecting rain water in a can, or crushing a cylindrical concentrated juice can, etc.).

See 3.10 for more practice.

1. Compute the differential dV and write it in the form
$$dV = (?)dr + (?)dh.$$
2. Show that we can write dV as the dot product
$$dV = (2\pi rh, ?) \cdot (dr, dh).$$
3. If we know $r = 3$ and $h = 4$, and we know that r could increase by $dr = 0.1$ and h could increase by about $dh = 0.2$, then use differentials to estimate how much V will increase.
4. If h is constant (so $dh = 0$), what is $\frac{dV}{dr}$? If r is constant, what is $\frac{dV}{dh}$?

The vector $(2\pi rh, ?)$ we call the gradient, or derivative, of V. Each entry in the gradient we call a partial derivative. The partial derivatives make up the whole derivative.

When h was held constant above, we got $\frac{dV}{dr} = 2\pi rh$. We call this the partial derivative of V with respect to r and we write $\frac{\partial V}{\partial r} = 2\pi rh$. This is the part of the differential we multiply by dr. Similarly, the partial derivative of V with respect to h, written $\frac{\partial V}{\partial h}$, is the part of the differential we multiply by dh. Using this partial derivative notation, we write the differential as

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh \quad \text{or as the dot product} \quad dV = \left(\frac{\partial V}{\partial r}, \frac{\partial V}{\partial h}\right) \cdot (dr, dh).$$

The above vector $\left(\frac{\partial V}{\partial r}, \frac{\partial V}{\partial h}\right)$ we call the gradient of V and write $\vec{\nabla} V(r,h)$.

Definition 4.1: Partial Derivative, Gradient. Given a function $f(x,y)$, we can write the differential df in the form $df = Mdx + Ndy$. The partial derivative of f with respect to x, written $\frac{\partial f}{\partial x}$ or f_x, is the portion of this differential that we multiply by dx, so $\frac{\partial f}{\partial x} = M$. Similarly the partial derivative of f with respect to y, written $\frac{\partial f}{\partial y}$ is the portion of this differential that we multiply by dy, so $\frac{\partial f}{\partial y} = N$. Symbolically, we can write the differential as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Different disciplines use different notations for the partial derivative. Four common uses are $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = f_x = D_x f$.

Notice that $\frac{\partial f}{\partial x}$ is precisely the derivative of f with respect to x when we assume that the other variables are constant (so $dy = 0$).

The gradient of f at (x,y), written $\vec{\nabla} f(x,y)$, is the vector $\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. The gradient of f is a vector field that returns a vector $\vec{\nabla} f$ at each point (x,y). Using this vector, we can write the differential of f as the dot product

$$df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (dx, dy) = \vec{\nabla} f \cdot (dx, dy).$$

We can extend this definition to functions with any number of inputs. For example, for the function $f(x,y,z)$, we have

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot (dx, dy, dz) \\ &= \vec{\nabla} f \cdot (dx, dy, dz). \end{aligned}$$

Problem 4.5 The volume of a box is $V(x,y,z) = xyz$.

1. Compute the differential dV and write it in the form
$$dV = (?)dx + (?)dy + (?)dz.$$
2. Show that we can write dV as the dot product (fill in the blanks)
$$dV = (yz, ?, ?) \cdot (dx, dy, dz).$$
3. If the current measurements are $x = 2$, $y = 3$, and $z = 5$, and we know that $dx = .01$, $dy = .02$, and $dz = .03$, then estimate the change in volume.
4. Compute $\frac{\partial V}{\partial x}$. Then also state $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$. Read the definition before this problem if you need help with the notation.

The vector $(yz, ?, ?)$ is the derivative, or gradient. The entries we call the partial derivatives. The partial derivatives make up the whole derivative.

Problem 4.6 Let's compute differentials, partial derivatives, and gradients.

1. Let $f(x,y) = 3x^2 + 2xy$. Compute the differential df. Then give $\frac{\partial f}{\partial x}$ and f_y, and finish by stating the gradient $\vec{\nabla} f(x,y)$.
2. Let $f(x,y) = e^{-2x}\cos(3y)$. Compute the differential df. Then give $\frac{\partial f}{\partial x}$ and f_y, and finish by stating the gradient $\vec{\nabla} f(x,y)$.
3. Let $g(r,s,t) = r^2s^3 + 4rt^2$. Compute dg. State $D_r g$, then g_s, and then $\frac{\partial g}{\partial t}$. Then give $\vec{\nabla} g$.

Let's now return to the topic of finding the slope of a function $z = f(x,y)$ at a point P in a given direction $\vec{u}$.

Problem 4.7 Suppose that our rover is located at point $P = (x,y)$ on a hill whose elevation is given by $z = f(x,y)$. The rover will be moving in the direction parallel to $\vec{u}$. We have already shown that the slope of the hill at P in the direction $\vec{u} = (dx, dy)$ is given by
$$\frac{dz}{\sqrt{(dx)^2 + (dy)^2}}.$$

1. Prove that this slope can be written, using gradients, as
$$\vec{\nabla} f(P) \cdot \frac{\vec{u}}{|\vec{u}|}.$$
2. Use the above fact to compute the slope of a hill given by $f(x,y) = x^2 + 3xy$ at $P = (2,-1)$ in the direction $\vec{u} = (3,4)$. (We call this the directional derivative of f at P in the direction $\vec{u}$, written $D_{\vec{u}} f(P)$.

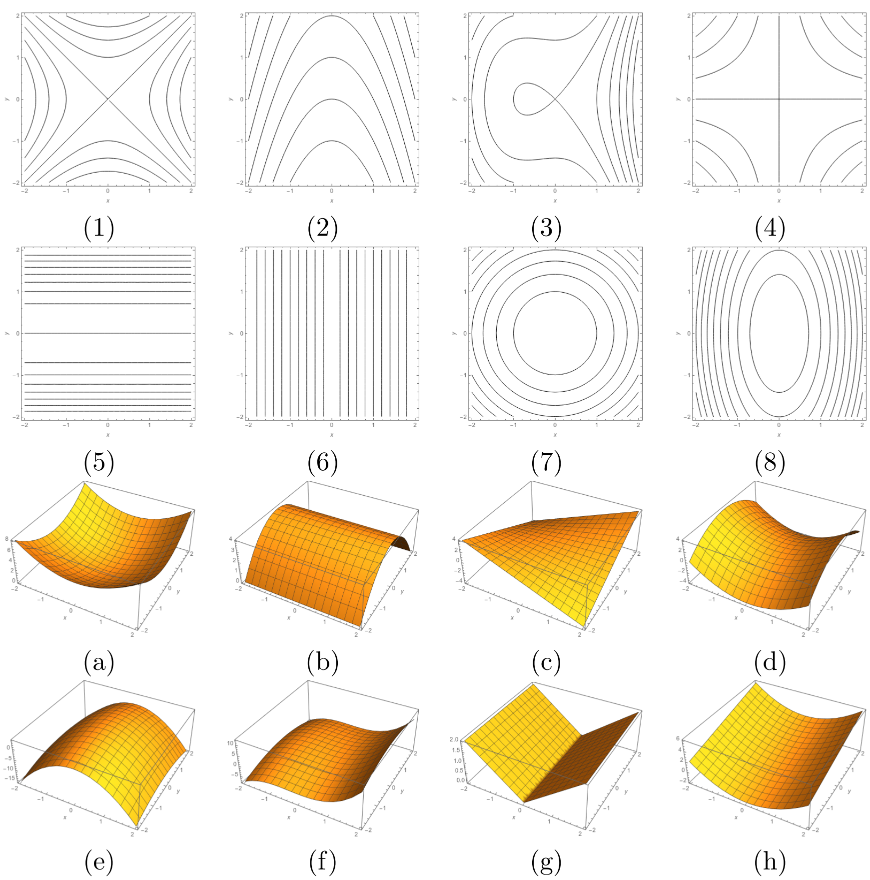
Definition 4.2. The directional derivative of f in the direction of the vector $\vec{u}$ at a point P is defined to be
$$D_{\vec{u}} f(P) = \vec{\nabla} f \cdot \frac{\vec{u}}{|\vec{u}|}, \quad \text{or} \quad D_{\hat{u}} f(P) = \vec{\nabla} f \cdot \hat{u}$$
if $\hat{u}$ is a unit vector. We dot the gradient of f with a unit vector in the direction of $\vec{u}$. The partial derivative of f with respect to x is precisely the directional derivative of f in the $(1,0)$ direction. Similarly, the partial derivative of f with respect to y is precisely the directional derivative of f in the $(0,1)$ direction. This definition extends to higher dimensions.

Problem 4.8 Suppose our rover is located at a point P on a hill whose elevation is given by $z = f(x, y)$. Recall that the directional derivative of $\vec{f}$ at P in the direction $\vec{u}$ is the dot product $D_{\vec{u}}f(P) = \vec{\nabla}f(P) \cdot \frac{\vec{u}}{|\vec{u}|}$. Also recall that we can compute dot products using the law of cosines $\vec{\nabla}f(P) \cdot \vec{u} = |\vec{\nabla}f(P)||\vec{u}| \cos\theta$, where θ is the angle between $\vec{\nabla}f(P)$ and $\vec{u}$.

1. What angle should θ be to obtain the largest slope (directional derivative)?
2. State a vector $\vec{u}$ that yields the largest directional derivative.
3. When $\vec{u}$ is parallel to $\vec{\nabla}f(P)$, show that $D_{\vec{u}}f(P) = |\vec{\nabla}f(P)|$.
4. Which direction points in the direction of greatest decrease?

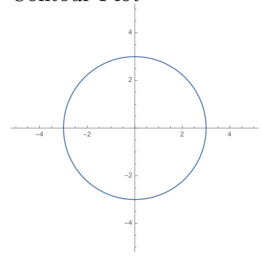
Problem 4.9 Match each contour plot with the appropriate surface plot. Some will be easy to pair up, while others a little more difficult. Record which pairings took you more time, and then come ready to explain how you made the final decisions, without needing software. It's perfectly fine to use the Mathematica notebook provided to the right to help you graph things, but your goal on this problem is make the pairings without needing software.

You can use Mathematica to check if you are correct. Download ContourSurfaceGradient.nb. The functions used to create the plots to the left are as follows:
(1) $f = x^2 - y^2$
(2) $f = x^2 + y$
(3) $f = x^3 + x^2 - y^2$
(4) $f = xy$
(5) $f = 4 - y^2$
(6) $f = |x|$
(7) $f = x^2 + y^2$
(8) $f = 4 - 4x^2 - y^2$

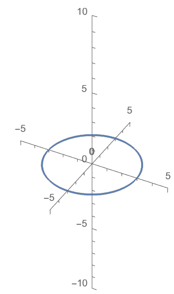


Problem 4.10 A computer chip has been disconnected from electricity and sitting in cold storage for quite some time. The chip is connected to power, and a few moments later the temperature (in Celsius) at various points (x, y) on the chip is measured. From these measurements, statistics is used to create a temperature function $z = f(x, y)$ to model the temperature at any point on the chip. Suppose that this chip's temperature function is given by the equation $z = f(x, y) = 9 - x^2 - y^2$. (This could just as easily have been the elevation of a rover at a point (x, y) on a hill.) We'll be creating a 3D model of this function in this problem, so you'll want to place all your graphs on the same x, y, z axes. The points in the plane with temperature $f(x, y) = 0$ satisfy $0 = 9 - x^2 - y^2$, or equivalently $x^2 + y^2 = 9$. These points lie on a circle of radius 3, so we'll draw that circle in the xy plane (the start of our contour plot) and also in 3D by plotting a circle of radius 3 at height $z = 0$ (the start of our surface plot). These two plots are shown to the right.

Contour Plot



Surface Plot



1. What is the temperature at $(0, 0)$, $(1, 2)$, and $(-4, 3)$?
2. Which points in the plane have temperature $z = 5$? Add this contour (level curve) to your 2D contour plot. Then at height $z = 5$, add the same curve to the surface plot.
3. Repeat the above for $z = 8$, $z = 9$, and $z = 1$. What's wrong with letting $z = 10$?
4. Describe the 3D surface that you created with your plot. Add any extra features to your 3D surface plot to convey the 3D image you constructed.

You can use Mathematica to check if you are correct. Download ContourSurfaceGradient.nb. See Thomas's calculus 14.1: 37-48, for more practice.

In the previous problem we started by constructing a contour plot. We picked a value for the output and then constructed the curve in the plane that gave all the points with this height. We call such a curve a level curve.

Definition 4.3: Level Curve, Contour. The level curve of a function $f(x, y)$, corresponding to the constant c, is the set of points in the xy-plane such that $f(x, y) = c$, the output is constant. Level curves provide a cross section of the surface plot with the plane $z = c$. Many names are given to level curves, such as contours, isotherms (constant temperature), isobars (constant pressure), and more. The key idea is that the output of the function should stay at the same level (be constant, same, iso, equal, etc.).

Problem 4.11 Consider the function $z = f(x, y) = x^2 - 4$.

1. Construct a contour plot by graphing several level curves. If you end up with several lines parallel to an axes, you are doing this correctly.
2. We now construct the 3D surface plot. Let $y = 0$ and then graph the curve $z = x^2 - 4$ on a 3D axes. Now let $y = 1$ and add to your plot the resulting curve. Then let y equal some other constant, and add to your plot the graph of the resulting object. If you find yourself drawing the same object, just shifted left or right along the y-axis, then you are doing this correctly.
3. Now let $x = 0$ and add to your graph the curve $z = -4$ (it should be a line in the yz-plane). Then let $x = 2$ and plot the corresponding curve. Repeat this for several values of x until you have made a 3D surface plot that you are happy with.

You can use Mathematica to check if you are correct. Download ContourSurfaceGradient.nb.

The elevation encountered by a rover is given by one function $z = f(x, y)$. The path the rover follows is given by a parameterization $\vec{r}(t) = (x(t), y(t))$. As the rover moves around, the height changes. The function f tells us the height based on position, while the other function $\vec{r}$ tells us position based on time. Using function composition, we can combine these two functions to get a new function $f(\vec{r}(t))$ that gives us the height based on time. These functions are like a chain of events. Changing t causes position (x, y) to change, which in turn causes the height z to change. The chain rule helps us see how to compute the derivative of a function that is composed of several smaller pieces. We'll see below that the chain rule, when written in differential form, is just direct substitution.

Problem 4.12 A rover moves on a hill where elevation is given by $z = f(x, y) = 9 - x^2 - y^2$. The rover's path is parametrized by $\vec{r}(t) = (2\cos t, 3\sin t)$.

1. At time $t = 0$, what is the rover's position $\vec{r}(0)$, and what is the elevation $f(\vec{r}(0))$ at that position? Then find the positions and elevations at $t = \pi/2$, $t = \pi$, and $t = 3\pi/2$ as well.

2. In the plane, draw the rover's path for $t \in [0, 2\pi]$. Then, on the same 2D graph, include a contour plot of the elevation function f. Include the level curves that pass through the points in part 1. Along each level curve drawn, state the elevation of the curve.

   If you end up with an ellipse and several concentric circles, then you've done this right.

3. As the rover follows its elliptical path, the elevation is rising and falling. At which t does the elevation reach a maximum? A minimum? Explain, using your graph.

4. As the rover moves past the point at $t = \pi/4$, is the elevation increasing or decreasing? In other words, is $\dfrac{df}{dt}$ positive or negative? Use your graph to explain.

Notice above that we wanted $\frac{df}{dt}$, the rate of change of temperature with respect to time, even though the function $f(x, y)$ does not explicitly have t as an input. The proper notation would be $\frac{d(f \circ r)}{dt}$, but this is so cumbersome that it's generally avoided.

Problem 4.13 Consider again $f(x, y) = 9 - x^2 - y^2$ and $\vec{r}(t) = (2\cos t, 3\sin t)$, which means $x = 2\cos t$ and $y = 3\sin t$.

1. At the point $\vec{r}(t)$, we'd like a formula for the elevation $f(\vec{r}(t))$. What is the elevation of the rover at any time t? [In $f(x, y)$, replace x and y with what they are in terms of t.]

2. Compute df/dt (the derivative as you did in first-semester calculus).

3. Construct a graph of $z = f(\vec{r}(t))$ (so t on the horizontal axis, and z on the vertical axis). From your graph, at what time values do the maxima and minima occur?

4. Compute $\frac{df}{dt}$ at $t = \pi/4$?

Let's repeat the above, but first compute differentials before substitution.

Problem 4.14 Let $f(x, y) = 9 - x^2 - y^2$ and $(x, y) = \vec{r}(t) = (2\cos t, 3\sin t)$.

1. Compute the differential df in terms of x, y, dx, and dy.
2. Compute dx and dy in terms of t and dt.
3. Use substitution to write df in terms of t and dt. Then divide by dt to obtain $\frac{df}{dt}$. Did you get the same answer as the previous problem?
4. Use your work above to state both $\vec{\nabla} f(x, y)$ and $\frac{d\vec{r}}{dt}$. How could you combine these two vectors to get $\frac{df}{dt}$?

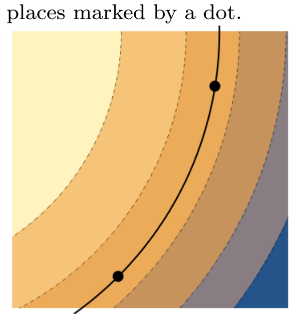
Problem 4.15: Chain Rule Suppose that $f(x, y)$ represents an arbitrary differentiable function and $\vec{r}(t) = (x, y)$ is a parametrization of a differentiable curve. Explain why

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \quad \text{and} \quad \frac{df}{dt} = \vec{\nabla} f \cdot \frac{d\vec{r}}{dt}.$$

Then obtain similar formulas for functions of the form $f(x, y, z)$ and curves parametrized by $\vec{r}(t) = (x, y, z)$.

Problem 4.16 Suppose a rover moves along the level curve of a function $f(x, y)$ following the path $\vec{r}(t) = (x, y)$. An example of such a scenario is shown to the right. Label the dots A and B (it doesn't matter which you label A or B). Our goal is to prove that the gradient of f is normal to level curves.

A rover moves along the solid level curve below, stopping at the two places marked by a dot.



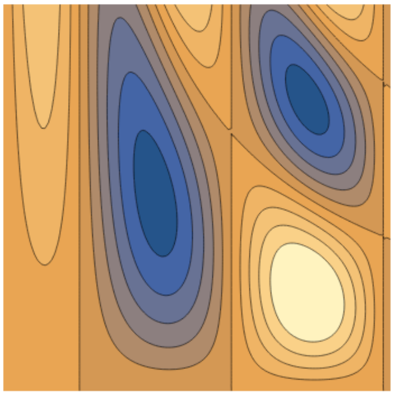
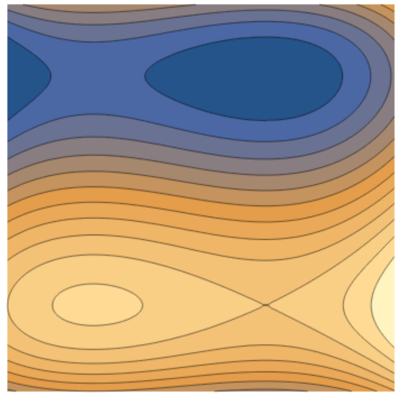
1. At each dot in the picture on the right, draw a vector that represents a possible option for $\frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$.
2. Suppose $\vec{r}(0) = A$ and $\vec{r}(1) = B$. If we know that $f(\vec{r}(0)) = 7$, then what is $f(\vec{r}(1))$? Explain.
3. As the rover moves along $\vec{r}(t)$, how much does f change? Use this to give a value for $\frac{df}{dt}$?
4. Explain why $\vec{\nabla} f$ and $\frac{d\vec{r}}{dt}$ are orthogonal at any point along the level curve. (Hint: Look at the result of the previous part, together with the definition of orthogonal.)
5. Draw a vector that points in the same direction as $\vec{\nabla} f$, and use your work above to explain why the gradient of f must be normal to the level curve.

