Multivariable Calculus

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Introduction

This course may be like no other course in mathematics you have ever taken. We'll discuss in class some of the key differences, and eventually this section will contain a complete description of how this course works. For now, it's just a skeleton.

I received the following email about 6 months after a student took the course:

> Hey Brother Woodruff,
>
> I was reading *Knowledge of Spiritual Things* by Elder Scott. I thought the following quote would be awesome to share with your students, especially those in Math 215 :)
>
>> Profound [spiritual] truth cannot simply be poured from one mind and heart to another. It takes faith and diligent effort. Precious truth comes a small piece at a time through faith, with great exertion, and at times wrenching struggles.

Elder Scott's words perfectly describe how we acquire mathematical truth, as well as spiritual truth.

Contents

Chapter 1

Review

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Give a summary of the ideas you learned in 112, including graphing, derivatives (product, quotient, power, chain, trig, exponential, and logarithm rules), and integration (u-sub and integration by parts).
2. Compute the differential dy of a function and use it to approximate the change in a function.
3. Explain how to perform matrix multiplication and compute determinants of square matrices.
4. Illustrate how to solve systems of linear equations, including how to express a solution parametrically (in terms of t) when there are infinitely solutions.
5. Extend the idea of differentials to approximate functions using parabolas, cubics, and polynomials of any degree.

You'll have a chance to teach your examples to your peers prior to the exam.

1.1 Review of First Semester Calculus

1.1.1 Graphing

This semester we'll be using Mathematica to visualize concepts, rapidly perform computations, and learn to program some more complicated algorithms in a computer algebra system. During this review unit, we'll learn to use Mathematica to perform many of the basic computations from first semester calculus.

Start by installing Mathematica on your computer. You'll find the instructions in I-Learn. When you create a WolframID, make sure you use your byui.edu email address. You should be able to download, install, and register the software for free. Once you have done so, you can complete the following problem.

We'll need to know how the graphs of several basic functions. When we start drawing functions in 3D, we'll have to piece together infinitely many 2D graphs. Knowing the basic shape of graphs will help us do this.

Problem 1.1 Use Mathematica to write a block of code that will plot a function. You should start with the command `''f= x^2''` and then use the

"Plot" command to graph the function. Then use your block of code to provide a sketch of the following functions:

$$x^2, x^3, x^4, \frac{1}{x}, \sin x, \cos x, \tan x, \sec x, \arctan x, e^x, \ln x.$$

Bring your laptop to class, or a printout of your work, so that we can see the commands you used. I have a projector that will display your work, whether it's a printout or a laptop.

1.1.2 Derivatives

In first semester calculus, one of the things you focused on was learning to compute derivatives. You'll need to know the derivatives of basic functions (found on the end cover of almost every calculus textbook). Computing derivatives accurately and rapidly will make learning calculus in high dimensions easier. You'll want to be familiar with the power rule, sum rule, product rule, quotient rule, and chain rule, as well as implicit differentiation.

Problem 1.2 Compute the derivative of $e^{\sec x}\cos(\tan(x)+\ln(x^2+4))$. Show each step in your computation, making sure to show what rules you used.

See sections 3.2-3.6 in Thomas's for more practice with derivatives. The later problems in 3.6 review of most of the entire differentiation chapter.

Problem 1.3 If $y(p) = \dfrac{e^{p^3}\cot(4p+7)}{\tan^{-1}(p^4)}$ find dy/dp. Again, show each step in your computation, making sure to show what rules you used.

Problem 1.4 Given $c^2 = a^2 + b^2$ and that a, b, c are all functions of t, use implicit differentiation to compute $\frac{dc}{dt}$ in terms of a, b, $\frac{da}{dt}$, and $\frac{db}{dt}$.

The following problem will help you review some of your trigonometry, inverse functions, as well as implicit differentiation.

Problem 1.5 Use implicit differentiation to explain why the derivative of $y = \arcsin x$ is $y' = \dfrac{1}{\sqrt{1-x^2}}$. [Rewrite $y = \arcsin x$ as $x = \sin y$, differentiate both sides, solve for y', and then write the answer in terms of x].

See sections 3.7-3.9 in Thomas's for more examples involving inverse trig functions and implicit differentiation.

Problem 1.6 Compute $\frac{dy}{dx}$ if we know $5 = x^2 + 3xy - y^3$.

1.1.3 Integrals

Each derivative rule from the front cover of your calculus text is also an integration rule. In addition to these basic rules, we'll need to know three integration techniques. They are (1) u-substitution, (2) integration-by-parts, and (3) integration by using software. There are many other integration techniques, but we will not focus on them. If you are trying to compute an integral to get a number while on the job, then software will almost always be the tool you use. As we develop new ideas in this and future classes (in engineering, physics, statistics, math), you'll find that u-substitution and integrations-by-parts show up so frequently that knowing when and how to apply them becomes crucial.

Problem 1.7 Compute $\int x\sqrt{x^2+4}dx$.

For practice with u-substitution, see section 5.5 and 5.6. in Thomas's For practice with integration by parts, see section 8.1.

Problem 1.8 Compute $\int x \sin 2x dx$.

Problem 1.9 Compute $\int \arctan x dx$.

Problem 1.10 Compute $\int x^2 e^{3x} dx$.