We've obtained two important facts, namely

- the chain rule which states $\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = \vec{\nabla} f \cdot \frac{d\vec{r}}{dt}$, and
- gradients are normal to level curves.

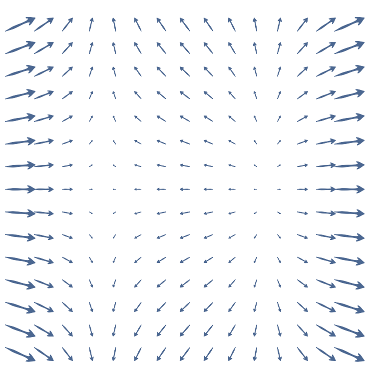
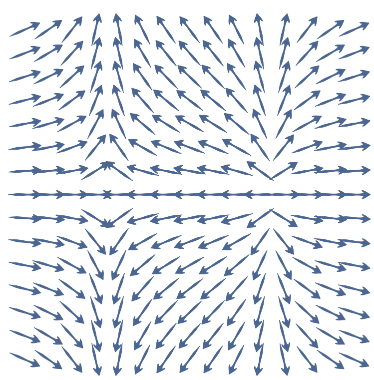
Let's use all our knowledge up to now to draw gradients on a given contour plot, and vice versa to construct a contour plot from a given gradient field plot.

Problem 4.17 Consider the contour plots below. Print this page.

1. On both plots, add lots of vectors to the plot, illustrating the direction in which the gradient points (it's fine to keep all the vectors quite short so they don't overlap).
2. In each plot, label a spot using the letter A where the gradient will be quite long, and a spot B where the gradient will be quite short.
3. In each plot, locate all the points where the gradient is zero. Explain.
4. Where are the local maximums and local minimums? Explain.

Problem 4.18 The gradient field of a function $f(x, y)$ is shown below on the left. Note that this plot shows the relative sizes of the gradients. Since the directions attached to the smaller vectors can be difficult to see, The plot on the right shows just the directions (all vectors have the same size). This second plot we call a direction field plot. Print this page.

1. Mark two points on the plot where the gradient is zero.
2. For each point you marked, classify it as either a local maximum, a local minimum, or neither. Explain.
3. Now add a few level curves of $f(x, y)$ to the map above. When you're done, you'll have constructed a contour plot of f from the gradient field.

Definition 4.4: Critical Point.

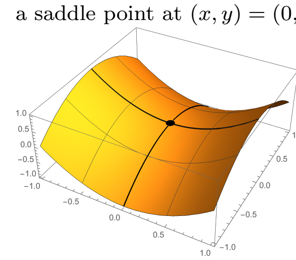
Our work above showed the following must hold for differentiable functions:

- A local maximum or local minimum must occur at a critical point.
- Near a local maximum, gradient vectors point towards the critical point.
- Near a local minimum, gradient vectors point away from the critical point.
- Some critical points are neither a local maximum nor local minimum.

A saddle point (x, y) is a critical point that corresponds to a maximum in one direction, but a minimum in another direction. The surface plot to the right provides an example of a saddle point.

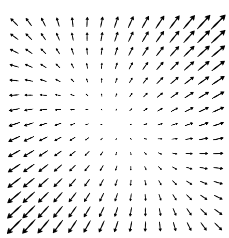
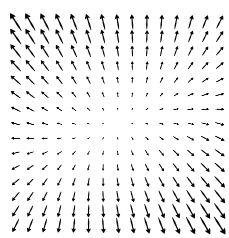
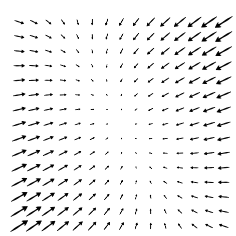
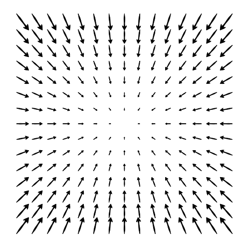
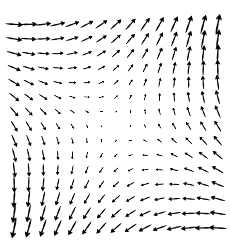
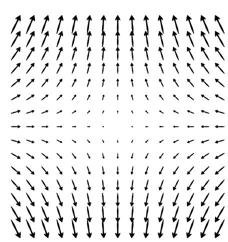
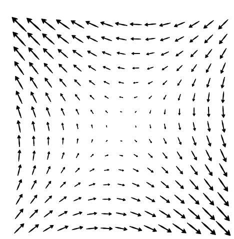
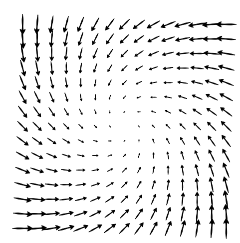
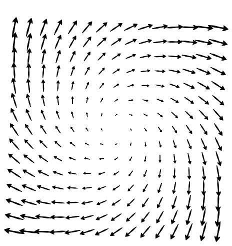
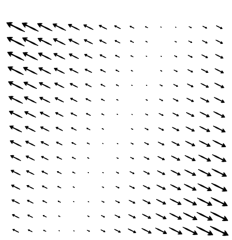
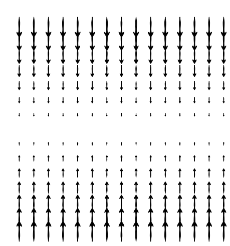
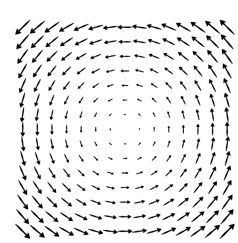
We now introduce a way to determine, without having to construct a gradient field plot, whether or not a critical point corresponds to a local maximum, local minimum, or saddle point. At every critical point there are two numbers, called eigenvalues, that answer this question for us. We'll learn how to compute these eigenvalues very soon, and they'll start appearing in your upper division courses. First, let's tackle a problem that helps us see how these numbers are useful.

The function $f(x, y) = x^2 - y^2$ has a saddle point at $(x, y) = (0, 0)$.



Letting $y = 0$ produces $z = x^2$, and letting $x = 0$ produces $z = -y^2$. Both are parabolas, but $(x, y) = (0, 0)$ is the location of a minimum for the first, but maximum for the second.

Problem 4.19 Below are several gradient field plots. Each plot is centered at a critical point of a function. The viewing window has been zoomed in to analyze the gradient near the critical point. Below each plot are two numbers, the eigenvalues, that correspond to the critical point.

 Eigenvalues: 4, 2	 Eigenvalues: 2, 1	 Eigenvalues: -3, -1	 Eigenvalues: -4, -3
 Eigenvalues: 5, -4	 Eigenvalues: 2, -1	 Eigenvalues: 3, -2	 Eigenvalues: $-1 \pm i$
 Eigenvalues: $2 \pm 3i$	 Eigenvalues: 1, 0	 Eigenvalues: -2, 0	 Eigenvalues: $0 \pm i$

Analyze the vector fields above and look for patterns between the behavior of the field and the eigenvalues. Look for ways you can use the eigenvalues to help you determine if the critical points corresponds to a maximum, minimum, or saddle point. List any and all patterns you see. Make as many conjectures as you can. Write these conjectures down, using complete sentences.

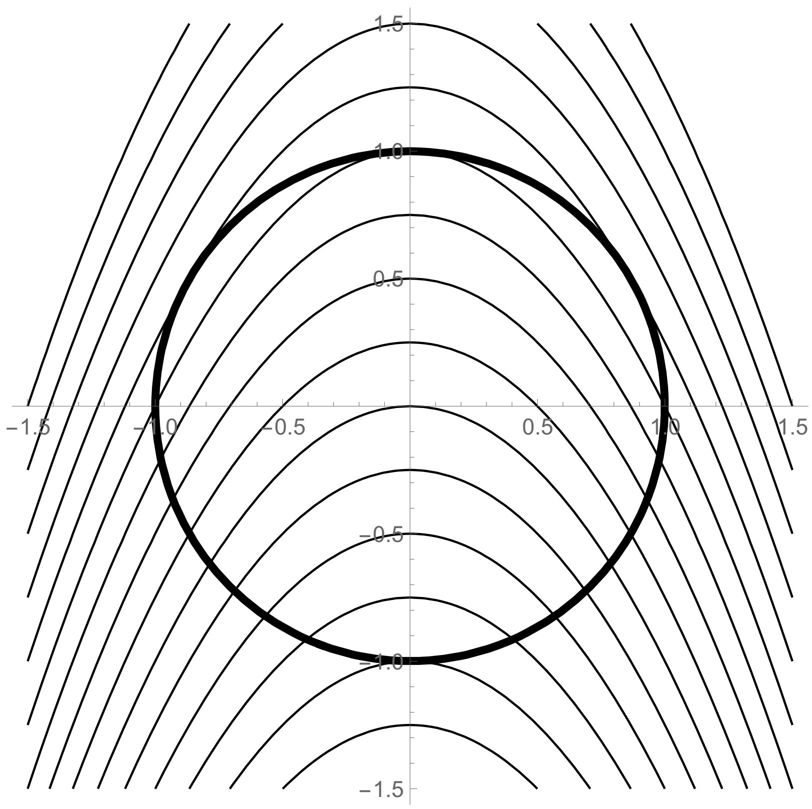
If you need some questions to help get you started, here are some things you might consider. You do not need to answer all of these questions. Your goal is to make lots of conjectures, as many as you can. The questions below all have a definite answer. There are more questions that could be asked. We've only touched the tip of an iceberg when it comes to eigenvalues.

- If the vectors all point towards the center, what can I tell about the eigenvalues? What if instead the vectors all point away from the center?
- What does a positive eigenvalue mean? What does a negative eigenvalue mean? What does a zero eigenvalue mean? What does a complex eigenvalue mean?
- What happens if one eigenvalue is positive, and the other is negative? What if they are both positive? What if they are both negative?
- If I want to find maxes, what do I look for? How do I find mins? How do I find saddle points?
- If a vector field involves rotation of some kind, what does that mean about the eigenvalues?
- Are there situations where it might be impossible, from the gradient field picture alone, to determine if the center point (the critical point) corresponds to a maximum, minimum, or saddle? Are the eigenvalues enough?

Many of the previous problems focused on finding maximum and minimum values in an entire region. The solution technique changes, however, when we restrict ourselves to a specific path, called a constraint. The next few problems examine how we optimize (find maxes and mins) of a function (elevation, profit, force, density, etc.) subject to a constraint (path followed, budget, limited resources, limited space, etc.). If you are economics student, this topic may be the key reason why you were asked to take multivariate calculus. In the business world, we often want to optimize something (profit, revenue, cost, utility, etc.) subject to some constraint (a limited budget, a demand curve, warehouse space, employee hours, etc.). An aerospace engineer will build the best wing that can withstand given forces. Everywhere in the engineering world, we often seek to create the "best" thing possible, subject to some outside constraints. Lagrange discovered an extremely useful method for answering this question. Today we call his method "Lagrange Multipliers."

Rather than introduce Cobb-Douglass production functions (from economics) or sheer-stress calculations (from engineering), we'll work with simple examples that illustrate the key points. We'll look for the extreme elevations (optimize elevation) of a Mars rover moving along a specific path (the constraint).

Problem 4.20 Print this page. Suppose a rover travels around the circle $g(x,y) = x^2 + y^2 = 1$. The elevation of the surrounding terrain is $f(x,y) = x^2 + y + 4$. The plot below shows the rover's path (the constraint $g(x,y) = 1$), placed on the same grid as a contour plot of the elevation (the function $f(x,y)$ we wish to optimize).



Each level curve above represents a difference in elevation of 0.25 m. Our goal is to find the maximum and minimum elevation reached by the rover as it travels around the circle. We will optimize $f(x,y)$ subject to the constraint $g(x,y) = 1$.

Lagrange Multiplier problems will appear as "Optimize $f(x,y)$ subject to the constraint $g(x,y) = c$ for some constant c.

1. Label each level curve with its elevation. Print this page if you have not.
2. At which (x,y) point does the rover encounter the minimum elevation? What is the minimum elevation?
3. Suppose the rover is currently at the point $(0,1)$ on its circular path. As the rover moves left, will the elevation rise or fall? What if the rover moves right? Is $(0,1)$ the location of a local maximum or local minimum?
4. On your graph, place a dot(s) where the rover reaches a maximum elevation. What is the maximum elevation? Explain.
5. Rather than visually inspecting level curves, let's examine the gradients $\vec{\nabla} f$ and $\vec{\nabla} g$ to see how these gradients compare at maximums and minimums. On the graph above, draw $\vec{\nabla} f$ at lots of places on your contour plot. At lots of points on the circle, with a different color, draw $\vec{\nabla} g$. Make sure you draw both gradients at all the points corresponding to local maxes and mins. Make a conjecture about the relationship between $\vec{\nabla} f$ and $\vec{\nabla} g$ at the local maximums and local minimums?

Theorem 4.5 (Lagrange Multipliers)**.** *Suppose f and g are continuously differentiable functions. Suppose that we want to find the maximum and minimum values of $f(x, y)$ subject to the constraint $g(x, y) = c$ (where c is some constant). If a local maximum or minimum occurs, it must occur at a spot where the gradient of f and the gradient of g point in the same, or opposite, directions. This means the gradient of g must be a multiple of the gradient of f. To find the (x, y) locations of the maximum and minimum values (if they exist), we solve the system of equations that result from*

$$\vec{\nabla} f = \lambda \vec{\nabla} g, \quad \textit{and} \quad g(x, y) = c$$

where λ is the proportionality constant. The locations of maximum and minimum values of f will be among the solutions of this system of equations.

Let's redo the previous problem, this time using gradients and the system

$$\vec{\nabla} f = \lambda \vec{\nabla} g, \quad \text{and} \quad g(x, y) = c.$$

Problem 4.21 A rover travels around the circle $x^2 + y^2 = 1$ where the elevation of the region nearby is given by $z = x^2 + y + 4$. Our goal is to find the maximum and minimum elevations reached by the rover on its circular path.

1. What function $f(x, y)$ do we wish to optimize? What is the constraint $g(x, y) = c$ (state both $g(x, y)$ and the constant c)?

The most common error on this problem is to divide both sides of $2x = \lambda 2x$ by x (which could be zero). If you did this, you'll only get 2 ordered pairs. Instead, subtract all variables to one side, and then factor out an x. NEVER DIVIDE BY ZERO or something that could be zero.

2. Compute $\vec{\nabla} f$ and $\vec{\nabla} g$. Explain why the system of equations $\vec{\nabla} f = \lambda \vec{\nabla} g$ and $g(x, y) = c$ is equivalent to the system of equations

$$2x = \lambda 2x, \quad 1 = \lambda 2y, \quad x^2 + y^2 = 1.$$

3. Solve the system of equations above to obtain 4 ordered pairs (x, y).

4. At each ordered pair, find the elevation. What is the maximum elevation obtained, and where does it occur? What is the minimum elevation obtained, and where does it occur?

Use the following Mathematica notebook to check your work: LagrangeMultipliers.nb.

The above example used a function of the form $f : \mathbb{R}^2 \to \mathbb{R}$, where the domain consisted of points in the plane $\mathbb{R}^2$. The exact same process can be used on functions of the form $f : \mathbb{R}^n \to \mathbb{R}$, where the domain can consist of several variables (such as space (x, y, z), time t, pressure p, density δ, velocity, acceleration, cost, etc.). Once the number of variables gets too large, visualizing the solution becomes more difficult. The next two problems focus on visualizing the level surfaces of a function of the form $f : \mathbb{R}^3 \to \mathbb{R}$.

Definition 4.6: Level surface of $f(x, y, z)$. Given a function $f(x, y, z)$, the level surface of f corresponding to a constant c is the set of points (x, y, z) in space $\mathbb{R}^3$ such that $f(x, y, z) = c$. This is the collection of inputs (x, y, z) that return the same, or level, output.

Problem 4.22 Suppose that an explosion occurs at the origin $(0, 0, 0)$. Heat from the explosion starts to radiate outwards. Suppose that a few moments after the explosion, the temperature at any point in space is given by $w = T(x, y, z) = 100 - x^2 - y^2 - z^2$.

Use the Mathematica notebook below to check your work: ContourSurfaceGradient.nb. You can access more problems on drawing level surfaces in 12.6:1-44 or 14.1:53-60.

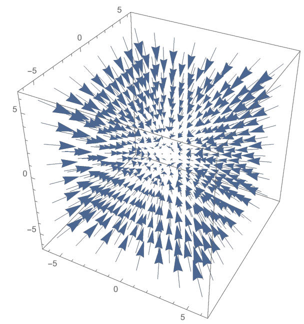
1. Which points in space have a temperature of 99? To answer this, replace $T(x, y, z)$ by 99 to get $99 = 100 - x^2 - y^2 - z^2$. Use algebra to simplify this to $x^2 + y^2 + z^2 = 1$. Draw this object.

2. Which points in space have a temperature of 96? of 84? Draw the surfaces.

3. What is the temperature at $(3, 0, -4)$? Draw the set of points that have this same temperature.

4. The 4 surfaces you drew above are called level surfaces. If you walk along a level surface, what happens to your temperature?

5. To the right is a picture of the gradient field $\vec{\nabla} f$. From the gradient field plot alone, explain what happens to the temperature as you move away from origin.



Above is a picture of the 3D gradient field of $f(x, y, z)$. All the vectors point towards the origin.

Problem 4.23 Consider the function $w = f(x, y, z) = x^2 + z^2$. This function has an input y, but notice that changing the input y does not change the output of the function.

1. Draw a graph of the level surface $w = 4$. [When $y = 0$ you can draw one curve. When $y = 1$, you draw the same curve. When $y = 2$, again you draw the same curve. This kind of graph we call a cylinder, and is important in manufacturing where you extrude an object through a hole.]

2. Graph the surface $9 = x^2 + z^2$ (so the level surface $w = 9$).

3. Graph the surface $16 = x^2 + z^2$.

4. Use the Mathematica notebook ContourSurfaceGradient.nb to construct a plot of the 3D gradient field of f. Add to that plot the level surfaces you drew above. What relationships hold between the gradient vectors and the level surfaces?

For Lagrange multipliers, the fact that the level curves of f and g meet tangentially at maximums and minimums was crucial. Because gradients are normal to level curves, we located maxes and mins by comparing the two gradients. In higher dimensions, we need level surfaces to meet tangentially, which means we need to compare tangent planes.

The next few problems focus on using differentials and the gradient to obtain equations of tangent lines and tangent planes to curves and surfaces. Let's first review how differential notation gives an equation of the tangent line to $y = f(x)$ at a point $x = c$, and then generalize.

Example 4.7: Tangent Lines. Consider the function $y = f(x) = x^2$.

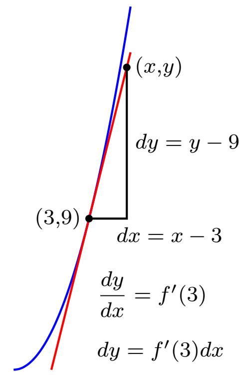
1. The derivative is $f'(x) = 2x$. When $x = 3$ this means the derivative is $f'(3) = 6$ and the output y is $y = f(3) = 9$.

2. The tangent line passes through the point $P = (3, 9)$. Let $Q = (x, y)$ be any other point on the tangent line. The vector from P to Q gives us differentials as

$$(dx, dy) = \vec{PQ} = \underbrace{(x, y)}_{\vec{Q}} - \underbrace{(3, 9)}_{\vec{P}} = (\underbrace{x - 3}_{dx}, \underbrace{y - 9}_{dy}).$$

   This vector tells us that on our tangent line, for a change in x of $dx = x - 3$, we know the change in y is $dy = y - 9$.