Problem 1.11 Use Mathematica to write a block of code that computes both the derivative and integral of a function. Then test your code on several function that you know the derivatives and integrals of. Bring either your laptop, or a printout of your code, to share with the class.

1.2 Differentials

The derivative of a function gives us the slope of a tangent line to that function. We can use this tangent line to estimate how much the output (y values) will change if we change the input (x-value). If we rewrite the notation $\frac{dy}{dx} = f'$ in the form $dy = f'dx$, then we can read this as "A small change in y (called dy) equals the derivative (f') times a small change in x (called dx)."

Definition 1.1. We call dx the differential of x. If f is a function of x, then the differential of f is $df = f'(x)dx$. Since we often write $y = f(x)$, we'll interchangeably use dy and df to represent the differential of f.

We will often refer to the differential notation $dy = f'dx$ as "a change in the output y equals the derivative times a change in the input x."

Problem 1.12 Let $f(x) = x^2 \ln(3x+2)$ and $g(t) = e^{2t}\tan(t^2)$. Compute the derivatives $\frac{df}{dx}$ and $\frac{dg}{dt}$, and then state the differentials df and dg. If you skipped reading the definition of a differential, you'll find it is given directly above this problem.

See 3.10:19-38.

This problem will help you see how the notion of differentials is used to develop equations of tangent lines. We'll use this same idea to develop tangent planes to surfaces in 3D and more.

Problem 1.13 Consider the function $y = f(x) = x^2$. This problem has multiple steps, but each is fairly short.

See 3.11:39-44. Also see problems 3.11:1-18. The linearization of a function is just an equation of the tangent line where you solve for y.

1. State the derivative of y with respect to x and the differential of y.

2. Give an equation of the tangent line to $f(x)$ at $x = 3$.

3. Draw a graph of $f(x)$ and the tangent line on the same axes. Place a dot at the point $(3, 9)$ and label it on your graph. Place another dot on the tangent line up and to the right of (3,9). Label the point (x, y), as it will represent any point on the tangent line.

4. From the point $(3, 9)$ to the point (x, y), the change in x, or run, is $dx = x - 3$. The change in y, or rise, is what? Use this to state the slope of the line connecting $(3, 9)$ and (x, y).

5. We already know the slope of the tangent line is the derivative $f'(3) = 6$. We also know the slope from the previous part. Set these two slope values equal, and verify that this gives an equation of the tangent line to $f(x)$ at $x = 3$.

Problem 1.14 The manufacturer of a spherical storage tank needs to create a tank with a radius of 5 m. Recall that the volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$. No manufacturing process is perfect, so the resulting sphere will have a radius of 5 m, plus or minus some small amount dr. The actual radius will be $5 + dr$. Find the differential dV. Then use differentials to estimate the change in the volume of the sphere if the actual radius is 5.02 m instead of the planned 5 m.

See 3.11:45-62.

Problem 1.15 A forest ranger needs to estimate the height of a tree. The ranger stands 50 feet from the base of tree and measures the angle of elevation to the top of the tree to be about 60°.

1. If this angle of 60° is correct, then what is the height of the tree?

2. If the ranger's angle measurement could be off by as much as 5°, then how much could his estimate of the height be off? Use differentials to give an answer.

If your answer here is quite large (much larger than the height of the tree), then look back at your work and see if using radians instead of degrees makes a difference. Why does it? Feel free to ask in class.

1.3 Matrices

We will soon discover that matrices represent derivatives in high dimensions. When you use matrices to represent derivatives, the chain rule is precisely matrix multiplication. For now, we just need to become comfortable with matrix multiplication.

We perform matrix multiplication "row by column". Wikipedia has an excellent visual illustration of how to do this. See Wikipedia for an explanation. See texample.net for a visualization of the idea.

The links will open your browser and take you to the web.

Problem 1.16 Compute the following matrix products.

For extra practice, make up two small matrices and multiply them. Use Sage or Wolfram Alpha to see if you are correct (click the links to see how to do matrix multiplication in each system).

- $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 6 & 1 \end{bmatrix}$

Problem 1.17 Compute the product $\begin{bmatrix}3 & 2 & 1\\ 0 & 1 & -4\end{bmatrix}\begin{bmatrix}-1 & 3 & 0\\ 2 & -1 & 0\\ 0 & 1 & 2\end{bmatrix}$. Then use Mathematica to compute the product as well (note that there is a difference between * and . in Mathematica). Come to class ready to show us how to perform matrix multiplication in Mathematica.

Most of higher dimensional calculus can quickly be developed from differential notation. Once we have the language of vectors and matrices at our command, we will develop calculus in higher dimensions by writing $d\vec{y} = Df(\vec{x})d\vec{x}$. Variables will become vectors, and the derivative will become a matrix.