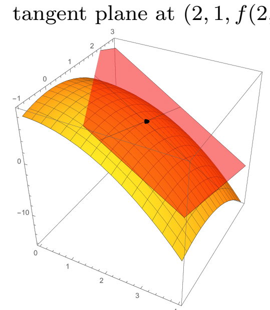
3. Differential notation states that a change in the output dy equals the derivative times a change in the input dx. In symbols, we have the equation $dy = f'(3)dx$. We then replace dx, dy, and $f'(3)$ with what we know they equal from the parts above to obtain the tangent line's equation
$$\underbrace{y-9}_{dy} = \underbrace{6}_{f'(3)}\underbrace{(x-3)}_{dx}.$$



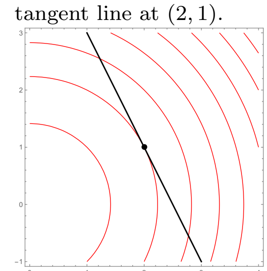
In first semester calculus, differential notation is $dy = f'dx$. At $x = c$, the line passes through the point $P = (c, f(c))$. If $Q = (x, y)$ is any other point on the line, then the vector $\vec{PQ} = (x - c, y - f(c))$ tells us that when $dx = x - c$ we have $dy = y - f(c)$. Substitution give us an equation for the tangent line tangent line as
$$\underbrace{y-f(c)}_{dy} = f'(c)\underbrace{(x-c)}_{dx}.$$
This equation tells us that a change in the output $(y - f(c))$ equals the derivative times a change in the input $(x - c)$. We now repeat this for the next problem, where the output is z and input is (x, y), where differential notation gives
$$dz = \vec{\nabla} f \cdot (dx, dy) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Problem 4.24 Consider the function $z = f(x, y) = 9 - x^2 - y^2$. If you haven't yet, read the example above. We'll be finding an equation of the tangent plane to f at $(x, y) = (2, 1)$

See 14.6: 9-12 for more practice. Here is the surface with the tangent plane at $(2, 1, f(2, 1))$.



Here is the contour plot with the tangent line at $(2, 1)$.



1. Compute the partial derivatives f_x and f_y, and the differential dz. At the point $(x, y) = (2, 1)$, evaluate the partial derivatives and the function $z = f(x, y)$.
2. One point on the tangent plane to the surface at $(2, 1)$ is the point $P = (2, 1, f(2, 1))$. Let $Q = (x, y, z)$ be another point on this plane. Use the vector $\vec{PQ}$ obtain dz when $dx = x - 2$ and $dy = y - 1$.
3. We'd like an equation of the tangent plane to $f(x, y)$ when $x = 2$ and $y = 1$. Differential notation tells us that
$$\underbrace{z-?}_{dz} = (-4)\underbrace{(x-?)}_{dx} + (?)\underbrace{(y-?)}_{dy}$$
Fill in the blanks to get an equation of the tangent plane.
4. The level curve of f that passes through $(2, 1)$ has no change in height, so $dz = 0$. Use this fact to give an equation of the tangent line to this level curve at $(2, 1)$.

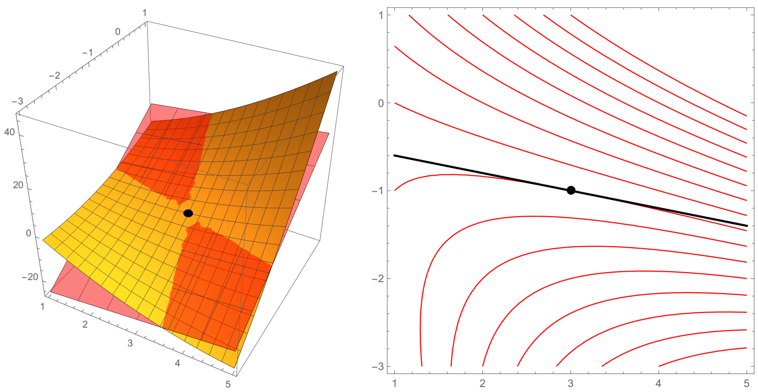
Problem 4.25 Let $z = f(x, y) = x^2 + 4xy + y^2$. At the point $P = (x, y) = (3, -1)$, we'll give an equation of the tangent plane to the surface and an equation of the tangent line to the level curve of f that passes through this point.

See 14.6: 9-12 for more practice.

1. Give an equation of the tangent plane at $P = (x, y) = (3, -1)$. [Hint: Find f_x, f_y, dx, dy, and then dz, all at $(x, y) = (3, -1)$. Then substitute, as done in the previous problem.]

2. The level curve of f that passes through P is a curve in the plane. Give an equation of the tangent line to this curve at P. [Hint: Since we're on a level curve, what does dz equal? The equation is almost identical to the previous part, with one minor change based on what dz equals.]

The tangent plane and tangent line you just found are shown below.

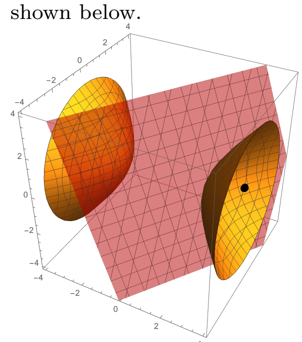


Problem 4.26 Consider the function $f(x, y, z) = -x^2 + y^2 + z^2$, and the level surface which passes through the point $(3, 2, -1)$. This level surface is shown to the right, along with the tangent plane to the surface through the point $(3, 2, -1)$. Use the differential

$$df = f_x dx + f_y dy + f_z dz \quad \text{or} \quad df = \vec{\nabla} f(a, b, c) \cdot (dx, dy, dz).$$

to give an equation of the tangent plane to this surface at the point $(3, 2, -1)$. [Hint: Start by explaining why $df = 0$. Then we have $dx = x - 3$, $dy = y - ?$, and $dz = ?$. Don't forget to evaluate the partials at the correct point.]

The surface $-4 = -x^2 + y^2 + z^2$ and tangent plane at $(3, 2, -1)$ are shown below.



We call this surface a hyperboloid of two sheet.

Review Joe wants to to find the tangent line to $y = x^3$ at $x = 2$. He knows the derivative is $y = 3x^2$, and when $x = 2$ the curve passes through 8. So he writes an equation of the tangent line as $y - 8 = 3x^2(x - 2)$. What's wrong? What part of the general formula $y - f(c) = f'(c)(x - c)$ did Joe forget? See [1] for an answer.

We can summarize our work above with tangent lines and planes as follows:

- The tangent plane to $f(x, y)$ at $(a, b, f(a, b))$ has equation

$$z - f(a, b) = \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) \quad \text{or} \quad z - f(a, b) = \vec{\nabla} f(a, b) \cdot (x - a, y - b).$$

- The tangent line to the level curve of $f(x, y)$ at (a, b) has equation

$$0 = \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) \quad \text{or} \quad 0 = \vec{\nabla} f(a, b) \cdot (x - a, y - b).$$

- The tangent plane to the level surface of $f(x, y, z)$ at (a, b, c) has equation

$$0 = \frac{\partial f}{\partial x}(a, b, c)(x - a) + \frac{\partial f}{\partial y}(a, b, c)(y - b) + \frac{\partial f}{\partial z}(a, b, c)(z - c) \quad \text{or} \quad 0 = \vec{\nabla} f(a, b, c) \cdot (x - a, y - b, z - c).$$

[1] Joe forgot to replace x with 2 in the derivative. The equation should be $y - 8 = 12(x - 2)$. The notation $f'(c)$ is the part he forgot. He used $f'(x) = 3x^2$ instead of $f'(2) = 12$.

Have you noticed since the semester started, when we compute a differential we are really just taking partial derivatives, multiplying them by scalars, and then summing the results. The tangent plane problems are exactly this as well, where we compute each of the partial derivatives, multiply them by a differential (such as $dx = x - 3$), and then sum results. This process occurs so often, in so many different settings, that mathematicians gave it a name. We could keep saying, "Take the things you have, multiply each by a scalar, and then sum the result," or we could invent a word that says to do all this.

Definition 4.8: Linear Combination and Matrix Notation. Given n vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ and n scalars $c_1, c_2, \cdots, c_n$ the linear combination of these vectors using these scalars is the sum

$$\sum_{i=1}^{n} c_1\vec{v}_i = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

Matrix notation and products were invented to organize linear combinations into a visually appealing compact form. We place each vector in the column of a matrix, and then place the corresponding scalars into a single column vector after the matrix. The linear combination above, in matrix form, becomes the matrix product

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \begin{bmatrix} \begin{pmatrix} \vec{v}_1 \end{pmatrix} & \begin{pmatrix} \vec{v}_2 \end{pmatrix} & \cdots & \begin{pmatrix} \vec{v}_n \end{pmatrix} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

The differential of a function is always a linear combination of the partial derivatives of the function (using the differentials of the input variables as the scalars). The table below shows the differential of several types of functions, written as a linear combination of the partials, and also written as a matrix product.

Function	Linear Combination	Matrix Product
$y = f(x)$	$dy = \left(\frac{df}{dx}\right) dx$	$dy = \left[\frac{df}{dx}\right] (dx)$
$\vec{r}(t) = (x, y)$	$d\vec{r} = \frac{d\vec{r}}{dt} dt = \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} dt$	$d\vec{r} = \left[\frac{d\vec{r}}{dt}\right] (dt) = \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} (dt)$
$f(x, y)$	$df = f_x dx + f_y dy$	$df = \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$
$f(x, y, z)$	$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$	$df = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$	$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} dr + \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix} d\theta$	$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} & \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix} \end{bmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}$

Notice that the partial derivatives are the columns of the first matrix, and the second matrix is just a column vector of differentials. The matrix of partials, we call the derivative, or total derivative, of the function.

Definition 4.9: Derivative. The derivative (or total derivative) of a function is a matrix whose columns are the partial derivatives of the function. The partial derivatives we insert into the columns of the matrix in the same order in which the variables are listed for the function. Some examples follow.

- For the function $f(x)$, the derivative is $Df(x) = \begin{bmatrix} f_x \end{bmatrix} = \begin{bmatrix} \frac{df}{dx} \end{bmatrix}$.
- For the function $f(x,y)$, the derivative is $Df(x,y) = \begin{bmatrix} f_x & f_y \end{bmatrix}$.
- For the function $f(r,s,t)$, the derivative is $Df(r,s,t) = \begin{bmatrix} f_r & f_s & f_t \end{bmatrix}$.
- For the function $\vec r(u,v)$, the derivative is $D\vec r(u,v) = \begin{bmatrix} \vec r_u & \vec r_v \end{bmatrix}$.

We've added some new definitions, so let's practice.

Problem 4.27 For each function below, (a) compute and label all relevant partial derivatives. Then (b) write the differential df as a linear combination of the partial derivatives, and then (c) write df as a matrix product. Finish by (d) stating the total derivative Df of the function.

If you haven't yet, then please go back and see 14.3: 1-40 in Thomas's Calculus for more practice. I strongly suggest you practice until you can compute partial derivatives with ease.

1. $f(x,y) = x^2y$ [Clearly label all 4 things you were asked to find, namely (a) all partials, (b) df as a linear combination, (c) df as a matrix product, and (d) the derivative Df.]
2. $f(x,y) = x^2 + 2xy + 3y^2$
3. $f(x,y,z) = 3xz - x^2y$

Your textbook has lots of examples to help you with partial derivatives in section 14.3. However, the textbook leaves out the actual derivative (putting the parts into a single matrix). The exercise below has 6 problems, with solutions, that you can use as extra practice for total derivatives. Complete the exercise below before moving on.

Exercise For each function, compute the total derivative. See [2] for answers.

1. $f(x,y) = 9 - x^2 + 3y^2$
2. $\vec r(t) = (t, \cos t, \sin t)$
3. $f(x,y,z) = xy^2z^3$
4. $\vec r(u,v) = (u^2, v^2, u - v)$
5. $\vec F(x,y) = (-y + 3x, x + 4y)$
6. $\vec T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$.

The gradient of a function $f(x,y)$ is itself a function. When we compute the partial derivatives of the gradient, we obtain vectors instead of numbers. The next problem has you examine the differential, partials, and derivative of the gradient of a function. We'll soon see that the derivative of the gradient is precisely the key to classifying maximums and minimums of a function.

Problem 4.28 The function $f(x,y) = x^2 + 3xy + 2y^2$ has the gradient $\vec\nabla f = (2x + 3y, 3x + 4y)$. This is the vector field

$$\vec F = (2x + 3y, 3x + 4y).$$

[2] The derivatives of each function are shown below.

1. $Df(x,y) = \begin{bmatrix} -2x & 6y \end{bmatrix}$
2. $D\vec r(t) = \begin{bmatrix} 1 \\ -\sin t \\ \cos t \end{bmatrix}$
3. $Df(x,y,z) = \begin{bmatrix} y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{bmatrix}$
4. $D\vec r(u,v) = \begin{bmatrix} 2u & 0 \\ 0 & 2v \\ 1 & -1 \end{bmatrix}$
5. $D\vec F(x,y) = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}$
6. $D\vec T(r,\theta,z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

1. Find the differential $d\vec{F}$ and write it as the linear combination
$$d\vec{F} = \begin{pmatrix} ? \\ ? \end{pmatrix} dx + \begin{pmatrix} ? \\ ? \end{pmatrix} dy.$$
2. Rewrite the above differential as a matrix product.
3. Clearly label the two partial derivatives $\frac{\partial \vec{F}}{\partial x}$ and $\vec{F}_y$.
4. State the total derivative $D\vec{F}(x, y)$.
5. The function $f(x, y) = xy^2$ has gradient $\vec{F} = (y^2, 2xy)$. Repeat the above to obtain the differential of $\vec{F}$ (as a linear combination, and in matrix form), the partials of $\vec{F}$, and the derivative $D\vec{F}(x, y)$.

We also write the derivative of the gradient as $D^2 f(x, y)$, or $D\vec{\nabla} f(x, y)$, and call the resulting matrix the Hessian of f. Some people use the notation $\vec{\nabla}^2 f$ for the Hessian, though this notation also gets use for the Laplacian $\vec{\nabla} \cdot (\vec{\nabla} f)$, which is a very different quantity.

In the previous problem we were computing partial derivatives of partial derivatives. Informally, we call these second partials, or second-order partial derivatives. Here is a formal definition of the notation.

Definition 4.10: Second-Order Partial Derivatives. A second-order partial derivative of f is a partial derivative of one of the partial derivatives of f. The second-order partial of f with respect to x and then y is the quantity $\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$, so we first compute the partial of f with respect to x, and then compute the partial of the result with respect to y. Alternate notations exist, for example the same second-order partial above we can write as
$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = (f_x)_y = f_{xy} = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}.$$
The subscript notation f_{xy} is easiest to write. Sometimes we'll use subscript notation to mean something other than a partial derivative (like the x or y component of a vector), at which point we use the fractional partial derivative notation to avoid confusion.

Did you notice the swap in order between the fractional notation and the subscript notation? Remember
$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x}.$$

Problem 4.29 Consider $f(x, y, z) = xy^2z^3$ and $g(x, y) = x\cos(xy)$.

1. First compute $\vec{\nabla} f$. Then compute f_{xy} and $\frac{\partial^2 f}{\partial z \partial y}$. Explain how you got these. End by computing the entire second derivative $D\vec{\nabla} f(x, y, z)$ (it is a 3 by 3 matrix with all 9 second partials placed inside).
2. Compute g_x and then g_{xy}. Then compute g_y followed by g_{yx}.

Problem 4.30: Mixed Partials Agree Complete the following:

1. Let $f(x, y) = 3xy^3 + e^x$. Compute the four second partials
$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$
2. For $f(x, y) = x^2\sin(y) + y^3$, compute both f_{xy} and f_{yx}.
3. Make a conjecture about a relationship between f_{xy} and f_{yx}. Then use your conjecture to quickly compute f_{xy} if
$$f(x, y) = 3xy^2 + \tan^2(\cos(x))(x^{49} + x)^{1000}.$$

We're finally prepared to return to the topic of eigenvalues, which we found let us classify extreme values as the location of local maximums or local minimums.

Definition 4.11. Let A be a square matrix, as $A = \left[\begin{pmatrix} a \\ b \end{pmatrix} \quad \begin{pmatrix} c \\ d \end{pmatrix}\right] = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. The eigenvalues λ and eigenvectors $\vec{x}$ of A are solutions λ and $\vec{x} \neq \vec{0}$ to the equation $A\vec{c} = \lambda\vec{c}$, effectively replacing the matrix product (linear combination) with scalar multiplication.

The identity matrix I is a square matrix with 1's on the diagonal and zeros everywhere else, so in 2D we have $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. To find the eigenvalues, we rewrite $A\vec{x} = c\vec{x}$ in the form $A\vec{x} - c\vec{x} = \vec{0}$ or $A\vec{x} - \lambda I\vec{x} = \vec{0}$, which becomes

$$(A - \lambda I) = \vec{0}.$$

We need to find the values λ so that

$$\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{bmatrix} a-\lambda & c \\ b & d-\lambda \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The matrix $\begin{bmatrix} a-\lambda & c \\ b & d-\lambda \end{bmatrix}$ contains two vectors $(a-\lambda, b)$ and $(c, d-\lambda)$. Where does the area of the parallelogram formed by these two vectors show up in the formula $(a-\lambda)(d-\lambda) - bc = 0$?

A linear algebra course will show that λ satisfies

$$(a-\lambda)(d-\lambda) - bc = 0,$$

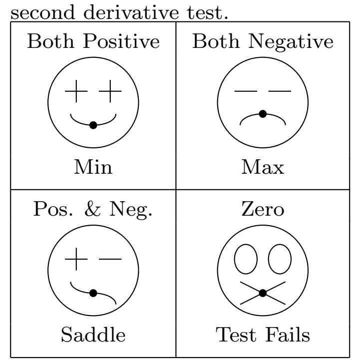
and the proof of this fact is directly connected to the area of the parallelogram formed by the vectors $(a-\lambda, b)$ and $(c, d-\lambda)$.

Theorem 4.12 (The Second Derivative Test)**.** *Let $f(x,y)$ be a function so that all the second partial derivatives exist and are continuous. The second derivative of f, written D^2f and sometimes called the Hessian of f, is a square matrix. Suppose $P = (a,b)$ is a critical point of f, meaning $\vec{\nabla} f(a,b) = (0,0)$.*

- *Suppose all the eigenvalues of $D^2f(a,b)$ are positive. Then at all points (x,y) sufficiently near P, the gradient $\vec{\nabla} f(x,y)$ points away from P. The function has a local minimum at P.*
- *Suppose all the eigenvalues of $D^2f(a,b)$ are negative. Then at all points (x,y) sufficiently near P, the gradient $\vec{\nabla} f(x,y)$ points inwards towards P. The function has a local maximum at P.*
- *Suppose the eigenvalues of $D^2f(a,b)$ differ in sign. Then at some points (x,y) near P, the gradient $\vec{\nabla} f(x,y)$ points inwards towards P. However, at other points (x,y) near P, the gradient $\vec{\nabla} f(x,y)$ points away from P. The function has a saddle point at P.*
- *If the largest or smallest eigenvalue of f equals 0, then the second derivative tests yields no information.*

Because the second derivative is always symmetric (why is it?), in a linear algebra course we can prove that the eigenvalues of D^2f must always be real numbers.

Once you have the eigenvalues of the second derivatives, the following chart is a simple visual aid to help you remember the second derivative test.



Example 4.13. Consider $f(x,y) = x^2 - 2x + xy + y^2$. The gradient is $\vec{\nabla} f(x,y) = (2x - 2 + y, x + 2y)$. The critical points of f occur where the gradient is zero. We need to solve the system $2x - 2 + y = 0$ and $x + 2y = 0$, which is equivalent to solving $2x + y = 2$ and $x + 2y = 0$. Double the second equation, and then subtract it from the first to obtain $0x - 3y = 2$, or $y = -2/3$. The second equation says that $x = -2y$, or that $x = 4/3$. So the only critical point is $(4/3, -2/3)$.

The second derivatives is $D^2f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The second derivative is constant, so $D^2f(4/3, -2/3)$ is the same as $D^2f(x, y)$. To find the eigenvalues we solve

In this example, the second derivative is constant, so the point $(4/3, -2/3)$ does not change the matrix. In general, the critical point will affect your matrix.

$$(2 - \lambda)(2 - \lambda) - (1)(1) = 0.$$

Expanding the left hand side gives $4 - 4\lambda + \lambda^2 - 1 = 0$. Simplifying and factoring gives us $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$. The eigenvalues are $\lambda = 1$ and $\lambda = 3$. Since both numbers are positive, we know the gradient points outwards away from the critical point. The critical point $(4/3, -2/3)$ corresponds to a local minimum of the function. The local minimum is the output $f(4/3, -2/3) = (4/3)^2 - 2(4/3) + (4/3)(-2/3) + (-2/3)^2$.