Problem 1.18 Consider the equations $x = r\cos\theta$ and $y = r\sin\theta$ and assume that x, y, r, θ are all functions of t. Then compute the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$. You should be able to write your answer in the form

$$\begin{aligned}\frac{dx}{dt} &= (?)\frac{dr}{dt} + (?)\frac{d\theta}{dt}\\ \frac{dy}{dt} &= (?)\frac{dr}{dt} + (?)\frac{d\theta}{dt}.\end{aligned}$$

Then rewrite the equations above in terms of matrices, by writing

$$\begin{bmatrix}dx/dt\\ dy/dt\end{bmatrix} = \begin{bmatrix}? & ?\\ ? & ?\end{bmatrix}\begin{bmatrix}dr/dt\\ d\theta/dt\end{bmatrix}, \quad \text{or in the differntial form} \quad \begin{bmatrix}dx\\ dy\end{bmatrix} = \begin{bmatrix}? & ?\\ ? & ?\end{bmatrix}\begin{bmatrix}dr\\ d\theta\end{bmatrix}.$$

1.3.1 Determinants

Determinants measure area, volume, length, and higher dimensional versions of these ideas. Determinants will appear as we study cross products and when we get to the high dimensional version of u-substitution.

Associated with every square matrix is a number, called the determinant, which is related to length, area, and volume, and we use the determinant to generalize volume to higher dimensions. Determinants are only defined for square matrices.

Definition 1.2. The determinant of a 2×2 matrix is the number

$$\det\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc.$$

We use vertical bars next to a matrix to state we want the determinant, so $\det A = |A|$.

The determinant of a 3×3 matrix is the number

$$\begin{aligned}\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} &= a\det\begin{vmatrix}e & f\\ h & i\end{vmatrix} - b\det\begin{vmatrix}d & f\\ g & i\end{vmatrix} + c\det\begin{vmatrix}d & e\\ g & h\end{vmatrix}\\ &= a(ei - hf) - b(di - gf) + c(dh - ge).\end{aligned}$$

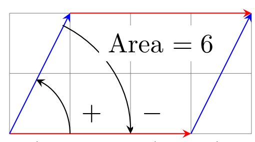
Notice the negative sign on the middle term of the 3×3 determinant. Also, notice that we had to compute three determinants of 2 by 2 matrices in order to find the determinant of a 3 by 3.

Problem 1.19 Compute $\begin{vmatrix}1 & 2\\ 3 & 4\end{vmatrix}$ and $\begin{vmatrix}1 & 2 & 0\\ -1 & 3 & 4\\ 2 & -3 & 1\end{vmatrix}$.

For extra practice, create your own square matrix (2 by 2 or 3 by 3) and compute the determinant by hand. Then use software to check your work. Do this until you feel comfortable taking determinants.

What good is the determinant? The determinant was discovered as a result of trying to find the area of a parallelogram and the volume of the three dimensional version of a parallelogram (called a parallelepiped) in space. If we had a full semester to spend on linear algebra, we could eventually prove the following facts that I will just present here with a few examples.

Consider the 2 by 2 matrix $\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}$ whose determinant is $3\cdot 2 - 0\cdot 1 = 6$. Draw the column vectors $\begin{bmatrix}3\\ 0\end{bmatrix}$ and $\begin{bmatrix}1\\ 2\end{bmatrix}$ with their base at the origin (see figure 1.1). These two vectors give the edges of a parallelogram whose area is the determinant 6. If I swap the order of the two vectors in the matrix, then the determinant of $\begin{bmatrix}1 & 3\\ 2 & 0\end{bmatrix}$ is -6. The reason for the difference is that the determinant not only keeps track of area, but also order. Starting at the first vector, if you can turn counterclockwise through an angle smaller than 180° to obtain the second vector, then the determinant is positive. If you have to turn clockwise instead, then the determinant is negative. This is often termed "the right-hand rule," as rotating the fingers of your right hand from the first vector to the second vector will cause your thumb to point up precisely when the determinant is positive.



$$\begin{vmatrix}3 & 1\\ 0 & 2\end{vmatrix} = 6 \text{ and } \begin{vmatrix}1 & 3\\ 2 & 0\end{vmatrix} = -6$$

Figure 1.1: The determinant gives both area and direction. A counter clockwise rotation from column 1 to column 2 gives a positive determinant.

For a 3 by 3 matrix, the columns give the edges of a three dimensional parallelepiped and the determinant produces the volume of this object. The sign of the determinant is related to orientation. If you can use your right hand and place your index finger on the first vector, middle finger on the second vector, and thumb on the third vector, then the determinant is positive. For example, consider the matrix $A = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$. Starting from the origin, each column represents an edge of the rectangular box $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$ with volume (and determinant) $V = lwh = (1)(2)(3) = 6$. The sign of the determinant is positive because if you place your index finger pointing in the direction (1,0,0) and your middle finger in the direction (0,2,0), then your thumb points upwards in the direction (0,0,3). If you interchange two of the columns, for example $B = \begin{bmatrix}0 & 1 & 0\\ 2 & 0 & 0\\ 0 & 0 & 3\end{bmatrix}$, then the volume doesn't change since the shape is still the same. However, the sign of the determinant is negative because if you point your index finger in the direction (0,2,0) and your middle finger in the direction (1,0,0), then your thumb points down in the direction (0,0,-3). If you repeat this with your left hand instead of right hand, then your thumb points up.

Problem 1.20 Compute the determinant of the matrix $\begin{bmatrix}-2 & 3\\ 5 & 4\end{bmatrix}$. Use your answer to find the area of the triangle with vertices $(0,0)$, $(-2,5)$, and $(3,4)$.

Problem 1.21 Find the area of a triangle with vertices $(-3, 1)$, $(-2, 5)$, and $(3, 4)$, using the determinant of an appropriate matrix. [Hint: if you shift all the points so one of them is at the origin, then this is extremely similar to the previous problem. If you search on the web, most of the search results will tell you to use a 3 by 3 matrix, where they add a column or row of all 1's, but don't explain why you should. Avoid this. Instead, you should be able to do this problem with just a 2 by 2 matrix and a determinant. You can use the 3 by 3 matrix method from online, provided you are prepared to explain why it works.]

1.4 Solving Systems of equations

Problem 1.22 Solve the following linear systems of equations.

For additional practice, make up your own systems of equations. Use Wolfram Alpha to check your work.

- $\begin{cases} x + y &= 3 \\ 2x - y &= 4 \end{cases}$
- $\begin{cases} -x + 4y &= 8 \\ 3x - 12y &= 2 \end{cases}$

Problem 1.23 Find all solutions to the linear system $\begin{cases} x + y + z &= 3 \\ 2x - y &= 4 \end{cases}$. Since there are more variables than equations, this suggests there is probably not just one solution, but perhaps infinitely many. One common way to deal with solving such a system is to let one variable equal t, and then solve for the other variables in terms of t. Do this three different ways.

This link will show you how to specify which variable is t when using Wolfram Alpha.

- If you let $x = t$, what are y and z. Write your solution in the form
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ ? \\ ? \end{bmatrix}.$$
where you replace the ?'s above with what y and z equal in terms of t.
- If you let $y = t$, what are x and z (in terms of t).
- If you let $z = t$, what are x and y.

1.5 Higher Order Approximations

When you ask a calculator to tell you what $e^{.1}$ means, your calculator uses an extension of differentials to give you an approximation. The calculator only uses polynomials (multiplication and addition) to give you an answer. This same process is used to evaluate any function that is not a polynomial (so trig functions, square roots, inverse trig functions, logarithms, etc.) The key idea needed to approximate functions is illustrated by the next problem.

Problem 1.24 Let $f(x) = e^x$. We already know how to find an equation of the tangent line to $f(x)$ at $x = 0$. We just need to find the slope $f'(0)$ and the value $f(0)$ of the function at $x = 0$. So the tangent line is just a first degree polynomial $P_1(x) = a + bx$ so that $P_1(0) = f(0)$ and $P_1'(0) = f'(0)$. In other words, the tangent line passes through the same point and has the same slope as $f(x) = e^x$ does at $x = 0$. What we'd like to do now is find polynomials of degree 2, 3, and higher, that approximate the function $f(x)$.

1. Find a second degree polynomial $P_2(x) = a+bx+cx^2$ so that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$. In other words, give me a parabola that passes through the same point, has the same slope, and has the same concavity as $f(x) = e^x$ does at $x = 0$. The polynomial has three unknowns, namely a, b, and c. We have three equations, namely $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$. So just compute the needed derivatives, plug in $x = 0$, and you should be able to use these three equations to find the unknowns.
2. Find a third degree polynomial $P_3(x) = a + bx + cx^2 + dx^3$ so that $P_3(0) = f(0)$, $P_3'(0) = f'(0)$, $P_3''(0) = f''(0)$, and $P_3'''(0) = f'''(0)$. In other words, give me a cubic that passes through the same point, has the same slope, the same concavity, and the same third derivative as $f(x) = e^x$ does at $x = 0$. Make sure you can show that $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$.
3. Now compute $e^{.1}$ with a calculator. Then compute both $P_2(.1)$ and $P_3(.1)$. Which of them is closer to $e^{.1}$?

Problem 1.25 Now let $f(x) = \sin x$. Find a 7th degree polynomial of the form

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_7x^7$$

so that the function and the polynomial have the same value and same first seven derivatives when evaluated at $x = 0$. This will require you to compute 8 derivatives of both $f(x)$ and $P(x)$, plug in $x = 0$ to create 8 equations, and then use those equations to determine the unknown coefficients $a_0, a_1, a_2, \ldots, a_7$ (many of which are zero). Once you've got this polynomial, evaluate the polynomial at $x = 0.3$. How close is this value to your calculator's estimate of $\sin(0.3)$?

The polynomial you are creating is often called a Taylor polynomial. (I'm giving you the name so that you can search online for more information if you are interested.)

As a check, you should get $a_7 = -\frac{1}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1} = -\frac{1}{7!}$.

Problem 1.26 Let's look at one more example of how to use a high degree polynomial to approximate a function. Consider the function $f(x) = \ln(x + 1)$. Find a 10th degree polynomial of the form

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{10}x^{10}$$

so that the function and the polynomial have the same value and same first ten derivatives when evaluated at $x = 0$. Once you've got this polynomial, use your calculator to compute $\ln(1.2)$ and $P(.2)$.

1.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 2

Vectors

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Define, draw, and explain what a vector is in 2 and 3 dimensions.
2. Add, subtract, and multiply (scalar, dot product, cross product) vectors. Be able to illustrate each operation geometrically, where possible.
3. Use vector products to find angles, length, area, projections, and work.
4. Use vectors to give equations of lines and planes, and be able to draw lines and planes in 3D.