Problem 4.31 Consider the function $f(x, y) = x^2 + 4xy + y^2$. If you have not yet read the example above, do so now.

See 14.7 for more practice.

1. Find the critical points of f by finding when $Df(x, y)$ is the zero matrix.
2. Find the eigenvalues of D^2f at any critical points.
3. Label each critical point as a local maximum, local minimum, or saddle point, and state the value of f at the critical point.

Let's now return to a Lagrange multiplier problem, where we have a constraint that limits the values over which we want to optimize the function.

Problem 4.32 Consider the curve $xy^2 = 54$ (draw it). The distance from each point on this curve to the origin is a function that must have a minimum value. Find a point (a, b) on the curve that is closest to the origin.

See 14.8 for more practice.

[Hint: The distance from (x, y) to the origin is $\sqrt{x^2 + y^2}$. This distance is minimized when $f(x, y) = x^2 + y^2$ is minimized. We'll use $f(x, y) = x^2 + y^2$ as the function we wish to minimize. What's the constant c and function g so that our constraint is $g(x, y) = c$? Remember to solve the system $\vec{\nabla} f = \lambda \vec{\nabla} g$ and $g = c$, but realize that here the value λ is not an eigenvalue.]

The Lagrange multiplier process can all be automated with software. The key piece needing human intervention is the selection of the function f and constraint $g = c$. The following problem has you practice selecting f, g, and c, and then using software to provide a quick solution to the problem.

Problem 4.33: Mathematica For each scenario below, (1) identify the function $f(x, y)$ to be optimized along with the constant c and function g in the constraint $g(x, y) = c$, (2) write the system of equations that results from $\vec{\nabla} f = \lambda \vec{\nabla} g$ and $g(x, y) = c$, (3) give the solution to this system (use software), (4) determine which points correspond to maxes and which to mins, and (5) produce a relevant plot to verify your conclusions are accurate. Please use the Mathematica notebook LagrangeMultipliers.nb to make this fast (look for the "All Code in One Block" section). To present in class, in addition to a list of the things mentioned above, be prepared to show us how you used Mathematica to perform the computations.

1. An rover travels along a circle of radius 5, centered at the origin. The elevation of the surrounding hill is give by $z = 4x^2 - 4xy + y^2$. What are the highest and lowest elevations reached by the rover.

2. Find the dimensions of the rectangle of largest possible area that will fit inside of the ellipse $\frac{x^2}{9}+\frac{y^2}{25}=1$. [The constraint $g=c$ is sitting in the sentence, but the function f requires you to write a formula for area.]

3. Find dimensions of the box of largest possible volume that lies above the plane $z=0$ and below the paraboloid $z=9-x^2-y^2$. Note that $V=lwh=(2x)(2y)z=4xyz$ is the function we wish to optimize. The Mathematica notebook has a section for functions of 3 variables.

We will finish the chapter by focusing on the second derivative test, so optimizing a function without a constraint.

Problem 4.34 Consider the function $f(x,y)=x^3-3x+y^2-4y$.

1. Find the critical points of f by finding when $Df(x,y)$ is the zero matrix.

2. Find the eigenvalues of D^2f at any critical points. [Hint: First compute D^2f. Since there are two critical points, evaluate the second derivative at each point to obtain 2 different matrices. Then find the eigenvalues of each matrix.]

3. Label each critical point as a local maximum, local minimum, or saddle point, and state the value of f at the critical point.

Problem 4.35 Consider the function $f(x,y)=6x^2-2x^3+3y^2+6xy$. The function has two critical points $(0,0)$ and $(1,-1)$. At each of these points, evaluate the second derivative and then find the corresponding eigenvalues. Use these eigenvalues to classify each critical point as the location of a local maximum, local minimum, or saddle point.

The second derivative test process can also be automated with software. The key piece needing human intervention is (1) determining whether or not there is a constraint (if there is, then use Lagrange multipliers) and (2) the selection of the function f. The following problem has you use technology to rapidly answer several questions where the second derivative test is useful.

Problem 4.36: Mathematica For each scenario below, (1) identify the function $f(x,y)$ to be optimized, (2) list all critical points, (3) state the second derivative at each critical point together with the corresponding eigenvalues, (4) determine if the function has a maximum, minimum, or saddle at each critical point, and (5) produce a relevant plot to verify your conclusions are accurate. Please use the Mathematica notebook 2ndDerTest.nb to make this fast (look for the "All Code in One Block" section). To present in class, in addition to a list of the things mentioned above, be prepared to show us how you used Mathematica to perform the computations.

1. Let $f(x,y)=x^3+3xy+y^3$. Find all local extreme values of f.

2. Find the largest box in the first octant (all variables are positive) that can fit under the paraboloid $z=9-x^2-y^2$. The volume of such a box is given by $V=lwh=xyz=xy(9-x^2-y^2)$. [Hint: There are 9 critical points. Why can you ignore every single one of the points except one?]

3. Find three numbers whose sum is 9 and whose sum of squares is minimized.

In this final optional problem, we'll derive the version of the second derivative test that is found in most multivariate calculus texts. The test given below only works for functions of the form $f : \mathbb{R}^2 \to \mathbb{R}$. The eigenvalue test you have been practicing will work with a function of the form $f : \mathbb{R}^n \to \mathbb{R}$, for any natural number n.

Problem: Optional Suppose that $f(x, y)$ has a critical point at (a, b).

1. We know that $D^2f(a, b) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$, where all partials are evaluated at (a, b). Prove that the eigenvalues of $D^2f(a, b)$ are given by

$$\lambda = \frac{(f_{xx} + f_{yy}) \pm \sqrt{(f_{xx} + f_{yy})^2 - 4(f_{xx}f_{yy} - f_{xy}^2)}}{2}.$$

From the spectral theorem, we know that the eigenvalues will always be real numbers.

2. Let $D = f_{xx}f_{yy} - f_{xy}^2$.
   - If $D < 0$, explain why the eigenvalues differ in sign.
   - If $D = 0$, explain why zero is an eigenvalue.
   - If $D > 0$, explain why the eigenvalues must have the same sign.
   - If $D > 0$, and $f_{xx} > 0$, explain why f has a local minimum at (a, b).
   - If $D > 0$, and $f_{xx} < 0$, explain why f has a local maximum at (a, b).

Different signs $\Rightarrow$ saddle point

$\lambda = 0 \Rightarrow$ test fails

Both positive $\Rightarrow$ minimum
Both negative $\Rightarrow$ maximum

The sign of f_{xx} determines the concavity of f in the x direction.

3. The only critical point of $f(x, y) = x^2 + 3xy + 2y^2$ is at $(0, 0)$. Does this point correspond to a local maximum, local minimum, or saddle point? Find D from part 2 to answer the question.

4.2 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 5

Integration

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

Please use the Mathematica notebook Integration.nb to help you visualize regions in 2D and 3D, as well as compute integrals throughout this chapter.

1. Set up and compute single, double, and triple integrals to obtain lengths, areas, and volumes. Connect these to the differentials dx, ds, dA, and dV.
2. Explain how to compute the mass of a wire, planar region, or solid object, if the density is known. Connect this to the differential dm.
3. Find the average value of a function over a region. Use this to compute the center-of-mass and centroid of a wire, planar region, or solid object.
4. Draw regions described by the bounds of an integral, and then use this drawing to swap the order of integration.
5. Obtain the cross product and use it to find a vector orthogonal to two given vectors, the area of a parallelogram, and the volume of a parallelepiped.
6. Appropriately use polar coordinates $dA = |r|drd\theta$, cylindrical coordinates $dV = |r|drd\theta dz$, and spherical coordinates $dV = |\rho^2 \sin\phi|d\rho d\theta d\phi$.

You'll have a chance to teach your examples to your peers prior to the exam.

5.1 Problems

In the previous chapter, we learned how to determine the slope of a hill along which our rover is moving. In this chapter, we will focus most of our efforts on answering the following question:

> What's the largest slope the rover can encounter before it tips over?

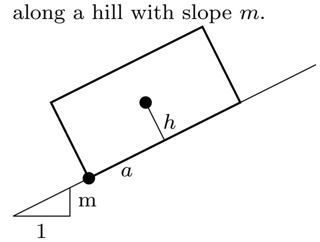
As the team in charge of moving the rover around, we need to make sure that the slope encountered never exceeds this number, and ideally stays quite far from it. The last chapter taught us that we can find the slope by computing the length of the gradient of the elevation function.

The key to making sure the rover does not tip requires we know how to find the center-of-mass of the rover. This center-of-mass changes any time the rovers arms move, the rover picks up an object, the solar panels shift to capture more light, etc. In this chapter, we'll focus on looking at center-of-mass at one instant in time, rather than tracking it as time changes, to keep the problem simpler.

The first problem connects the rover's center-of-mass to the maximum slope the rover can encounter. This maximum slope occurs precisely when the center-of-mass is directly above the point of the rover that is most downhill.

Problem 5.1 When sitting on flat ground, the rover has a center-of-mass that is h units above the ground. Assume this center-of-mass is a units from each side of the rover. Currently the rover is traveling along a contour of the surface. One side of the rover is lower than the other side, and the slope on the mountain is m (see the picture to the right). The tipping slope m_c (critical slope) is the largest possible slope for which the rover will not tip over.

The rover (a box) is moving through the page as it travels along a hill with slope m.



The distance from the center-of-mass to the hill is h units. The distance from the edge of the rover to the spot directly below this center-of-mass (when the rover is on a flat plane) is a. The dots represent the center-of-mass and the point of the rover furthest downhill.

1. Draw several pictures using different slopes to illustrate (1) a slope that would cause the rover to tip over, (2) a slope that would not cause the rover to tip over, and (3) the tipping slope.
2. Find a formula to connect m_c, h, and a. [Hint: Look for similar triangles in your picture.]
3. If you know m_c, what is h? If you know h, what is m_c?

The problem above completely solves the problem of tipping for a stationary rover, provided we know the center-of-mass of the rover. The rest of this chapter focuses on locating the center-of-mass. Let's start by analyzing several ways of computing an average, for example the average of a bunch of test scores.

Problem 5.2 Suppose a class takes a test and there are three scores of 70, five scores of 85, one score of 90, and two scores of 95. We will calculate the average class score, $\bar{s}$, four different ways to emphasize four ways of thinking about the averages. We are emphasizing the pattern of the calculations in this problem, rather than the final answer, so it is important to write out each calculation completely, without doing any simplifying, in the form $\bar{s} = \underline{\qquad}$ before calculating the number $\bar{s}$.

1. Compute the average by adding 11 numbers together and dividing by the number of scores. Write down the whole computation before doing any arithmetic.

   $\bar{s} = \frac{\sum \text{values}}{\text{number of values}}$

2. Compute the numerator of the fraction in the previous part by multiplying each score by how many times it occurs, rather than adding it in the sum that many times. Again, write down the calculation for $\bar{s}$ before doing any arithmetic.

   $\bar{s} = \frac{\sum(\text{value}\cdot\text{weight})}{\sum \text{weight}}$

3. Compute $\bar{s}$ by splitting up the fraction in the previous part into the sum of four numbers. This is called a "weighted average" because we are multiplying each score value by a weight.

   $\bar{s} = \sum(\text{value} \cdot (\% \text{ of stuff}))$

4. Another way of thinking about the average $\bar{s}$ is that $\bar{s}$ is the number so that if all 11 scores were the same value $\bar{s}$, you'd have the same sum of scores. Write this way of thinking about these computations by taking the formulas for $\bar{s}$ in the first two parts and multiplying both sides by the denominator.

   $(\text{number of values})\bar{s} = \sum \text{values}$
   $(\sum \text{weight})\bar{s} = \sum(\text{value} \cdot \text{weight})$

In the next problem, we generalize the above ways of thinking about averages from a discrete situation to a continuous situation. We did this in first-semester calculus when we computed the average value of f using integrals.

Problem 5.3 Suppose the price of a stock is \$10 for two days. Then the price of the stock jumps to \$20 for three days. Our goal is to determine the average price of the stock over the total 5 day period.

1. Why is the average stock price not \$15? Use any of the methods from the previous problem to show that the average price is $\bar{f} = \$16$.

2. The function $f(t) = \begin{cases} 10 & 0 \leq t < 2 \\ 20 & 2 \leq t \leq 5 \end{cases}$ models the price of the stock for the 5-day period. The graph to the right shows both f and $\bar{f}$. Show that the area under f for $0 \leq t \leq 5$ is 80. Then show that the area under $\bar{f}$ for $0 \leq t \leq 5$ is also 80.

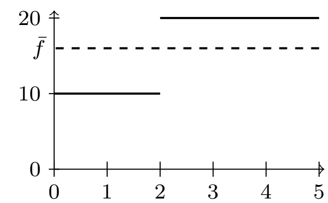
3. The average value of a function over an interval $[a, b]$ is a constant value $\bar{f}$ so that the areas under both f and $\bar{f}$ are equal, which means

$$\int_a^b \bar{f}dx = \int_a^b f dx.$$

Explain why the formula above can be rewritten in the form

$$\bar{f} = \frac{\displaystyle\int_a^b f dx}{\displaystyle\int_a^b dx} \quad \text{or} \quad \bar{f} = \int_a^b f \frac{dx}{\int_a^b dx}.$$

4. The formulas for $\bar{f}$ in the previous part resemble at least one of the ways of calculating averages from Problem 5.2. Which ones and why?



The solid line shows the graph of f while the dashed line shows the average price $\bar{f}$.

The quantity $\bar{f}$ we call the average value of f over $[a, b]$. It was crucial to proving the Fundamental Theorem of Calculus from first-semester calculus.

Let's return to the rover. If we know the mass and center-of-mass of each part of the rover, we can use weighted averages to combine these values and obtain the center-of-mass of the the entire rover.

Problem 5.4 Consider a simplified rover with a bottom and a top.

- The bottom part of the rover has a volume of V_1 m^3, a constant density (mass per volume) of δ_1 g/m^3, and a center-of-mass located at $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$.

- The top part of the rover has a volume of V_2 m^3, a constant density (mass per volume) of δ_2 g/m^3, and a center-of-mass located at $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$.

Complete the following:

1. Give the masses m_1 and m_2 of the bottom and top of the rover. Explain.

2. What proportion of the total mass comes from the bottom of the rover?

3. Explain why the center-of-mass of the entire rover is

$$(\bar{x}_1, \bar{y}_1, \bar{z}_1)\frac{m_1}{m_1 + m_2} + (\bar{x}_2, \bar{y}_2, \bar{z}_2)\frac{m_2}{m_1 + m_2} = \left(\frac{\bar{x}_1(m_1) + \bar{x}_2(m_2)}{m_1 + m_2}, \frac{\bar{y}_1(m_1) + \bar{y}_2(m_2)}{m_1 + m_2}, \frac{\bar{z}_1(m_1) + \bar{z}_2(m_2)}{m_1 + m_2}\right).$$

4. The rover picks up an additional object. The object's mass is m_3 with center-of-mass $(\bar{x}_3, \bar{y}_3, \bar{z}_3)$. Modify the formula above to give the center-of-mass of the rover, together with the new object. Try writing the formula using summation notation.

Of course, a rover consists of many parts (not just a top and bottom). Each little part has a little mass dm and a center-of-mass. We can predict these quantities prior to building the rover, before we can weigh anything. We just need the length ds, area dA, or volume dV of a small part, together with the material's density δ (mass per length, area, or volume, as appropriate).

- For thin wires, we get little masses dm by multiplying little lengths ds by a density δ with units of mass per length.
- For thin plates, we get little masses dm by multiplying little areas dA by a density δ with units of mass per area.
- For solid objects, we get little masses dm by multiplying little volumes dV by a density δ with units of mass per volume.

In all three cases, we can obtain the total mass m by adding up the little masses with an integral. The difference between the three cases will be whether we use a single, double, or triple integral. Often the density δ will be constant throughout an entire object. However, composite materials exist where density $\delta(x, y, z)$ can vary throughout an object. We can then compute the center-of-mass using the average value formulas from above. Let's look at some examples.

Problem 5.5 Consider a thin rod (like a drive shaft or thinner) that lies along the z-axis for $a \leq z \leq b$. The rod is made out of a single material whose density is given by the constant δ g/m (mass per length).

1. A small part of the rod has length dz. Compute $\int_a^b dz$, and explain what physical quantity this integral computes.
2. A small bit of the rod has mass $dm = \delta dz$. Compute the total mass by computing $\int_a^b \delta dz$. Remember that δ is a constant.
3. Guess the location of the average z-value of the rod (the center-of-mass).
4. Compute and simplify the integral formula below, to validate your guess.

$$\bar{z} = \frac{\displaystyle\int_a^b z dm}{\displaystyle\int_a^b dm} = \frac{\displaystyle\int_a^b z\delta dz}{\displaystyle\int_a^b \delta dz}.$$

Then explain, if you can, why these integrals give $\bar{z}$.

Problem 5.6 Suppose again we have a thin rod lying on the z-axis for $a \leq z \leq b$. However, this time the rod is more like an antenna and the rod gets thinner as we move up the rod. This means the density $\delta(z)$ is now a function of z. Let's use, for simplicity, the linear density function $\delta(z) = b - z$.

1. What is the density of the rod at a? What is the density of the rod at b? Construct a rough sketch of a rod that could have this type of density function.
2. A small bit of the rod has mass $dm = \delta(z)dz$. Compute the total mass by computing $\int_a^b \delta(z)dz = \int_a^b (b - z)dz$.

3. Is the location of $\bar{z}$ closer to $z = a$ or $z = b$? Explain.

4. We know that the z-coordinate of the center-of-mass is given by

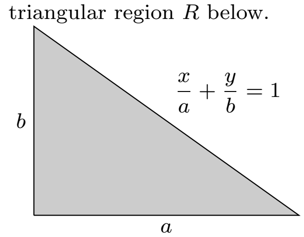
$$\bar{z} = \frac{\displaystyle\int_a^b z dm}{\displaystyle\int_a^b dm} = \frac{\displaystyle\int_a^b z\delta dz}{\displaystyle\int_a^b \delta dz}.$$

   Compute the integrals above by hand (show this). Then simplify your answer (feel free to use software) and verify that $\bar{z} = \frac{2a+b}{3}$.

The previous two problems showed the computations for a rod that lies on an axis. This works great for a drive shaft, or antenna, or any part of the rover that consists of a straight thin rod. The next problem repeats these computations for a portion of the rover that is a thin flat plate, such as a solar panel, an armored plate, or any object which is best described by thinking of the area, multiplied by some tiny thickness.

Problem 5.7 Consider the triangular region R in the first quadrant that lies under the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$, shown to the right. If you would rather work with numbers instead of variables, feel free to let $a = 5$ and $b = 7$ in this problem.

A metal plate occupies the triangular region R below.



1. Compute the double integral $\displaystyle\int_0^a \int_0^{b(1-\frac{x}{a})} dydx$. What physical quantity of the region R does this integral give?

2. The density of the metal plate is δ g/m^2. Set up a double integral formula to compute the mass of the region using this density.

3. The center-of-mass in the x-direction is given by the formula

$$\bar{x} = \frac{\displaystyle\iint_R x dm}{\displaystyle\iint_R dm} = \frac{\displaystyle\int_0^a \int_0^{b(1-\frac{x}{a})} x\delta dydx}{\displaystyle\int_0^a \int_0^{b(1-\frac{x}{a})} \delta dydx}.$$

   Assuming δ is constant, compute this integral and show that $\bar{x} = \frac{a}{3}$.

   Feel free to use software.

4. Set up an integral formula, like the one above, to compute $\bar{y}$. Show the integral formula you used, and then state the value $\bar{y}$ obtained.