You'll have a chance to teach your examples to your peers prior to the exam.

2.1 Vectors and Lines

Learning to work with vectors will be a key tool we need for our work in high dimensions. Let's start with some problems related to finding distance in 3D, drawing in 3D, and then we'll be ready to work with vectors.

To find the distance between two points (x_1, y_1) and (x_2, y_2) in the plane, we create a triangle connecting the two points. The base of the triangle has length $\Delta x = (x_2 - x_1)$ and the vertical side has length $\Delta y = (y_2 - y_1)$. The Pythagorean theorem gives us the distance between the two points as $\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Problem 2.1 The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in 3-dimensions is $\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. Construct an appropriate picture and show how to use the Pythagorean theorem repeatedly to prove this fact about distance in 3D.

Problem 2.2 Find the distance between the two points $P = (2, 3, -4)$ and $Q = (0, -1, 1)$. Then give an equation of the sphere passing though point Q whose center is at P. See 12.1:41-58.

Problem 2.3 For each of the following, construct a rough sketch of the set of points in space (3D) satisfying: See 12.1:1-40.

1. $2 \leq z \leq 5$
2. $x = 2, y = 3$
3. $x^2 + y^2 + z^2 = 25$

Definition 2.1. A vector is a magnitude in a certain direction. If P and Q are points, then the vector $\vec{PQ}$ is the directed line segment from P to Q. This definition holds in 1D, 2D, 3D, and beyond. If $V = (v_1, v_2, v_3)$ is a point in space, then to talk about the vector $\vec{v}$ from the origin O to V we'll use any of the following notations:

Most textbooks use a bold font to write vectors. When writing vectors by hand, it's common to use an arrow above a letter to represent that it's a vector.

$$\vec{v} = \mathbf{v} = \vec{OV} = \langle v_1, v_2, v_3 \rangle = (v_1, v_2, v_3) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$
$$= \underbrace{v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}}_{\text{common in engineering}} = \underbrace{v_1\hat{\mathbf{x}} + v_2\hat{\mathbf{y}} + v_3\hat{\mathbf{z}}}_{\text{common in physics}}.$$

We call v_1, v_2, and v_3 the x, y, and z components of the vector, respectively.

Note that (v_1, v_2, v_3) could refer to either the point V or the vector $\vec{v}$. The context of the problem we are working on will help us know if we are dealing with a point or a vector.

Definition 2.2. Let $\mathbb{R}$ represent the set real numbers. Real numbers are actually 1D vectors. Let $\mathbb{R}^2$ represent the set of vectors (x_1, x_2) in the plane. Let $\mathbb{R}^3$ represent the set of vectors (x_1, x_2, x_3) in space. There's no reason to stop at 3, so let $\mathbb{R}^n$ represent the set of vectors $(x_1, x_2, \ldots, x_n)$ in n dimensions.

In first semester calculus and before, most of our work dealt with problem in $\mathbb{R}$ and $\mathbb{R}^2$. Most of our work now will involve problems in $\mathbb{R}^2$ and $\mathbb{R}^3$. We've got to learn to visualize in $\mathbb{R}^3$.

Definition 2.3. The **magnitude**, or **length**, or **norm** of a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. It is just the distance from the point (v_1, v_2, v_3) to the origin.

A **unit vector** is a vector whose length is one unit. We commonly place a hat above unit vectors, as in $\hat{v}$ or $\hat{\mathbf{v}}$, The standard unit vectors are vectors of length one that point in the positive x, y, and z directions, namely

$$\mathbf{i} = \langle 1, 0, 0 \rangle = \hat{\mathbf{x}}, \quad \mathbf{j} = \langle 0, 1, 0 \rangle = \hat{\mathbf{y}}, \quad \mathbf{k} = \langle 0, 0, 1 \rangle = \hat{\mathbf{z}}.$$

Note that in 1D, the length of the vector $\langle -2 \rangle$ is simply $|-2| = \sqrt{(-2)^2} = 2$, the distance to 0. Our use of the absolute value symbols is appropriate, as it generalizes the concept of absolute value (distance to zero) to all dimensions.

Definition 2.4. Suppose $\vec{x} = \langle x_1, x_2, x_3 \rangle$ and $\vec{y} = \langle y_1, y_2, y_3 \rangle$ are two vectors in 3D, and c is a real number. We define vector addition and scalar multiplication as follows:

- Vector addition: $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ (add component-wise).
- Scalar multiplication: $c\vec{x} = (cx_1, cx_2, cx_3)$.

Problem 2.4 Consider the vectors $\vec{u} = (1, 2)$ and $\vec{v} = \langle 3, 1 \rangle$. Draw $\vec{u}$, $\vec{v}$, $\vec{u} + \vec{v}$, and $\vec{u} - \vec{v}$ with their tail placed at the origin. Then draw $\vec{v}$ with its tail at the head of $\vec{u}$.

See 12.2:23-24.

Problem 2.5 Consider the vector $\vec{v} = (3, -1)$. Draw $\vec{v}$, $-\vec{v}$, and $3\vec{v}$. Suppose a donkey travels along the path given by $(x, y) = \vec{v}t = (3t, -t)$, where t represents time. Draw the path followed by the donkey. Where is the donkey at time $t = 0, 1, 2$? Put markers on your graph to show the donkey's location. Then determine how fast the donkey is traveling.

See 11.1: 3,4.

In the previous problem you encountered $(x, y) = (3t, -t)$. This is an example of a function where the input is t and the output is a vector (x, y). For each input t, you get a single vector output (x, y). Such a function we call a **parametrization** of the donkey's path. Because the output is a vector, we call the function a **vector-valued function**. Often, we'll use the variable $\vec{r}$ to represent the radial vector (x, y), or (x, y, z) in 3D which points from the origin outwards. So we could rewrite the position of the donkey as $\vec{r}(t) = (3, -1)t$. We use $\vec{r}$ instead of r to remind us that the output is a vector.

Problem 2.6 Suppose a horse races down a path given by the vector-valued function $\vec{r}(t) = (1, 2)t + (3, 4)$. (Remember this is the same as writing $(x, y) = (1, 2)t + (3, 4)$ or similarly $(x, y) = (1t + 3, 2t + 4)$.) Where is the horse at time $t = 0, 1, 2$? Put markers on your graph to show the horse's location. Draw the path followed by the horse. Give a unit vector that tells the horse's direction. Then determine how fast the horse is traveling.

See 12.2: 1.

Problem 2.7 Consider the two points $P = (1, 2, 3)$ and $Q = (2, -1, 0)$. Write the vector $\vec{PQ}$ in component form (a, b, c). Find the length of vector $\vec{PQ}$. Then find a unit vector in the same direction as $\vec{PQ}$. Finally, find a vector of length 7 units that points in the same direction as $\vec{PQ}$.

See 12.2: 9,17,25,33 and surrounding.

Problem 2.8 A raccoon is sitting at point $P = (0, 2, 3)$. It starts to climb in the direction $\vec{v} = \langle 1, -1, 2 \rangle$. Write a vector equation $(x, y, z) = (?, ?, ?)$ for the line that passes through the point P and is parallel to $\vec{v}$. [Hint, study problem 2.6, and base your work off of what you saw there. It's almost identical.]

Then generalize your work to give an equation of the line that passes through the point $P = (x_1, y_1, z_1)$ and is parallel to the vector $\vec{v} = (v_1, v_2, v_3)$.

See 12.5: 1-12.

Make sure you ask me in class to show you how to connect the equation developed above to what you have been doing since middle school. If you can remember $y = mx + b$, then you can quickly remember the equation of a line. If I don't show you in class, make sure you ask me (or feel free to come by early and ask before class).

Problem 2.9 Let $P = (3, 1)$ and $Q = (-1, 4)$.

See 12.5: 13-20.

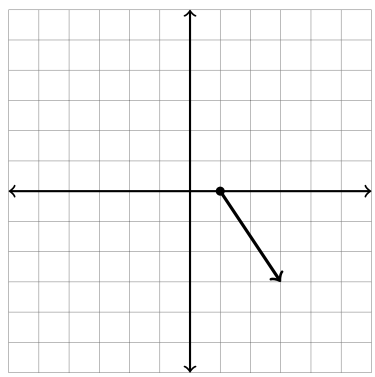
- Write a vector equation $\vec{r}(t) = (?, ?)$ for (i.e, give a parametrization of) the line that passes through P and Q, with $\vec{r}(0) = P$ and $\vec{r}(1) = Q$.
- Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is twice the speed of the first line.
- Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is one unit per second.

If you want to analyze how a river is flowing, one way to do so would be to construct a plot of the river and at each point in the river draw a vector to represent the velocity at that point. This create a collection of many vectors drawn all at once, where the base of each velocity vector is placed at the point where the velocity occurs. The next problem has you construct your first vector field. We'll come back to vector fields more as the semester progresses. Eventually vector fields will be one of the most important ideas in this course. I want you to see one now.

Problem 2.10: Vector Fields Consider the function $\vec{F}(x,y) = \langle 2x+y, -3x+2y\rangle$. This is a function where the input is a point (x,y) in the plane, and the output is the vector $\langle 2x+y, -3x+2y\rangle$. For example, if we input the point $(1,0)$, then the output is $\langle 2(1)+0, -3(1)+0\rangle$. To construct a vector field, you draw the output with its base located at the input. In the picture below, based at (1,0) we draw a vector that points right 2 and down 3.

1. Complete the table below and add the other 7 vectors to the graph.

(x,y)	$\langle 2x+y, -3x+2y\rangle$
$(1,0)$	$\langle 2,-3\rangle$
$(1,1)$	
$(1,-1)$	
$(0,1)$	
$(0,-1)$	
$(-1,0)$	
$(-1,1)$	
$(-1,-1)$	



2. Repeat the above for the vector field $\vec{F}(x,y) = (-2y, 3x)$, constructing a vector field plot consisting of 8 vectors.

2.2 The Dot Product

Now that we've learned how to add and subtract vectors, stretch them by scalars, and use them to find lines, it's time to introduce a way of multiplying vectors called the dot product. The dot product arises naturally when we try to find the angle between two vectors. We'll need to recall the law of cosines, stated below.