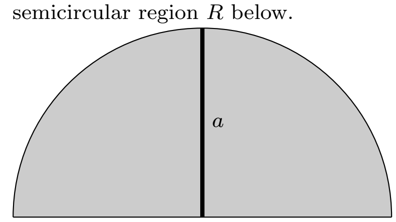
Recall that we have already shown for a region R described in polar coordinates by $0 \leq \theta \leq \alpha$ and $0 \leq r \leq r_1(\theta)$, that we can compute the area of this region using the formula

$$\iint_R dA = \int_0^\alpha \int_0^{r_1(\theta)} |r| dr d\theta.$$

The quantity $|r|$ is the stretch factor that tells us how much little areas $drd\theta$ in the polar plane need to multiplied by to obtain areas in the xy-plane (so $dA = |r|drd\theta$). This stretch factor we call the "Jacobian."

Problem 5.8 Consider the semicircular disc R that lies above the x-axis and below the circle of radius a, shown to the right. If you would rather work with numbers instead of variables, feel free to let $a = 5$ for this problem.

A metal plate occupies the semicircular region R below.



1. We know the area of R is $\frac{1}{2}\pi a^2$. Set up a double integral using polar coordinates to compute this area. Then compute the integral by hand and simplify your work to obtain the correct area.

2. Let's assume the density for this problem is $\delta = 1$, so that $dm = dA$. When the density is constant, we use the word "centroid" instead of "center-of-mass" to talk about the geometric center of the object. The centroid in the x-direction is given by the formula

$$\bar{x} = \frac{\iint_R x dA}{\iint_R dA} = \frac{\int_0^\pi \int_0^a \overbrace{(r\cos\theta)}^{x}\overbrace{rdrd\theta}^{dA}}{\int_0^\pi \int_0^a \underbrace{rdrd\theta}_{dA}}.$$

Compute the integrals above, by hand, to show that $\bar{x} = 0$.

If you struggled on part 1, the formula for the area appears as part of the formula to the left.

3. Set up an integral formula, like the one above, to compute $\bar{y}$. Show the integral formula you used, and then compute it to obtain $\bar{y}$.

Feel free to use software. You can find the correct answer on the back end cover of most engineering statics textbooks, or by searching the web for the centroid of a semicircular disc.

Rods and wires along with thin metal plates are easily tackled using single and double integrals. Other parts of the rover we'll tackle using triple integrals.

Problem 5.9 The triple integral $\int_0^5 \int_0^7 \int_0^{10-2x} dzdydx$ gives the volume of a solid domain D in space.

1. Draw the solid domain D described by the bounds of the integral above. This is the solid satisfying the inequalities $0 \le x \le 5$, $0 \le y \le 7$, and $0 \le z \le 10 - 2x$.

2. Let $\delta = 1$ so that $dm = \delta dV = 1dV$. The centroid of D has three coordinates $(\bar{x}, \bar{y}, \bar{z})$. The x-coordinate is given by the integral formula

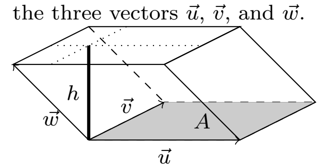
$$\bar{x} = \frac{\iiint_R x dV}{\iiint_R dV} = \frac{\int_0^5 \int_0^7 \int_0^{10-2x} (x) dzdydx}{\int_0^5 \int_0^7 \int_0^{10-2x} 1dzdydx}.$$

Compute this triple integral and simplify to show that $\bar{x} = \frac{5}{3}$.

3. Modify the above formula to obtain integral formulas for both $\bar{y}$ and $\bar{z}$. Then state the values of $\bar{y}$ and $\bar{z}$, either by using facts we've already proven or by computing the integrals directly. Use software to check.

We'll be working with triple integrals quite a bit in this chapter. When we try to change coordinates in 3 dimensions, we will need to be able to compute the volume of a parallelepiped (a 3D parallelogram) if we have the three vectors that define the edges. To find this volume, we need to tackle a couple of problems. First, we will need to compute the area of one face of the parallelepiped. Second, we'll need to compute the distance between that face and the opposing side of the parallelepiped, which is easy to do by projecting any vector connecting the two edges onto a vector that is normal to the face. It turns out that both problems, are addressed with the same process. The second problem, finding a vector orthogonal to two given vectors, is the simpler of the two.

Here is a parallelepiped formed by the three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.



The volume is found by obtaining the area A of one face, multiplied by the distance h between the base and the plane containing the opposing face.

Problem 5.10 Let $\vec{u} = (a, b, c)$ and $\vec{v} = (d, e, f)$. Our goal is to find a single nonzero vector (x, y, z) that is orthogonal to both $\vec{u}$ and $\vec{v}$, preferably with as few fractions as possible in the final answer.

1. Explain why we need to solve the system of equations
$$ax + by + cz = 0 \quad \text{and} \quad dx + ey + fz = 0.$$
2. To solve the system above, multiply the first equation by d and the second equation by $-a$ (assume for a moment that both a and d are not zero). Then add the two equation together to eliminate x. Solve for y in terms of z, and then x in terms of z, to show that every solution to this system can be written in the form
$$(x, y, z) = \left(\left(\frac{bf - ce}{ae - bd} \right) z, \left(\frac{cd - af}{ae - bd} \right) z, z \right).$$
3. The above solution has some complicated fractions. Why is $(x, y, z) = (bf - ce, cd - af, ae - bd)$ a solution to the system?

Definition 5.1: Cross Product. The cross product of the two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ is the vector
$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Let's tackle the problem of finding the area of a parallelogram in 3D. In 2D, recall that if we have the two vector $(u_1, u_2, 0)$ and $(v_1, v_2, 0)$, then the area of the parallelogram formed using these two vectors is $|u_1v_2 - u_2v_1|$. Take a second and look at the cross product formula above. Do you notice any similarities?

Problem 5.11 Let $\vec{u} = (a, b, c)$ and $\vec{v} = (d, e, f)$. Our goal is to find the area of a parallelogram whose edges are formed from these two vectors.

1. Draw a picture that contains 2 vectors labeled $\vec{u}$ and $\vec{v}$. In that picture, include $\vec{u}_{\|\vec{v}}$ and $\vec{u}_{\perp\vec{v}}$. Explain why the area we seek is $A = |\vec{u}_{\perp\vec{v}}||\vec{v}|$.
2. Compute the projection $\vec{u}_{\|\vec{v}}$ and then $\vec{u}_{\perp\vec{v}}$.
3. Computing the magnitude of $\vec{u}_{\perp\vec{v}}$ can be quite tedious by hand. Using software, we can quickly get
$$A = |\vec{u}_{\perp\vec{v}}||\vec{v}| = \sqrt{a^2\left(e^2 + f^2\right) - 2be(ad + cf) - 2acdf + b^2\left(d^2 + f^2\right) + c^2\left(d^2 + e^2\right)}.$$
The magnitude of the cross product is
$$|\vec{u}\times\vec{v}| = |(bf-ce, cd-af, ae-bd)| = \sqrt{(bf - ce)^2 + (cd - af)^2 + (ae - bd)^2}.$$
Show that the two quantities above are equal (so show $A = |\vec{u} \times \vec{v}|$).
4. In summary, to find the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$, compute the magnitude of ______________________.

Theorem 5.2. *The cross product $\vec{u} \times \vec{v}$ of $\vec{u}$ and $\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. The magnitude $|\vec{u} \times \vec{v}|$ is the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.*

Problem 5.12 Prove that for two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, we have

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta,$$

where θ is the angle between the two vectors.

Draw two vectors and the parallelogram they form. Add a right triangle where one of the angles is given by θ. Basic trigonometry, along with the fact that $|\vec{u} \times \vec{v}|$ gives the area of the parallelogram, will complete the proof.

Observation 5.3. Given two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, we have

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta \quad \text{and} \quad |\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta.$$

We'll return to more problems with the cross product later in this chapter. The next problem interchanges the order of the bounds of a double integral.

Problem 5.13 Consider the region R in the xy-plane that is below the line $y = x + 2$, above the line $y = 2$, and left of the line $x = 5$. We can describe this region by saying for each x with $0 \leq x \leq 5$, we want y to satisfy $2 \leq y \leq x + 2$. In set builder notation, we write

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 5, 2 \leq y \leq x + 2\}.$$

We use the symbols { and } to enclose sets and the symbol | for "such that". We read the above line as "R equals the set of (x, y) in the plane such that zero is less than x which is less than 5, and 2 is less than y which is less than $x + 2$." The iterated double integral $\int_0^5 \int_2^{x+2} dydx$ gives the area of this region.

1. Draw this region.
2. Describe the region R by saying for each y with $c \leq y \leq d$, we want x to satisfy $a(y) \leq x \leq b(y)$. In other words, find constants c and d, and functions $a(y)$ and $b(y)$, so that for each y between c and d, the x values must be between the functions $a(y)$ and $b(y)$. Write your answer using the set builder notation
$$R = \{(x, y) \mid c \leq y \leq d, a(y) \leq x \leq b(y)\}.$$
3. Finish setting up the iterated double integral $\int_?^? \int_?^? dxdy$.

[Hint: Draw the 4 curves given by $0 = x$, $x = 5$, $2 = y$ and $y = x + 2$. Then appropriately shade above, below, left, or right of each curve.]

Definition 5.4: Double and Iterated Integrals. Given a region R, we write $\iint_R f(x, y)dA$ for the **double integral** of f over R. We just have to state what the region R is to talk about a double integral. The formal definition of a double integrals involves slicing the region R up into tiny rectangles of area $dxdy$, multiplying each rectangle by a function f, and then summing over all rectangles. This process is repeated as the length and width of the rectangles shrinks to zero at similar rates, with the double integral being the limit of this process.

An **iterated integral** is an integral where we have actually specified the order of integration and given bounds for each integral. For double integrals there are two options, namely

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y)dydx \quad \text{and} \quad \int_c^d \int_{a(y)}^{b(y)} f(x, y)dxdxy.$$

We'll focus mostly on setting up iterated integrals.

To find the mass of a thin metal plate occupying a region R in the xy-plane, we add up the differentials $dm = \delta(x,y)dA$, where δ is the density (mass per area), to obtain the mass as

$$m = \iint_R \delta(x,y)dA = \iint_R \delta(x,y)dxdy = \iint_R \delta(x,y)dxdy.$$

Note that if $\delta(x,y) = 1$, then this reduces to the formula for the area of R. The next problem has you practice setting up several mass integrals.

Problem 5.14 For each region R below, draw the region. Then use the given density to set up an iterated double integral which would give the mass. You do not need to fully compute each integral, rather just set it up.

1. The region R is above the line $x + y = 1$ and inside the circle $x^2 + y^2 = 1$. The density is $\delta(x,y) = x$.
2. The region R is below the line $y = 8$, above the curve $y = x^2$, and to the right of the y-axis. The density is $\delta(x,y) = xy^2$.
3. The region R is bounded by $2x + y = 3$, $y = x$, and $x = 0$. The density is $\delta(x,y) = 7$.

Problem 5.15 Consider the iterated integral $\displaystyle\int_0^3 \int_x^3 e^{y^2} dydx$.

1. Write the bounds as two inequalities ($0 \leq x \leq 3$ and $? \leq y \leq ?$). Then draw and shade the region R described by these two inequalities.
2. Swap the order of integration from $dydx$ to $dxdy$. This forces you to describe the region using two inequalities of the form $c \leq y \leq d$ and $a(y) \leq x \leq b(y)$.
3. Use your new bounds to compute the integral by hand.
4. Why is the original integral $\displaystyle\int_0^3 \int_x^3 e^{y^2} dydx$ impossible to compute without first swapping the order of integration? [Hint: Try computing the inner integral $\int_x^3 e^{y^2} dy$ – why can't you?]

Problem 5.16 Compute by hand the iterated integral

First swap the order of the bounds. Then integrate

$$\int_0^{2\sqrt{\pi}} \int_{y/2}^{\sqrt{\pi}} \sin(x^2)dxdy.$$

In the previous problem we used Cartesian coordinates to compute the integral. The next problem is impossible to complete using Cartesian coordinates, though becomes completely doable if we swap to polar coordinates. Recall that earlier in the semester we showed that for a region R_{xy} in the xy-plane, we showed that little areas $drd\theta$ in the $r\theta$ plane were transformed to little regions in the xy-plane with area

$$dA = |r|drd\theta.$$

The stretch factor, or Jacobian, of this transformation is $|r|$. Provided we never let r become negative, this means

$$\iint_{R_{xy}} f(x,y)dxdy = \iint_{R_{r\theta}} f(r\cos\theta, r\sin\theta)r\ drd\theta.$$

To swap from Cartesian to polar coordinates, we just replace x with $r\cos\theta$, replace y with $r\sin\theta$, replace $dxdy$ or $dydx$ with $rdrd\theta$, and then use bounds for r and θ to set up an iterated integral. The next problem has you do this.

Problem 5.17 The double integral $\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} e^{x^2+y^2}dxdy$ computes the mass of a region in the plane with density $\delta = e^{x^2+y^2}$ that is bounded by the curves $y=0$, $y=\sqrt{2}$, $x=y$, and $x=\sqrt{4-y^2}$.

1. Draw the region described by these bounds. (Did you get a sector of a circle, something like a 1/8th of a pizza?)
2. Give bounds of the form $? \le \theta \le ?$ and $? \le r \le ?$ that describe the region using polar coordinates. (The new bounds are all constants.)
3. Convert the Cartesian integral to an integral in polar coordinates (don't forget the Jacobian).
4. Compute the integral by hand. Show your steps.

Problem 5.18 For each region R below, draw the region in the xy-plane. Then use the given density to set up an iterated double integral in polar coordinates which would give the mass. You do not need to fully compute each integral, rather just set it up. For example, if the region is the inside of the circle $x^2+y^2=9$, and the density is $\delta(x,y)=y$, then the mass is

$$m = \iint_R \delta dA = \int_0^{2\pi}\int_0^3 \underbrace{(r\sin\theta)}_{\delta=y}\underbrace{rdrd\theta}_{dA}.$$

1. The region R is the quarter disc in the first quadrant that lies inside the circle $x^2+y^2=25$. The density is $\delta(x,y)=x$.
2. The region R is bounded above by $y=\sqrt{9-x^2}$, bounded below by $y=x$, and bounded on the left by the y-axis. The density is $\delta(x,y)=xy^2$.
3. The region R is the inside of the cardioid $r=3+3\cos\theta$. The density is $\delta(x,y)=7$.

Just as we've used double integrals to compute the area and mass for regions in the plane, we can use triple integrals to compute volume and mass for solids in space. A triple integral is an integral of the form $\iiint_D dV$, where dV represents a small portion of volume of the solid region D. However, now there are six different possible orders of integration when we want to create iterated integrals. For example if we pick the order $dzdydx$, then to set up the integral we'll need $a \le x \le b$, $c(x) \le y \le d(x)$, and $e(x,y) \le z \le f(x,y)$. Note that the outermost bounds must be always be constant, whereas the innermost bounds can depend on all of the other variables.

Problem 5.19 Do not evaluate the three integrals below. Our focus is setting up the bounds for triple integrals.

1. The iterated triple integral $\int_{-1}^{1}\int_{0}^{4}\int_{0}^{y^2} dzdxdy$ gives the volume of the solid D that lies under the surface $z = y^2$, above the xy-plane, and bounded by the planes $y = -1$, $y = 1$, $x = 0$, and $x = 4$. Sketch this region.

2. Set up an iterated triple integral that gives the volume of the solid in the first octant that is bounded by the coordinate planes ($x = 0$, $y = 0$, $z = 0$), the plane $y + z = 2$, and the surface $x = 4 - y^2$, using the order of integration $dxdzdy$. Make sure you sketch the region.

3. Set up an integral to give the volume of the pyramid in the first octant that is below the planes $\frac{x}{3} + \frac{z}{2} = 1$ and $\frac{y}{5} + \frac{z}{2} = 1$. [Hint, don't let z be the inside bound. Try an order such as $dydxdz$.]

Problem 5.20 Consider the triangular wedge D that is in the first octant, bounded by the planes $\frac{y}{7} + \frac{z}{5} = 1$ and $x = 12$. In the yz-plane, the wedge forms a triangle that passes through the points $(0, 0, 0)$, $(0, 7, 0)$, and $(0, 0, 5)$.

1. Draw the solid.

2. Assume the density δ of the solid is constant. Recall that $\bar{x}$, the x-coordinate of the centroid, is given by the integral formula

$$\bar{x} = \frac{\iiint_D x\delta dV}{\iiint_D \delta dV}.$$

   Set up the corresponding integral formulas for $\bar{y}$ and $\bar{z}$ (if your answers look almost identical to the above, you are doing this correctly).

3. Actually compute the integrals for $\bar{y}$. Show your integration steps.

4. State $\bar{x}$ and $\bar{z}$ using symmetry or other arguments.

We've now found the mass and center-of-mass for straight wires, thin flat metal plates, and solid regions in space. Earlier in the semester we used

$$s = \int_C ds = \int_C \left|\frac{d\vec{r}}{dt}\right| dt$$

to obtain the length of a thin wire lying on the curve C with parametrization $\vec{r}(t)$. For such a wire, we use the differential

$$\underbrace{ds}_{\text{little distance}} = \underbrace{\left|\frac{d\vec{r}}{dt}\right|}_{\text{speed}} \underbrace{dt}_{\text{little time}}$$

instead of dx (little length in a straight rod), dA (little area in a thin metal plate), or dV (little volume in a solid). The differential ds can replace dx, dA, or dV in any of our previous formulas to help us determine, for a curved wire, the length, mass, center-of-mass, and more. The next problem has you set up several integrals that do this.

Problem 5.21 Consider a wire that lies along the curve C with parametrization $\vec{r}(t) = (5\cos t, 5\sin t)$ for $0 \leq t \leq \pi$.

1. Draw the curve, compute $\frac{d\vec{r}}{dt}$, and show that $ds = 5dt$.
2. Evaluate $\int_C ds$ to obtain the length of the wire.
3. Assuming the density is constant, why do we know $\bar{x} = 0$?
4. Set up an integral formula for $\bar{y}$ and compute the integrals involved to obtain $\bar{y}$, showing your integration steps.
5. If instead, the density is $\delta = xy^2 + 7$, then set up an integral formula to find $\bar{x}$. You don't need to compute the integral, rather just set it up.

Since the curve is half a circle, the length you obtain from integration should be half the circumference of the circle.

Problem 5.22 A sphere of radius a centered at the origin is described by the equation $x^2 + y^2 + z^2 = a^2$. A right circular cone whose tip is at the origin is given by $z^2 = x^2 + y^2$. You'll be setting up integrals on this problem. Don't worry about computing any integrals, rather focus on setting up the integrals.

1. Draw the surface $x^2 + y^2 + z^2 = a^2$ and then set up an iterated triple integral using Cartesian coordinates to compute the volume inside the sphere $x^2 + y^2 + z^2 = a^2$.
2. Draw the surface $z^2 = x^2 + y^2$ and then set up an iterated triple integral using Cartesian coordinates to compute the volume of the solid cone that lies above $z^2 = x^2 + y^2$ and below $z = h$.

Both of the integrals above are quite messy to actually compute using Cartesian coordinates. What we need are two new coordinate systems, called cylindrical and spherical coordinates. We'll introduce these coordinates, and then find the appropriate stretch factor (Jacobian) that will let us replace the differential $dV = dxdydz$ with an appropriate differential in the new coordinate system. Then we'll return to the integrals above and compute them using these new coordinate systems. First, let's revisit what we did with polar coordinates. This will remind us of the key things we will need to tackle a three dimensional change of coordinates.

Problem 5.23 Consider the polar change-of-coordinates $x = r\cos\theta$ and $y = r\sin\theta$. We can write this in vector form as

$$\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta).$$

1. Compute the differential $d\vec{T}$. Write it as a linear combination of the partial derivatives of $\vec{T}$, as well as a matrix product.
2. In your differential above, you should have a linear combination of two vectors. Find the area of the parallelogram formed by these two vectors. We call this the Jacobian of the polar transformation, written $\frac{\partial(x,y)}{\partial(r,\theta)}$.
3. Explain what the differential equation $dA = |r|drd\theta$ means.
4. The notation $\frac{\partial(x,y)}{\partial(r,\theta)}$ was invented to help remember where to insert the Jacobian in an integral formula. Consider the two formulas

$$\iint_{R_{xy}} f dxdy = \iint_{R_{r\theta}} f\frac{\partial(x,y)}{\partial(r,\theta)}drd\theta \quad \text{or} \quad \iint_{R_{xy}} f\frac{\partial(x,y)}{\partial(r,\theta)}dxdy = \iint_{R_{r\theta}} f drd\theta.$$

Which formula above is correct, and how does the notation help you remember this?

Recall that the for $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, we have

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Also recall that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$, and the magnitude of $\vec{u} \times \vec{v}$ is equal to the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.

Problem 5.24 Let $\vec{u} = (1, -2, 3)$ and $\vec{v} = (2, 0, -1)$. The paragraph before this problem reminds you of some key facts about the cross product.

See 12.4: 1-8.

1. Compute $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$. How are they related?
2. Compute and simplify both $\vec{u} \cdot (\vec{u} \times \vec{v})$ and $\vec{v} \cdot (\vec{u} \times \vec{v})$. Did you get zero for both? What fact about the cross product guarantees you get zero?
3. Compute the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.
4. Explain (without doing any computations if you can) why $\vec{u} \times (2\vec{u})$ must equal $(0, 0, 0)$.

Problem 5.25 Let $P = (a, b, c)$ be a point on a plane in 3D. Let $\vec{n} = (A, B, C)$ be a normal vector to the plane (so the angle between the plane and $\vec{n}$ is 90°). Let $Q = (x, y, z)$ be another point on the plane.

See 12.5: 21-28.

1. What is the angle between $\vec{PQ} = (x - a, y - b, z - c)$ and $\vec{n} = (A, B, C)$?
2. Explain why an equation of the plane through P with normal vector $\vec{n}$ is
$$A(x - a) + B(y - b) + C(z - c) = 0.$$
3. Consider the three points $R = (1, 0, 0)$, $S = (2, 0, -1)$, and $T = (0, 1, 3)$. Give an equation of the plane which passes through these three points.

Hint: Use an appropriate cross product to get a normal vector.

Problem 5.26 Let $P = (2, 0, 0)$, $Q = (0, 3, 0)$, and $R = (0, 0, 4)$.

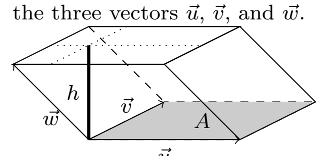
See 12.4: 15-18. Remember, the magnitude of the cross product gives the area of the parallelogram formed using the two vectors as the edges.

1. Find a vector that is orthogonal to both $\vec{PQ}$ and $\vec{PR}$.
2. Find the area of the triangle PQR. Construct a 3D graph of this triangle.
3. Give an equation of the plane through $P = (2, 0, 0)$, $Q = (0, 3, 0)$, and $R = (0, 0, 4)$.

The Jacobian of a change-of-coordinates in 2D is found by computing the area of the parallelogram formed by the partial derivatives of the transformation. In 3 dimensions, we'll need to find the volume of a parallelepiped formed by 3 partial derivatives, instead of just two. We'll now tackle this problem.

Problem 5.27 Consider three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$. We will find the volume of the parallelepiped formed by these three vectors. Let the base of the parallelogram be the face formed by $\vec{u}$ and $\vec{v}$. The height h is the distance between the planes contain the base to the plane containing the opposing side of the parallelepiped. The cross product will be your friend on this problem. Review the problems related to the cross product before proceeding.

Here is a parallelepiped formed by the three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.



The volume is found by obtaining the area A of one face, multiplied by the distance h between the base and the plane containing the opposing face.

1. Give a formula to compute the area of the base of the parallelogram.
2. Give a vector $\vec{n}$ that is normal to the base.
3. Use the projection formula, with the vectors $\vec{w}$ and $\vec{n}$ in an appropriate manner, to state the height of the parallelogram in terms of dot products.
4. Use your work above to explain why the parallelepiped's volume is
$$|(\vec{u} \times \vec{v}) \cdot \vec{w}|.$$

We call $(\vec{u} \times \vec{v}) \cdot \vec{w}$ the triple product of $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Problem 5.28 Suppose $\vec{u} = (a, b, c)$, $\vec{v} = (d, e, f)$, and $\vec{w} = (g, h, i)$.

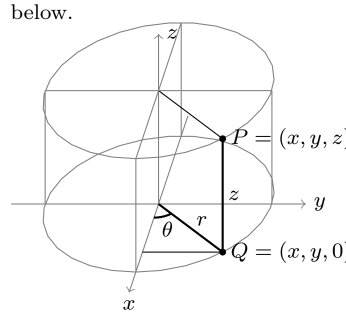
1. Compute and simplify $|(\vec{u} \times \vec{v}) \cdot \vec{w}|$ for these three vectors to obtain a formula for the volume of the parallelepiped formed by these three vectors.
2. Compute and simplify $|(\vec{v} \times \vec{w}) \cdot \vec{u}|$. What do you notice?

In the new coordinates chapter, we focused quite a bit on how to work with a two-dimensional change-of-coordinates. In particular, we've already seen examples of coordinate transformations with polar coordinates. In three dimensions, some common coordinate systems are cylindrical and spherical coordinates. The equations for these coordinate systems are shown below.

Cylindrical Coordinates	Spherical Coordinates
$x = r\cos\theta$ $y = r\sin\theta$ $z = z$	$x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$

Problem 5.29 Let $P = (x, y, z)$ be a point in space. This point lies on a cylinder of radius r, where the cylinder has the z axis as its axis of symmetry. The height of the point is z units up from the xy plane. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the ray $\vec{Q}$ and the x-axis is θ. See the figure to the right. Use the graph and the information above to explain why the equations for cylindrical coordinates are
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

A point $P = (x, y, z)$ lies on a cylinder of radius r whose axis of rotation is the z-axis, shown below.



Now that we have a new coordinate system, let's compute the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$, namely the stretch factor in the equation
$$dV = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} dz dr d\theta.$$
As one last reminder, here is the process for polar coordinates. For polar coordinates we have $x = r\cos\theta$ and $y = r\sin\theta$. The differential $d(x, y)$, when written as a linear combination of partial derivatives, becomes
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} dr + \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix} d\theta.$$
The area of the parallelogram formed by the partial derivatives is
$$|(\cos\theta)(r\cos\theta) - (\sin\theta)(-r\sin\theta)| = |r(\cos^2\theta + \sin^2\theta)| = |r|.$$
This gives the Jacobian as $\dfrac{\partial(x, y)}{\partial(r, \theta)} = |r|$. The only difference for cylindrical coordinates is that now we we'll use the triple product $(\vec{u} \times \vec{v}) \cdot \vec{w}$ to find the volume of the parallelepiped formed by the three partial derivatives.

Problem 5.30 Compute the Jacobian for cylindrical coordinates. The steps below are a guide, if needed.

1. For cylindrical coordinates we have $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. Write the differential $d(x, y, z)$ as the linear combination of partial derivatives
$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} dr + \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} d\theta + \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} dz.$$

2. Compute the volume of the parallelepiped formed by the three vectors (partial derivatives) above. Use the triple product $(\vec{u} \times \vec{v}) \cdot \vec{w}$.

3. Simplify your result and state $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$.

Problem 5.31 Consider the solid domain D in space that lies inside the right circular cylinder $x^2 + y^2 = a^2$ for $0 \leq z \leq h$.

1. Draw the domain D.

2. Set up an integral in Cartesian coordinates that gives the volume of D. (If some of your bounds involve something like $\sqrt{a^2 - x^2}$ or $\sqrt{a^2 - y^2}$, then you're doing this correctly - the bounds are quite gross.)

3. Set up an integral in cylindrical coordinates that gives the volume of D. (Your bounds should all be constants - this is one indication that we picked the correct coordinate system.)

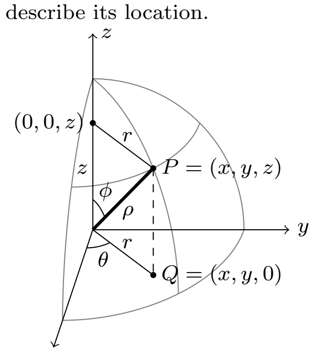
4. Compute the integral in cylindrical coordinates and simplify your work till you obtain $V = \pi a^2 h$, the volume inside a cylinder of height h and radius a.

Problem 5.32 Consider the solid domain D in space which is above the cone $z = \sqrt{x^2 + y^2}$ and below the paraboloid $z = 6 - x^2 - y^2$. Sketch the region by hand, and then use cylindrical coordinates to set up an iterated triple integral that would give the volume of the region. You'll need to find where the surfaces intersect, as their intersection will help you determine the appropriate bounds.

See Sage for a picture of the region.

Problem 5.33 Let $P = (x, y, z)$ be a point in space. This point lies on a sphere of radius ρ ("rho"), where the sphere's center is at the origin $O = (0, 0, 0)$. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the ray $\vec{Q}$ and the x-axis is θ, which some call the azimuth angle. The angle between the ray $\vec{P}$ and the z-axis is ϕ ("phi"), which some call the inclination angle, polar angle, or zenith angle. See the figure to the right. Use this information to explain why the equations for spherical coordinates are
$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.$$

A point $P = (x, y, z)$ lies on a sphere of radius ρ, where the two angles ϕ and θ (think latitude and longitude) are sufficient to describe its location.



There is some disagreement between different scientific fields about the notation for spherical coordinates. In some fields (like physics), ϕ represents the azimuth angle and θ represents the inclination angle, swapped from what

See Wikipedia or MathWorld for a discussion of conventions in different disciplines.

we see here. In some fields, like geography, instead of the inclination angle, the *elevation* angle is given — the angle from the xy-plane (lines of latitude are from the elevation angle). Additionally, sometimes the coordinates are written in a different order. You should always check the notation for spherical coordinates before communicating to others with them. As long as you have an agreed upon convention, it doesn't really matter how you denote them.

Problem 5.34 Let's compute the Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}$ for spherical coordinates.

1. For spherical coordinates we have

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

Write $d(x, y, z)$ as a linear combination of partial derivatives, so

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} d\rho + \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} d\phi + \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} d\theta.$$

2. Compute the volume of the parallelepiped formed by the three vectors (partial derivatives) above. Simplify your result to $\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = |\rho^2 \sin\phi|$.

Feel free to use software on this problem. You can do it all by hand, but you'll to use a Pythagorean identity several times to complete the simplification.

3. We'd like to remove the absolute values above and instead write

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin\phi.$$

Give bounds for ϕ that allow us to remove the absolute values.

Problem 5.35 Consider the solid domain D that lies inside the sphere $x^2 + y^2 + z^2 = a^2$. We call this region a ball of radius a (a sphere is a surface, whereas a ball is the solid region inside a sphere). We have already shown that the volume of this region, using Cartesian coordinates, is given by

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dzdydx.$$

1. Set up an integral, using cylindrical coordinates, to find the volume of D.

Don't forget the Jacobian r.

2. Set up an integral, using spherical coordinates, to find the volume of D. This will require you to give bounds for ρ, ϕ, and θ. (Did you forget the spherical Jacobian $\rho^2 \sin\phi$? If all your bounds are constants, then you know you've done this correctly and that spherical coordinates is the correct coordinate system to use.)
3. Compute the integral in spherical coordinates and simplify your work till you obtain $V = \frac{4}{3}\pi a^3$, the volume of a ball of radius a.

Problem 5.36 Consider the solid domain D in space that lies below the cone $z = \sqrt{x^2 + y^2}$, above the xy-plane, and inside the sphere $x^2 + y^2 + z^2 = 25$.

1. Provide a sketch of the domain D.
2. Set up an integral in cylindrical coordinates that gives the volume of D.

3. Set up an integral in spherical coordinates that gives the volume of D.
4. Set up an integral in the coordinate system of your choice that would give the z coordinate of the centroid of D.

Problem 5.37 Consider the solid domain D in space that lies inside a right circular cone whose height is h and the radius of the base is a. Set up and then compute (using software) appropriate integrals (pick a relevant coordinate system) to give both the volume V and centroid $(\bar{x}, \bar{y}, \bar{z})$ of the region. Place the origin at whatever point you deem most appropriate, and provide a picture.

We've been working with rods, wires, thin plates, and solid domains. For example, we could work with a circular wire, or a circular disc, or a ball. How do the centroid formulas change in each setting? The following problem has you examine these three setting, set up the corresponding integrals, use software to solve them, and then compare the locations of the centroids.

Problem 5.38 Consider the curve C that is the upper half of the circle $x^2+y^2 = 49$, the region R that lies above $y = 0$ and inside the circle $x^2+y^2 = 49$, and the solid domain D that lies inside the sphere $x^2+y^2+z^2 = 49$ and satisfies $y \geq 0$. Because of symmetry, for each region it is clear that $\bar{x} = \bar{z} = 0$.

1. Set up an integral formula to compute $\bar{y}$ for the curve C.
2. Set up an integral formula to compute $\bar{y}$ for the region R.
3. Set up an integral formula to compute $\bar{y}$ for the domain D.
4. Use software to compute all three integral formulas above, obtaining an exact value for the answer (not a numerical approximation).
5. For each object, state a general formulas for $\bar{y}$ if the radius is a (not 7).

You'll need a parametrization.

Problem 5.39 Let R be the region in the plane with $a \leq x \leq b$ and $g(x) \leq y \leq f(x)$.

1. Set up an iterated integral to compute the area of R. Then compute the inside integral. You should obtain a familiar formula from first-semester calculus.
2. Set up an iterated integral formula to compute $\bar{x}$ for the centroid. By computing the inside integrals, show that
$$\bar{x} = \frac{\int_a^b x(f-g)dx}{\int_a^b (f-g)dx}.$$
3. If the density depends only on x, so $\delta = \delta(x)$, set up an iterated integral formula to compute $\bar{y}$ for the center of mass. Compute the inside integral and show that
$$\bar{y} = \frac{\displaystyle\int_a^b \frac{1}{2}(f^2-g^2)\delta(x)dx}{\displaystyle\int_a^b (f-g)\delta(x)dx} = \frac{\displaystyle\int_a^b \overbrace{\frac{(f+g)}{2}}^{\tilde{y}} \overbrace{\delta(x)\underbrace{(f-g)dx}_{dA}}^{dm}}{\text{mass}}.$$

When we use double integrals to find centroids, the formulas for the centroid are similar for both $\bar{x}$ and $\bar{y}$. In other courses, you may see the formulas on the left, because the ideas are presented without requiring knowledge of double integrals. Integrating the inside integral from the double integral formula gives the single variable formulas.

In class, we'll analyze the integral formula on the left and show how you can set this up as a single integral using geometric reasoning. We'll discuss the quantities $\tilde{y}$, dm, and dA, as appropriate.

5.2 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Exam 2 Review

At the end of each chapter, the following words appeared.

> Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam.

I've summarized the objectives from each chapter below. For our in class review, please come to class with examples to help illustrate each idea below. You'll get a chance to teach another member of class the examples you prepared. If you keep the examples simple, you'll have time to review each key idea.

Optimization

1. For a function of the form $f(x, y)$ or $f(x, y, z)$, construct (by hand and with software) contour plots, surface plots, and gradient field plots.
2. Compute differentials, partial derivatives, and gradients.
3. Compute slopes (directional derivatives), tolerances (differentials), and equations of tangent planes.
4. Obtain and use the chain rule to analyze a function f along a parametrized path $\vec{r}(t)$. In particular, calculate slopes and locate maximums and minimums of f along $\vec{r}$.
5. Use Lagrange multipliers to locate and compute extreme values of a function f subject to a constraint $g = c$.
6. Apply the second derivative test, using eigenvalues, to locate local maximum and local minimum values of a function f over a region R.

Integration

1. Set up and compute single, double, and triple integrals to obtain lengths, areas, and volumes. Connect these to the differentials dx, ds, dA, and dV.
2. Explain how to compute the mass of a wire, planar region, or solid object, if the density is known. Connect this to the differential dm.
3. Find the average value of a function over a region. Use this to compute the center-of-mass and centroid of a wire, planar region, or solid object.
4. Draw regions described by the bounds of an integral, and then use this drawing to swap the order of integration.
5. Obtain the cross product and use it to find a vector orthogonal to two given vectors, the area of a parallelogram, and the volume of a parallelepiped.
6. Appropriately use polar coordinates $dA = |r|drd\theta$, cylindrical coordinates $dV = |r|drd\theta dz$, and spherical coordinates $dV = |\rho^2 \sin\phi|d\rho d\theta d\phi$.

Chapter 6

Boundaries

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Determine whether a vector field has a potential. Use a potential of a vector field to compute the work done by the field along a curve.

2. Use the del operator $\vec{\nabla}$ to compute divergence and curl of a vector field.

3. Use Green's theorem to compute the circulation of a vector field along a closed curve.

4. Draw parametrized surfaces and set up surface integrals to compute their surface area, mass, and center-of-mass.

5. Compute the flux of a vector field across an oriented surface.

6. Verify Stokes's theorem and the Divergence theorem.

You'll have a chance to teach your examples to your peers prior to the exam.

6.1 The Fundamental Theorem of Line Integrals

Many vector fields are the derivative of a function. When this occurs, computing work along a curve is extremely easy. All we have to know are the endpoints of the curve (the boundary of the curve), and the function f whose derivative is the vector field. This function we call a potential for the vector field.

Definition 6.1: Gradients and Potentials. A potential for the vector field $\vec{F}$ is a function f whose gradient equals $\vec{F}$, so $\vec{\nabla} f = \vec{F}$. Watch a YouTube Video.

Problem 6.1 Let's practice finding gradients and potentials. Watch a YouTube Video.

1. Let $f(x, y) = x^2 + 3xy + 2y^2$. Find $\vec{\nabla} f$. Then compute $D^2 f(x, y)$ (you should get a square matrix). What are f_{xy} and f_{yx}?

2. Consider the vector field $\vec{F}(x, y) = (2x + y, x + 4y)$. Find the derivative of $\vec{F}(x, y)$ (it should be a square matrix). Then find a function $f(x, y)$ whose gradient is $\vec{F}$ (i.e. $Df = \vec{F}$). What are f_{xy} and f_{yx}?

3. Consider the vector field $\vec{F}(x, y) = (2x + y, 3x + 4y)$. Find the derivative of $\vec{F}$. Why is there no function $f(x, y)$ so that $Df(x, y) = \vec{F}(x, y)$? [Hint: look at f_{xy} and f_{yx}.] See problem 4.30.

Based on your observations in the previous problem, we have the following key theorem.

Theorem 6.2. *Let $\vec{F}$ be a vector field that is everywhere continuously differentiable. Then $\vec{F}$ has a potential if and only if the derivative $D\vec{F}$ is a symmetric matrix. We say that a matrix is symmetric if interchanging the rows and columns results in the same matrix (so if you replace row 1 with column 1, and row 2 with column 2, etc., then you obtain the same matrix).*

Problem 6.2 For each of the following vector fields, start by computing the derivative. Then find a potential, or explain why none exists.

If you haven't yet, please watch this YouTube video.

1. $\vec{F}(x, y) = (2x - y, 3x + 2y)$
2. $\vec{F}(x, y) = (2x + 4y, 4x + 3y)$
3. $\vec{F}(x, y) = (2x + 4xy, 2x^2 + y)$

Problem 6.3 For each of the following vector fields, start by computing the derivative. Then find a potential, or explain why none exists.

If you haven't yet, please watch this YouTube video.

1. $\vec{F}(x, y, z) = (x + 2y + 3z, 2x + 3y + 4z, 2x + 3y + 4z)$
2. $\vec{F}(x, y, z) = (x + 2y + 3z, 2x + 3y + 4z, 3x + 4y + 5z)$
3. $\vec{F}(x, y, z) = (x + yz, xz + z, xy + y)$

Definition 6.3: Smooth Curve. A smooth curve is a a curve C with a continuously differentiable parameterization $\vec{r}(t)$ that is never the zero vector. This condition requires that changes in direction happen gradually (derivative is continuous), and it's not possible to stop (derivative can't be zero) and then back up. The two conditions together prevent the curve from having any cusps.