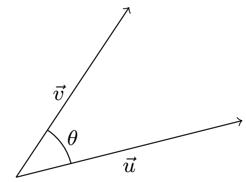
Theorem (The Law of Cosines)**.** *Consider a triangle with side lengths a, b, and c. Let θ be the angle between the sides of length a and b. Then the law of cosines states that*

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

If $\theta = 90°$, *then* $\cos\theta = 0$ *and this reduces to the Pythagorean theorem.*

Problem 2.11 Sketch in $\mathbb{R}^2$ the vectors $\langle -1,2\rangle$ and $\langle 3,5\rangle$. Then use the law of cosines to find the angle between the vectors. See 12.3: 9-12.

Problem 2.12 Consider the two vectors $\vec{u}$ and $\vec{v}$ in the plane (so $\vec{u}, \vec{v} \in \mathbb{R}^2$) shown in margin to the right.



1. Add the vector $\vec{u} - \vec{v}$ to the picture to the right.
2. Use the law of cosines to explain why $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$.

Notice that in your work on the previous problem, the fact that $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$ did not require ever referring to the fact that the vectors were in $\mathbb{R}^2$. This fact is true for vectors in general.

Problem 2.13 Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$ (which we write as $\vec{u}, \vec{v} \in \mathbb{R}^3$).

See page 693 if you are struggling.

1. First use the result of the previous problem to explain why
$$|\vec{u}||\vec{v}|\cos\theta = \frac{|\vec{u}|^2 + |\vec{v}|^2 - |\vec{u} - \vec{v}|^2}{2}.$$
2. Now use the coordinates (u_1, u_2, u_3) and (v_1, v_2, v_3) to simplify the right hand side of the equation above. For example, you'll replace $|\vec{u}|^2$ with $\left(\sqrt{u_1^2 + u_2^2 + u_3^2}\right)^2 = u_1^2 + u_2^2 + u_3^2$. For the difference $|\vec{u} - \vec{v}|$, you'll need to subtract coordinates and then compute the magnitude, which gives something like $|\vec{u} - \vec{v}| = \sqrt{(u_1 - v_1)^2 + \cdots}$. When you are done simplifying you should end up with something quite simple.

Definition 2.5: The Dot Product. Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$. We define the dot product of these two vectors to be
$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$
A similar definition holds for vectors in $\mathbb{R}^n$, where $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$. You just multiply corresponding components together and then add. It is the same process that we use in matrix multiplication.

With the definition of the dot product, we can rewrite the law of cosines as
$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta.$$

Problem 2.14 Use our new rule $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$ to find the angle between each pair of vectors below. If the angle is messy, first write the answer in terms of arccos and then use a calculator to approximate the angle.

See 12.3: 9-12.

1. $1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $-2\mathbf{i} + 1\mathbf{j} + 4\mathbf{k}$
2. $(1, 2, 3)$ and $(-2, 1, 0)$

In the previous problem, you should have found that one of the pairs of vectors had a dot product that was zero.

Definition 2.6. We say two vectors $\vec{u}$ and $\vec{v}$ are orthogonal when $\vec{u} \cdot \vec{v} = 0$.

Problem 2.15 Find two vectors orthogonal to $(1, 2)$. Then find 4 vectors orthogonal to $(3, 2, 1)$.

The dot product provides a really easy way to determine when two vectors meet at a right angle. The dot product is precisely zero when this happens. The next problem has you justify this fact.

Problem 2.16 Show that if two nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal, then the angle between them is $90°$. Then show that if the angle between them is $90°$, then the vectors are orthogonal.

See page 694.

Note: There are two things to show above. First, assume that the vectors are orthogonal (so their dot product is zero) and use this to compute the angle. Then second, assume that the angle between them is $90°$ and use this to compute the dot product.

Let's end this section by looking at some properties of the dot product.

Problem 2.17 Mark each statement true or false. Then make up an example to illustrate why you gave your answer. I have done the first as an example. You can assume that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ and that $c \in \mathbb{R}$.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

   Solution: This is true. If $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$, then we know $\vec{u} \cdot \vec{v} = (a, b) \cdot (c, d) = ac + bd$ and $\vec{v} \cdot \vec{u} = (a, b) \cdot (c, d) = ca + db$. Since $ab = ba$ and $cd = dc$, these two are clearly true.

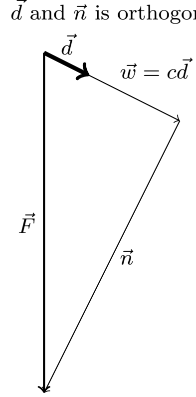
2. $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$.
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$.
4. $\vec{u} + (\vec{v} \cdot \vec{w}) = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{w})$.
5. $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$.
6. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

The last property above is extremely important, namley it connects the length of a vector to the dot product. We have now seen that we can compute both lengths and angles from the dot product. Any time you are working with either lengths or angles, there is a dot product hiding in the background. On a side note, in dimension 4 and higher, we define lenths and angles directly from the dot product.