If a vector field has a potential, then there is an extremely simple way to compute work. To see this, we must first review the fundamental theorem of calculus. The second half of the fundamental theorem of calculus states,

> If f is continuous on $[a, b]$ and F is an anti-derivative of f, then $F(b) - F(a) = \int_a^b f(x)dx$.

If we replace f with f', then an anti-derivative of f' is f, and we can write,

> If f is continuously differentiable on $[a, b]$, then
> $$f(b) - f(a) = \int_a^b f'(x)dx.$$

This last version is the version we now generalize.

Theorem 6.4 (The Fundamental Theorem of Line Integrals)**.** *Suppose f is a continuously differentiable function, defined along some open region containing the smooth curve C. Let $\vec{r}(t)$ be a parametrization of the curve C for $t \in [a, b]$. Then we have*

Watch a YouTube video.

$$f(\vec{r}(b)) - f(\vec{r}(a)) = \int_a^b Df(\vec{r}(t))D\vec{r}(t)\ dt.$$

Notice that if $\vec{F}$ is a vector field, and has a potential f, which means $\vec{F} = Df$, then we could rephrase this theorem as follows.

> Suppose $\vec{F}$ is a a vector field that is continuous along some open region containing the curve C. Suppose $\vec{F}$ has a potential f. Let A and B be the start and end points of the smooth curve C. Then the work done by $\vec{F}$ along C depends only on the start and end points, and is precisely
>
> $$f(B) - f(A) = \int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy.$$
>
> The work done by $\vec{F}$ is the difference in a potential.

If you are familiar with kinetic energy, then you should notice a key idea here. Work is a transfer of energy. As an object falls, energy is transferred from potential energy to kinetic energy. The total kinetic energy at the end of a fall is precisely equal to the difference between the potential energy at the top of the fall and the potential energy at the bottom of the fall (neglecting air resistance). So work (the transfer of energy) is exactly the difference in potential energy.

Problem 6.4: Proof of Fundamental Theorem Suppose $f(x, y)$ is continuously differentiable, and suppose that $\vec{r}(t)$ for $t \in [a, b]$ is a parametrization of a smooth curve C. Prove that $f(\vec{r}(b)) - f(\vec{r}(a)) = \int_a^b Df(\vec{r}(t))D\vec{r}(t)\, dt$. [Let $g(t) = f(\vec{r}(t))$. Why does $g(b) - g(a) = \int_a^b g'(t)dt$? Use the chain rule (matrix form) to compute $g'(t)$. Then just substitute things back in.]

The proof of the fundamental theorem of line integrals is quite short. All you need is the fundamental theorem of calculus, together with the chain rule.

Problem 6.5 For each vector field and curve below, find the work done by $\vec{F}$ along C (compute the integral $\int_C M dx + N dy$ or $\int_C M dx + N dy + P dz$).

Watch a YouTube video.

1. Let $\vec{F}(x, y) = (2x + y, x + 4y)$ and C be the parabolic path $y = 9 - x^2$ for x from -3 to 2. (See Sage for a picture.)

2. Let $\vec{F}(x, y, z) = (2x + yz, 2z + xz, 2y + xy)$ and C be the straight segment from $(2, -5, 0)$ to $(1, 2, 3)$. (See Sage for a picture.)

[Hint: If you parametrize the curve, then you've done the problem the HARD way. You don't need any parameterizations at all. Did you find a potential, and then plug in the end points?]

Problem 6.6 Let $\vec{F} = (x, 2yz, y^2)$. Let C_1 be the curve which starts at $(1, 0, 0)$ and follows a helical path $(\cos t, \sin t, t)$ to $(1, 0, 2\pi)$. Let C_2 be the curve which starts at $(1, 0, 2\pi)$ and follows a straight line path to $(2, 4, 3)$. Let C_3 be any smooth curve that starts at $(2, 4, 3)$ and ends at $(0, 1, 2)$.

See Sage—C_1 and C_2 are in blue, and several possible C_3 are shown in red.

- Find the work done by $\vec{F}$ along each path C_1, C_2, and C_3.
- Find the work done by $\vec{F}$ along the path C which follows C_1, then C_2, then C_3.
- If C is any path that starts at $(1, 0, 0)$ and ends at $(0, 1, 2)$, compute the work done by $\vec{F}$ along C.

If you are parameterizing the curves, you're doing this the really hard way. Are you using the potential of the vector field?

In the problem above, the path we took to get from one point to another did not matter. The vector field had a potential, which meant that the work done did not depend on the path traveled.

Definition 6.5: Conservative Vector Field. We say that a vector field is conservative if the integral $\int_C \vec{F}\cdot d\vec{r}$ does not depend on the path C. We say that a curve C is piecewise smooth if it can be broken up into finitely many smooth curves.

We now know how to draw a vector field provided someone tells us the equation. What we really need is to do the reverse. If we see vectors (forces, velocities, etc.) acting on something, how do we obtain an equation of the vector field? The spin field $\vec{F}=(-y,x)$ is directly related to the magnetic field that surrounds a wire with a current running through it. The following problem develops the gravitational vector field.

Problem 6.7: Radial fields Do the following:

Use Sage to plot your vector fields. See 16.2: 39-44 for more practice.

1. Let $P=(x,y,z)$ be a point in space. At the point P, let $\vec{F}_1(x,y,z)$ be the vector which points from P to the origin. Give a formula for $\vec{F}_1(x,y,z)$.
2. Give an equation of the vector field where at each point P in space, the vector $\vec{F}_2(P)$ is a unit vector that points towards the origin.
3. Give an equation of the vector field where at each point P in space, the vector $\vec{F}_3(P)$ is a vector of length 7 that points towards the origin.
4. Give an equation of the vector field where at each point P in space, the vector $\vec{F}(P)$ points towards the origin, and has a magnitude equal to G/d^2 where d is the distance to the origin, and G is a constant.

Now that we have a formula for the gravitation vector field, let's prove that this field has a potential. The following review problem is very similar.

Review Compute $\displaystyle\int \frac{x}{\sqrt{x^2+4}}dx$. See [1].

Definition 6.6: Simple Closed Curve. A closed curve is a curve C that starts and ends at the same point.

Problem 6.8 The gravitational vector field is directly related to the radial field $\vec{F}=\dfrac{(-x,-y,-z)}{(x^2+y^2+z^2)^{3/2}}$. Show that this vector field is conservative, by finding a potential for $\vec{F}$. Then compute the work done by an object that moves from $(1,2,-2)$ to $(0,-3,4)$ along ANY path that avoids the origin.

[See the review problem just before this if you're struggling with the integral.]

Problem 6.9 Suppose $\vec{F}$ is a gradient field. Let C be a piecewise smooth closed curve. Compute $\int_C \vec{F}\cdot d\vec{r}$ (you should get a number). Explain how you know your answer is correct.

---

[1] Let $u=x^2+4$, which means $du=2xdx$ or $dx=\frac{du}{2x}$. This means

$$\int \frac{x}{\sqrt{x^2+4}}dx=\int \frac{x}{\sqrt{u}}\frac{du}{2x}=\frac{1}{2}\int u^{-1/2}du=\frac{1}{2}\frac{u^{1/2}}{1/2}=\sqrt{u}=\sqrt{x^2+4}.$$

6.2 Parametric Surfaces

If someone gives us parametric equations for a curve in the plane or space, we can draw the curve. What we really need is the ability to obtain parametric equations of a curve that we can see. We've already seen that the parametric curve, given by the equations $x = 2\cos t$ and $y = 3\sin t$ or $\vec{r}(t) = (2\cos t, 3\sin t)$, is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. The next problem reviews a few parametrizations that we've learned to construct on our own.

Problem 6.10 For each planar curve below, give a parametrization of the curve. You can write your parametrization in the vector form $\vec{r}(t) = (?, ?)$, or in the parametric form $x =?,\ y =?$. Remember to give bounds for t of the form $? \leq t \leq ?$ so that we obtain precisely the requested portion of the curve.

Use Sage or Wolfram Alpha to visualize your parameterizations.

1. The top half (so $y \geq 0$) of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
2. The straight line segment from $(a, 0)$ to $(0, b)$.
3. The parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$.
4. The function $y = f(x)$ for $x \in [a, b]$.

The next two problems introduce parametric surfaces. Basically, parametric surfaces are a collection of lots of parametric curves along a surface.

Problem 6.11 A jet begins spiraling upwards to gain height. The position of the jet after t seconds is modeled by the equation $\vec{r}(t) = (2\cos t, 2\sin t, t)$. We could alternatively write this as $x = 2\cos t$, $y = 2\sin t$, $z = t$.

See Sage or Wolfram Alpha. The text has more practice in 13.1: 9-14, 19-22.

1. Construct a graph of this function by picking several values of t and plotting the resulting points $(2\cos t, 2\sin t, t)$.
2. Next to a few points on your graph, include the time t at which the jet is at this point on the graph. Include an arrow for the jet's direction.
3. Find the first and second derivative of $\vec{r}(t)$.
4. Compute the velocity and acceleration vectors at $t = \pi/2$. Place these vectors on your graph with their tails at the point corresponding to $t = \pi/2$.
5. Give an equation of the tangent line to this curve at $t = \pi/2$.

Problem 6.12 The jet from problem 6.11 is actually accompanied by several jets flying side by side. As all the jets fly, they leave a smoke trail behind them (it's an air show). The smoke from each jet spreads outwards to mix together, so that it looks like the jets are leaving wide sheet of smoke behind them as they spiral upwards. The position of two of the many other jets is given by $\vec{r}_3(t) = (3\cos t, 3\sin t, t)$ and $\vec{r}_4(t) = (4\cos t, 4\sin t, t)$. A function which represents the smoke stream from these jets is $\vec{r}(a, t) = (a\cos t, a\sin t, t)$ for $0 \leq t \leq 4\pi$ and $2 \leq a \leq 4$.

More practice in 16.5: 1-16.

1. Start by graphing the position of the three jets $\vec{r}(2, t) = (2\cos t, 2\sin t, t)$, $\vec{r}(3, t) = (3\cos t, 3\sin t, t)$, and $\vec{r}(4, t) = (4\cos t, 4\sin t, t)$.
2. Let $t = 0$ and graph the curve $r(a, 0) = (a, 0, 0)$ for $a \in [2, 4]$, which represents a segment along which the smoke has spread. Then repeat this for $t = \pi/2, \pi, 3\pi/2$.

3. Describe the resulting surface, and make sure you check your answer with technology (use the links to the side). See Sage or Wolfram Alpha.

We call the surface you drew above a parametric surface. The vector equation describing the smoke screen is a parametrization of this surface.

Definition 6.7: Parametric Surface, Parametrization of a surface. A parametrization of a surface is a collection of three equations to tell us the position

$$x = x(u, v), y = y(u, v), z = z(u, v)$$

of a point (x, y, z) on the surface. We call u and v parameters, and these parameters give us a two dimensional pair (u, v), the input, needed to obtain a specific location (x, y, z), the output, on the surface. We can also write a parametrization in vector form as

$$\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

We'll often give bounds on the parameters u and v, which help us describe specific portions of the surface. A parametric surface is a surface together with a parametrization.

We draw parametric surfaces by joining together many parametric space curves, as done in the previous problem. Just pick one variable, hold it constant, and draw the resulting space curve. Repeat this several times, and you'll have a 3D surface plot. Most of 3D computer animation is done using parametric surfaces. Woody's entire body in *Toy Story* is a collection of parametric surfaces. Car companies create computer models of vehicles using parametric surfaces, and then use those parametric surfaces to study collisions. Often the mathematics behind these models is hidden in the software program, but parametric surfaces are at the heart of just about every 3D computer model.

Problem 6.13 Consider the parametric surface $\vec{r}(u, v) = (u \cos v, u \sin v, u^2)$ for $0 \leq u \leq 3$ and $0 \leq v \leq 2\pi$. Construct a graph of this function. Remember, to do so we just let u equal a constant (such as 1, 2, 3) and then graph the resulting space curve where we let v vary. After doing this for several values of u, swap and let v equal a constant (such as 0, $\pi/2$, etc.) and graph the resulting space curve as u varies. [Hint: Did you get a satellite dish? Use the software links to the right to make sure you did this right.] See Sage or Wolfram Alpha.

We'll return to parametric surfaces in a bit, and use the parametrization to compute surface areas.

6.3 Operators

Recall that the gradient of a function f is the quantity

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f,$$

where in the last expression we let $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and then treat $\vec{\nabla} f$ as a "vector" $\vec{\nabla}$ times a scalar f. The quantity $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is an example of something we call an "operator," something that operates on functions.

Definition 6.8: Operator. An operator is a function whose input is a function itself. This allows us to say "operator on functions" instead of "function of functions."

We've already encountered several operators before this class. For example, the derivative operator $\frac{d}{dx}$ from first semester calculus takes a function such as $f(x) = x^2$ and returns a new function $\frac{d}{dx} f = 2x$. The integral operator $\int_a^b f dx$ takes a function f and return a real number. The gradient operator takes a function f and returns a vector of functions $\vec{\nabla} f = (f_x, f_y, f_z)$. This is just 3 examples of operators. Here are two more.

Definition 6.9: Divergence and Curl of a vector field. Consider the vector field $\vec{F}(x,y,z) = (M,N,P)$, where M, N, and P are functions of x, y, and z.

- The divergence of $\vec{F}$ is the scalar quantity

$$\begin{aligned}
\text{div}(\vec{F}) &= \vec{\nabla} \cdot \vec{F} \\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (M,N,P) \\
&= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \\
&= M_x + N_y + P_z.
\end{aligned}$$

- The curl of $\vec{F}$ is the vector quantity

$$\begin{aligned}
\text{curl}(\vec{F}) &= \vec{\nabla} \times \vec{F} \\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (M,N,P) \\
&= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, - \left[\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right], \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\
&= \left(P_y - N_z, - \left[P_x - M_z \right], N_y - M_x \right).
\end{aligned}$$

Problem 6.14 Compute the divergence and curl of each vector field below.

1. $\vec{F} = (2x, 3y^2, e^z)$
2. $\vec{F} = (-3y, 3x, 5z)$
3. $\vec{F} = (z - 3y, 3x, -x)$

In class, we'll talk about the physical meaning of each result above.

Problem 6.15 Suppose $f(x,y,z)$ is a twice continuously differentiable.

1. Compute the curl of the gradient of f, so compute $\vec{\nabla} \times \vec{\nabla} f$, and simplify the result as much as possible.
2. Which of the vector fields from the previous problem have a potential?
3. If a vector field $\vec{F} = (M,N,P)$ has a potential, then what is the curl of $\vec{F}$?

Problem 6.16 Suppose $\vec{F}(x,y,z) = (M,N,P)$ is a vector field and $f(x,y,z)$ is a function, both of which are twice continuously differentiable.

1. Compute the divergence of the curl of $\vec{F}$, so compute $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{F}\right)$, and simplify the result as much as possible.
2. Compute the divergence of the gradient of f, so compute $\vec{\nabla}\cdot\vec{\nabla}f$, and simplify the result as much as possible.

6.4 Green's Theorem

When a vector has no potential, there is a version of the fundamental theorem of calculus that simplifies the work computations.

Definition 6.10: Circulation, Simple closed curve. When a curve C is a closed curve (starts and ends at the same point), we call the work done by vector field $\vec{F}$ along C the circulation of $\vec{F}$ along C. A simple closed curve is a closed curve that does not cross itself.

Definition 6.11: Circulation Density and Flux Density (Divergence). Let $\vec{F}(x,y) = \langle M,N\rangle$ be a continuously differentiable vector field. At the point (x,y) in the plane, create a circle C_a of radius a centered at (x,y), oriented counterclockwise. The area inside of C_a is $A_a = \pi a^2$. The quotient $\frac{1}{A_a}\oint_{C_a}\vec{F}\cdot d\vec{r}$ is a circulation per area. The counterclockwise circulation density of $\vec{F}$ at (x,y) we define to be

$$\lim_{a\to 0}\frac{1}{A_a}\oint_{C_a}\vec{F}\cdot d\vec{r} = \lim_{a\to 0}\frac{1}{A_a}\oint_{C_a} Mdx+Ndy = \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y} = N_x - M_y.$$

We will not prove that the partial derivative expression $N_x - M_y$ is actually equal to the limit given here. That is best left to an advanced course.

In the definition above, we could have replaced the circle C_a with a square of side lengths a centered at (x,y) with interior area A_a. Alternately, we could have chosen any collection of curves C_a which "shrink nicely" to (x,y) and have area A_a inside. Regardless of which curves you chose, it can be shown that

$$N_x - M_y = \lim_{a\to 0}\frac{1}{A_a}\oint_{C_a} Mdx+Ndy.$$

To understand what the circulation density mean in a physical sense, think of $\vec{F}$ as the velocity field of some gas. The circulation density tells us the rate at which the vector field $\vec{F}$ causes objects to rotate around points. If circulation density is positive, then particles near (x,y) would tend to circulate around the point in a counterclockwise direction. The larger the circulation density, the faster the rotation. The velocity field of a gas could have some regions where the gas is swirling clockwise, and some regions where the gas is swirling counterclockwise.

We are now ready to state Green's Theorem. Ask me in class to give an informal proof as to why this theorem is valid.

Theorem 6.12 (Green's Theorem)**.** *Let $\vec{F}(x,y) = (M,N)$ be a continuously differentiable vector field, which is defined on an open region in the plane that contains a simple closed curve C and the region R inside the curve C. Then we can compute the counterclockwise circulation of $\vec{F}$ along C using*

$$\oint_C Mdx+Ndy = \iint_R N_x - M_y dA$$

Let's use this theorem to rapidly find circulation (work on a closed curve).

Problem 6.17 Consider the vector field $\vec{F} = (2x + 3y, 4x + 5y)$. Start by computing $N_x - M_y$. If C is the boundary of the rectangle $2 \leq x \leq 7$ and $0 \leq y \leq 3$, find the circulation of $\vec{F}$ along C. You should be able to reduce the integrals to facts about area. [If you tried doing this without Green's theorem, you would have to parametrize 4 line segments, compute 4 integrals, and then sum the results.]

See 16.4 for more practice. Try doing a bunch of these, as they get really fast.

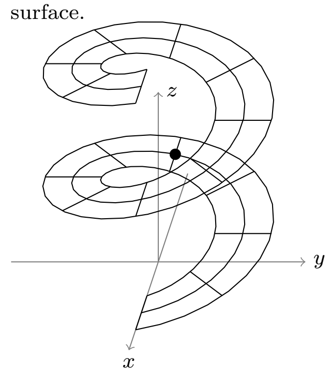
Problem 6.18 Consider the vector field $\vec{F} = (x^2 + y^2, 3x + 5y)$. Start by computing $N_x - M_y$. If C is the circle $(x - 3)^2 + (y + 1)^2 = 4$ (oriented counterclockwise), then find the circulation of $\vec{F}$ along C.

Problem 6.19 Repeat the previous problem, but change the curve C to the boundary of the triangular region R with vertexes at $(0, 0)$, $(3, 0)$, and $(3, 6)$.

6.5 Surface Integrals

Problem 6.20 Consider the parametric surface $\vec{r}(a, t) = (a \cos t, a \sin t, t)$ for $2 \leq a \leq 4$ and $0 \leq t \leq 4\pi$. We encountered this parametric surface early in the chapter when we considered a smoke screen left by multiple jets.

Here's a rough sketch of the surface.



1. Compute the differential $d\vec{r}$ which is the same as finding dx, dy, and dz. Write your answer as a linear combination of vectors, so as

$$d\vec{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} da + \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} dt$$

2. Suppose an object is on this surface at the point $\vec{r}(3, \pi) = (-3, 0, \pi)$ (the dot on the graph to the right). Evaluate the vectors above at this point and then draw both vectors on the plot with their tail at the point $\vec{r}(3, \pi)$. This will go best if you PRINT THIS PAGE.

3. Give vector equations for two tangent lines to the surface at $\vec{r}(3, \pi)$.

   [Hint: You've got the point as $\vec{r}(3, \pi)$, and you've got two different direction vectors as the columns of the matrix. Use the ideas from chapter 2 to get an equation of a line, or see the review problem above.]

In the previous problem, you should have noticed that the columns of your matrix are tangent vectors to the surface. Because we have two tangent vectors to the surface, we should be able to use them to construct a normal vector to the surface, and from that we can get the equation of a tangent plane.

Review If you know that a plane passes through the point $(1, 2, 3)$ and has normal vector $(4, 5, 6)$, then give an equation of the plane. See [2] for an answer.

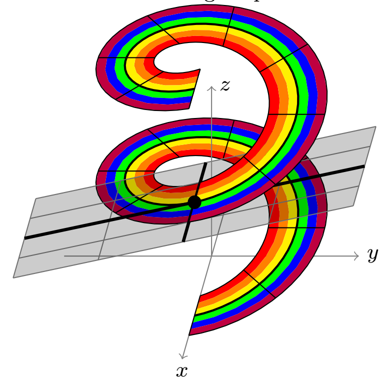
[2] An equation of the plane is $4(x - 1) + 5(y - 2) + 6(z - 3) = 0$. If (x, y, z) is any point in the plane, then the vector $(x - 1, y - 2, z - 3)$ is a vector in the plane, and hence orthogonal to $(4, 5, 6)$. The dot product of these two vectors should be equal to zero, which is why the plane's equation is $(4, 5, 6) \cdot (x - 1, y - 2, z - 3) = 0$.

Problem 6.21 Consider again the parametric surface

$$\vec{r}(a,t) = (a\cos t, a\sin t, t)$$

for $2 \leq a \leq 4$ and $0 \leq t \leq 4\pi$. We'd like to obtain an equation of the tangent plane to this surface at the point $\vec{r}(3, 2\pi)$. Once you have a point on the plane, and a normal vector to the surface, we can use the concepts in chapter 2 to get an equation of the plane. Give an equation of the tangent plane.

Here's a rough sketch of the surface with its tangent plane.



[Hint: To get the point, what is $\vec{r}(3, 2\pi)$? The columns of the matrix we obtain, when computing the differential $d\vec{r}$, give us two tangent vectors. How do we obtain a vector orthogonal to both these vectors?]

[Here's an alternate version of this problem, for Mario Kart fans. Mario and Luigi are booking it up rainbow road. About half way up, there is a glitch in the computer game and the road temporarily disappears. Instead of following the road, they instead are stuck on an infinite plane that meets the road tangentially where the glitch occurred. Give an equation of this plane.]

In first-semester calculus, we learned how to compute integrals $\int_a^b f dx$ along straight (flat) segments $[a, b]$. This semester, in the line integral unit, we learned how to change the segment to a curve, which allowed us to compute integrals $\int_C f ds$ along any curve C, instead of just along curves (segments) on the x-axis. The integral $\int_a^b dx = b - a$ gives the length of the segment $[a, b]$. The integral $\int_C ds$ gives the length s of the curve C.

This semester we've learned how to compute double integrals $\iint_R f dA$ along flat regions R in the plane. We'll now learn how to change the flat region R into a curved surface S, and then compute integrals of the form $\iint_S f d\sigma$ along curved surfaces. The differential $d\sigma$ stands for a little bit of surface area. We already know that $\iint_R dA$ gives the area of R. We'll define $\iint_S d\sigma$ so that it gives the surface area of S.

Problem 6.22 Consider the surface S given by $z = 9 - x^2 - y^2$, an upside down paraboloid that intersects the xy-plane in a circle of radius 3. A parametrization of the portion of this surface that lies above the xy-plane is

$$\vec{r}(x,y) = (x, y, 9 - x^2 - y^2) \quad \text{for} \quad -3 \leq x \leq 3, -\sqrt{9 - x^2} \leq y \leq \sqrt{9 - x^2}.$$

1. Draw the surface S. Add to your surface plot the parabolas given by $\vec{r}(x, 0)$, $\vec{r}(x, 1)$, and $\vec{r}(x, 2)$, as well as the parabolas given by $\vec{r}(0, y)$, $\vec{r}(1, y)$, and $\vec{r}(2, y)$. You should have an upside down paraboloid, with at least 6 different parabolas drawn on the surface. These parabolas should divide the surface up into a bunch of different patches. Our goal is to find the area of each patch, where each patch is almost like a parallelogram.

   See Sage for a solution.

2. Find both $\dfrac{\partial \vec{r}}{\partial x}$ and $\dfrac{\partial \vec{r}}{\partial y}$. Then at the point $(2, 1)$, draw both of these partial derivatives with their bases at $(2, 1)$. These vectors form the edges of a parallelogram. Add that parallelogram to your picture.

3. Show that the area of a parallelogram whose edges are the vectors $\dfrac{\partial \vec{r}}{\partial x}$ and $\dfrac{\partial \vec{r}}{\partial y}$ is $\sqrt{1 + 4x^2 + 4y^2}$. [Hint: think about the cross product.]

4. Notice in your work above that we drew parabolas by changing both x and y by 1 unit. If instead we had drawn parabolas at increments of .5 instead of 1, then we'd need to multiply our partial derivatives by .5 before finding the area of the parallelogram. If we use increments of dx and dy, then the edges of our parallelogram are the vectors $\vec{r}_x dx$ and $\vec{r}_y dy$. Find the area of this parallelogram.

In the previous problem, you showed that the area of the parallelogram with edges given by $\frac{\partial \vec{r}}{\partial x}dx$ and $\frac{\partial \vec{r}}{\partial y}dy$ is

$$d\sigma = \left|\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}\right| dxdy = |\vec{r}_x \times \vec{r}_y|\, dxdy.$$

This little bit of area approximates the surface area of a tiny patch on the surface. When we add all these areas up, we obtain the surface area.

Definition 6.13. Let S be a surface. Let $\vec{r}(u,v) = (x,y,z)$ be a parametrization of the surface, where the bounds on u and v form a region R in the uv plane. Then the surface area element (representing a little bit of surface) is

$$d\sigma = \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dudv = |\vec{r}_u \times \vec{r}_v|\, dudv.$$

The surface integral of a continuous function $f(x,y,z)$ along the surface S is

$$\iint_S f(x,y,z)d\sigma = \iint_R f(\vec{r}(u,v)) \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dudv.$$

If we let $f = 1$, then the surface area of S is simply

$$\sigma = \iint_S d\sigma = \iint_R \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dudv.$$

This definition tells us how to compute any surface integral. The steps are almost identical to the line integral steps.

1. Start by getting a parametrization $\vec{r}$ of the surface S where the bounds form a region R.
2. Find a little bit of surface area by computing $d\sigma = \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dudv$.
3. Multiply f by $d\sigma$, and replace each x, y, z with what they equals from the parametrization.
4. Integrate the previous function along R, your parameterization's bounds.

Example 6.14. Consider again the surface S given by $z = 9 - x^2 - y^2$, for $z \geq 0$. We used the parametrization

$$\vec{r}(x,y) = (x, y, 9 - x^2 - y^2) \quad \text{for} \quad -3 \leq x \leq 3, -\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2}.$$

to obtain $d\sigma = |\vec{r}_x \times \vec{r}_y|\, dxdy = \sqrt{4x^2 + 4y^2 + 1}dxdy$. This means that the surface area is

$$\sigma = \iint_S d\sigma = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{4x^2 + 4y^2 + 1}dydx.$$

At this point we now have an iterated double integral. As the region described by the integral is a circle, we can swap to polar coordinates to simplify the computations. The bounds are $0 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$, which means

$$\sigma = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{4(x^2+y^2)+1}\ dydx = \int_0^{2\pi} \int_0^3 \sqrt{4r^2+1}\ rdrd\theta.$$

Problem 6.23 Consider again the surface S from the example above. A different parametrization of this surface is

$$\vec{r}(r,\theta) = (r\cos\theta, r\sin\theta, 9-r^2), \quad \text{where } 9-r^2 \geq 0.$$

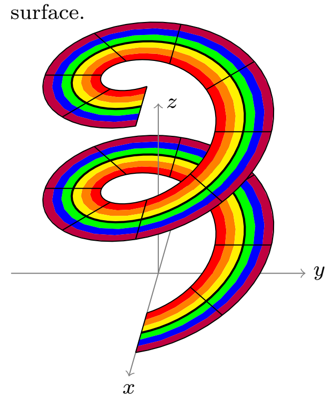
1. Give a set of inequalities for r and θ that describe the region $R_{r\theta}$ over which we need to integrate.
2. Find the surface area element $d\sigma = |\vec{r}_r \times \vec{r}_\theta|\, drd\theta$. Simplify your work to show that $d\sigma = r\sqrt{4r^2+1}\; drd\theta$.
3. Set up the surface integral $\iint_S d\sigma$ as an iterated double integral over $R_{r\theta}$, and then actually compute the integral by hand.

Problem 6.24 Consider the parametric surface

$$\vec{r}(a,t) = (a\cos t, a\sin t, t) \quad \text{for} \quad 2 \leq a \leq 4 \text{ and } 0 \leq t \leq 4\pi.$$

Find $\vec{r}_a$ and $\vec{r}_t$. Then compute the surface area element $d\sigma = |\vec{r}_a \times \vec{r}_t| dadt$. Set up an iterated integral for the surface area. Don't compute the integral.

Here's a rough sketch of the surface.



Problem 6.25 If a surface S is parametrized by $\vec{r}(x,y) = (x,y,f(x,y))$, show that $d\sigma = \sqrt{1+f_x^2+f_y^2}\; dxdy$ (compute a cross product). If $\vec{r}(x,z) = (x,f(x,z),z)$, what does $d\sigma$ equal (compute a cross product - you should see a pattern)? Use the pattern you've discovered to quickly compute $d\sigma$ for the surface $x = 4-y^2-z^2$, and then set up an iterated double integral that would give the surface area of S for $x \geq 0$.

Problem 6.26 Consider the sphere $x^2+y^2+z^2 = a^2$. We'll find $d\sigma$ using two different parameterizations.

1. Consider the rectangular parametrization $\vec{r}(x,y) = (x,y,\sqrt{a^2-x^2-y^2})$. Compute $d\sigma$? [Hint, use the previous problem.] Why can this parametrization only be use if the surface has positive z-values?
2. Consider the spherical parametrization

$$\vec{r}(\phi,\theta) = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi).$$

Show that

$$d\sigma = (a^2|\sin\phi|)d\phi d\theta = (a^2\sin\phi)d\phi d\theta,$$

where we can ignore the absolute values if we require $0 \leq \phi \leq \pi$. Along the way, you'll show that

$$\vec{r}_\phi \times \vec{r}_\theta = a^2\sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

We can compute mass, average value, centroids, and center of mass for surfaces. We just replace dA with $d\sigma$, and all the formulas are the same.

Problem 6.27 Consider the hemisphere $x^2+y^2+z^2 = a^2$ for $z \geq 0$.

1. Set up a formula that would give $\bar{z}$ for the centroid of the hemisphere. I suggest you use a spherical parametrization, as then the bounds are fairly simple, and we know $d\sigma = (a^2 \sin\phi) d\phi d\theta$ from the previous problem.
2. Compute both the integrals in your formula. The combine your work to show that $\bar{z} = \frac{a}{2}$.
3. One of the integrals you computed gave the surface area of a hemisphere of radius a. Which is it? Use that result to give the surface area of a sphere of radius a.

Problem 6.28 Consider the surface S that is the portion of the cone $x^2 = y^2 + z^2$ with $1 \leq x \leq 4$.

1. Give a parametrization of the cone, including bounds.
2. Use your parametrization to compute the surface area element $d\sigma$.
3. Compute the surface area of S. Yes, actually compute the integral.
4. Setup a formula that would give the center of mass $\bar{x}$ of the cone if the density is $\delta(x, y, z) = x$. Don't spend any time computing the integrals.

6.6 Flux

Flux (often represented with Φ) is a measure of flow across a surface. This might be the flow of water across a net. Alternately, it might be the flow of light through a solar panel. For simplicity, let's start by assuming $\vec{F}$ represents the velocity of a fluid (the units are meters per second). When we want to find the flux of a vector field across a surface, we must state in which direction $\hat{n}$ we want to compute the flux. This direction gives an orientation to the surface, differentiating between the two sides of the surface. The flux of $\vec{F}$ across the surface S (oriented using the normal vector $\hat{n}$) is a measure of the amount of the fluid per unit time that flows across S in the direction of $\hat{n}$. To compute this flux, we need to know the projection of $\vec{F}$ onto the normal $\hat{n}$ to the surface. The next problem has you prove that little bits of flux are given by $d\Phi = \vec{F} \cdot \hat{n} d\sigma$.

Problem 6.29 Let $\vec{F}(x, y, z)$ be a vector field and $\vec{n}(u, v)$ be a unit normal vector to a surface S given by parametrization $\vec{r}(u, v)$ for $a \leq u \leq b$ and $c \leq v \leq d$.

1. Use the projection formula, and simplify the result, to prove that the projection of $\vec{F}$ onto $\hat{n}$ is given by $(\vec{F} \cdot \hat{n})\hat{n}$.
2. What is the scalar component of $\vec{F}$ in the direction of $\hat{n}$?
3. Let $d\sigma$ be the surface area of a small portion of the surface. What does $\vec{F} \cdot \hat{n} d\sigma$ measure?
4. If the units on $\vec{F}$ are m/s, then what are the units of $\vec{F} \cdot \hat{n} d\sigma$?

Adding up little bits of flux gives us the following formula for the flux of a vector field $\vec{F}$ across a surface S:

$$\text{Flux} = \Phi = \iint_S \vec{F} \cdot \vec{n} d\sigma.$$

The next problem will help us simplify the computation of $\vec{n} d\sigma$.

Problem 6.30 Consider again the surface $z = 9 - x^2 - y^2$.

1. Using the parametrization $\vec{r}(x, y) = (x, y, 9 - x^2 - y^2)$, find a unit normal vector $\vec{n}$ to the surface so that $\vec{n}$ points upwards away from the z-axis. State what $d\sigma$ equals, as well as $\vec{n}d\sigma$. Explain how you know the normal vector you give is pointing upwards away from the z axis.

2. Using the parametrization $\vec{r}(r, \theta) = (r\cos\theta, r\sin\theta, 9 - r^2)$, find a unit normal vector $\vec{n}$ to the surface so that $\vec{n}$ points downwards towards the z-axis. State what $d\sigma$ equals, as well as $\vec{n}d\sigma$. Explain how you know the normal vector you give is pointing downwards towards the z axis.

In the problem above, we showed that $\vec{n}d\sigma = \pm(\vec{r}_x \times \vec{r}_y)dxdy$ and that $\vec{n}d\sigma = \pm(\vec{r}_r \times \vec{r}_\theta)drd\theta$. We no longer need to find the magnitude of the cross product, but we must determine the correct sign to put on our cross product. This shows us that we can write flux as

$$\text{Flux} = \Phi = \iint_S \vec{F} \cdot \vec{n}d\sigma = \iint_{R_{uv}} \vec{F} \cdot (\pm\vec{r}_u \times \vec{r}_v)dudv.$$

Problem 6.31 Consider the cone $z^2 = x^2 + y^2$ and vector field $\vec{F} = (2x + 3y, x - 2y, yz)$. Set up an iterated integral that would give the flux of $\vec{F}$ outwards (away from the z-axis) for the portion of the cone between $z = 1$ and $z = 3$. [Hint: Start by parameterizing the cone by using a polar parametrization

$$x = r\cos\theta, y = r\sin\theta, z = ?.$$

You should obtain bounds for r and θ that are constants. Compute the normal vector and look at the third component to determine if it points up or down. Then just plug everything into the formula.]

When the surface is flat, often you can determine the normal vector without having to perform any cross products. We'll now compute a flux of a vector field outwards across the 6 faces of a cube.

Problem 6.32 Find the flux of $\vec{F} = (x + y, y, z)$ outward across the surface of the cube in the first quadrant bounded by $x = 2, y = 3, z = 5$. The cube has 6 surfaces, so we have to compute the flux across all 6 surfaces. Fill in the table below to complete the flux across each surface, and then compute each integral to find the total flux.

Surface	$\vec{r}(u, v)$	$\vec{n}$	$\vec{F}(\vec{r}(u, v))$	$\vec{F} \cdot \vec{n}$	Flux
Back $x = 0$	$\langle 0, y, z\rangle$	$\langle -1, 0, 0\rangle$	$\vec{F}(0, y, z) = \langle y, y, z\rangle$	$-y$	$\iint_{\text{Back}} -yd\sigma = -\bar{y}\sigma = -(\frac{3}{2})(15)$
Front $x = 2$	$\langle 2, y, z\rangle$		$\vec{F}(2, y, z) = \langle 2 + y, y, z\rangle$		
Left $y = 0$					0 (Why?)
Right $y = 3$	$\langle x, 3, z\rangle$	$\langle 0, 1, 0\rangle$	$\vec{F}(x, 3, z) = \langle x + 3, 3, z\rangle$	3	30 (Why?)
Bottom $z = 0$					
Top $z = 3$					

You should be able to complete each integral by considering centroids and surface area of each of the 6 different flat surfaces. Show that the total flux is 90.

In the double integral chapter, we learned a way to greatly simplify work computations when working with simple closed curves (a closed curve is the boundary of a planar region). Green's theorem stated that $\int_C \vec{F} \cdot d\vec{r} = \iint_R N_x - N_y dA$. A similar theorem, called the divergence theorem, connects the outward flux of a vector field across a closed surface S containing the solid D using the formula

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint\int_D \vec{\nabla} \cdot \vec{F} dV.$$

The divergence of $\vec{F}$ is the quantity $\text{div}(\vec{F}) = M_x + N_y + P_z$.

Problem 6.33 Consider the exact same vector field and box as the previous problem. So we have the vector field $\vec{F} = (x + y, y, z)$ and S is the surface of the cube in the first quadrant bounded by $x = 2, y = 3, z = 5$.

1. Compute the divergence of $\vec{F}$, which is $\text{div}(\vec{F}) = M_x + N_y + P_z$.
2. The divergence theorem states that if S is a closed surface (has an inside and an outside), and the inside of the surface is the solid domain D, then the flux of $\vec{F}$ outward across S equals the triple integral

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \vec{\nabla} \cdot \vec{F} dV.$$

Use the divergence theorem to compute the flux of $\vec{F}$ across S. [Hint: Just as the area is found by adding up little bits of area, which is what we mean by $A = \iint_R dA$, the volume is found by adding up little bits of volume.]

Problem 6.34 In problem 6.26, we found

$$\vec{n} d\sigma = \vec{r}_\phi \times \vec{r}_\theta d\phi d\theta = a^2 \sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) d\phi d\theta$$

for a sphere of radius a. Use this to compute the outward flux of

$$\vec{F} = \frac{\langle -x, -y, -z\rangle}{(x^2 + y^2 + z^2)^{3/2}}$$

across a sphere of radius a. You should get a negative number since the vector field has all arrows pointing in. [Hint: Remember that for a sphere of radius a we have $a^2 = x^2 + y^2 + z^2$. When you perform the dot product of $\vec{F}$ and $\vec{n}$, you'll save yourself a lot of time if you remember that $\vec{u} \cdot \vec{u} = |\vec{u}|^2$; the dot product of a vector with itself is the length squared.]

Problem 6.35 Repeat the previous problem, but this time don't use the formula from problem 6.26. In fact, you don't even need to parametrize the surface. Instead, if you are at the point (x, y, z) on a sphere of radius a, give a formula for the outward pointing unit normal vector $\vec{n}$. Give this formula by only using a geometric argument. Then find the outward flux of $\vec{F} = \dfrac{\langle -x, -y, -z\rangle}{(x^2 + y^2 + z^2)^{3/2}}$ across a sphere of radius a. You should find that $\vec{F} \cdot \vec{n}$ simplifies to a constant, so that you never actually have to compute $d\sigma$. Then you can use known facts about the surface area of a sphere.