2.2.1 Projections and Work

Suppose a heavy box needs to be lowered down a ramp. The box exerts a downward force of say 200 Newtons, which we could write in vector notation as $\vec{F} = \langle 0, -200 \rangle$. If the ramp was placed so that the box needed to be moved right 6 m, and down 3 m, then we'd need to get from the origin $(0, 0)$ to the point $(6, -3)$. This displacement can be written as $\vec{d} = \langle 6, -3 \rangle$. The force $\vec{F}$ acts straight down, rather than with the displacement. Our goal in this section is to find out how much of the force $\vec{F}$ acts in the direction of the displacement. This will tell us precisely the force needed to prevent the box from sliding down the ramp (neglecting friction). We are going to break the force $\vec{F}$ into two components, one component in the direction of $\vec{d}$, and another component orthogonal to $\vec{d}$.

In the diagram below, we have $\vec{F} = \vec{w} + \vec{n}$ where $\vec{w}$ is parallel to $\vec{d}$ and $\vec{n}$ is orthogonal to $\vec{d}$.



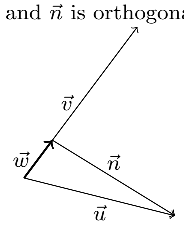
Problem 2.18 Read the preceding paragraph. Rather than working with the specific numbers given in that paragraph, please use $\vec{F}$ and $\vec{d}$ to represent any vector, so that when we are done with this problem we'll have a symbolic solution.

We want to write $\vec{F}$ as the sum of two vectors $\vec{F} = \vec{w} + \vec{n}$, where $\vec{w}$ is parallel to $\vec{d}$ and $\vec{n}$ is orthogonal to $\vec{d}$. Since $\vec{w}$ is parallel to $\vec{d}$, we can write $\vec{w} = c\vec{d}$ for some unknown scalar c. This means that $\vec{F} = c\vec{d} + \vec{n}$. Use the fact that $\vec{n}$ is orthogonal to $\vec{d}$ to show that $c = \dfrac{\vec{F}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}$.

[Hint: Dot each side of $\vec{F} = c\vec{d} + \vec{n}$ with $\vec{d}$ and distribute. You'll need to use the fact that $\vec{n}$ and $\vec{d}$ are orthogonal to remove $\vec{n}\cdot\vec{d}$ from the problem. This should turn the vectors into numbers, so you can use division and solve for c directly. Don't spent more than 10 minutes on this problem.]

Problem 2.19 Consider the vectors $\vec{u}$ and $\vec{v}$ in the diagram to the right. We can write $\vec{u}$ as the sum of a vector that is parallel to $\vec{v}$ (called $\vec{w}$ below) and a vector that is orthogonal to $\vec{v}$ (called $\vec{n}$ below). This gives us $\vec{u} = \vec{w} + \vec{n}$.

In the diagram below, we have $\vec{u} = \vec{w} + \vec{n}$ where $\vec{w}$ is parallel to $\vec{v}$ and $\vec{n}$ is orthogonal to $\vec{v}$.



Notice the right angle where vectors $\vec{n}$ and $\vec{w}$ meet.

1. Let θ be the angle between $\vec{u}$ and $\vec{v}$. Use right triangle trigonometry to explain why the length of $\vec{w}$ is given by $|\vec{w}| = |\vec{u}|\cos\theta$.
2. Now that we know the length of $\vec{w}$, explain why $\vec{w} = (|\vec{u}|\cos\theta)\dfrac{\vec{v}}{|\vec{v}|}$. See problem 2.7 if you need help.
3. We have a formula that connects the dot product to the cosine of the angle between two vectors. Show the steps that transform the equation above into the equation

$$\vec{w} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|}.$$

Can you explain why this also means

$$\vec{w} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}?$$

The previous two problems give us the definition of a projection.

Definition 2.7. The projection of $\vec{F}$ onto $\vec{d}$, written $\text{proj}_{\vec{d}}\vec{F}$, is defined as

$$\text{proj}_{\vec{d}}\vec{F} = \underbrace{\left(\frac{\vec{F}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\right)\vec{d}}_{\text{quick computation method}} = \underbrace{\left(\frac{\vec{F}\cdot\vec{d}}{|\vec{d}|}\right)\frac{\vec{d}}{|\vec{d}|}}_{\substack{\text{geometric method}\\ \text{magnitude times direction}}}.$$

When we wish to write $\vec{F}$ as the sum of a vector parallel to $\vec{d}$ plus a vector orthogonal to $\vec{d}$, the projection of $\vec{F}$ onto $\vec{d}$ is precisely the portion of $\vec{F}$ that is parallel to $\vec{d}$.

Problem 2.20 Let $\vec{u} = (-1, 2)$ and $\vec{v} = (3, 4)$. Draw $\vec{u}$, $\vec{v}$, and $\text{proj}_{\vec{v}}\vec{u}$. Then draw a line segment from the head of $\vec{u}$ to the head of the projection.

See 12.3:1-8 (part d).

Now let $\vec{u} = (-2, 0)$ and keep $\vec{v} = (3, 4)$. Draw $\vec{u}$, $\vec{v}$, and $\text{proj}_{\vec{v}}\vec{u}$. Then draw a line segment from the head of $\vec{u}$ to the head of the projection.

One final application of projections pertains to the concept of work. Work is the transfer of energy. If a force F acts through a displacement d, then the most basic definition of work is $W = Fd$, the product of the force and the displacement. This basic definition has a few assumptions.

- The force F must act in the same direction as the displacement.
- The force F must be constant throughout the entire displacement.
- The displacement must be in a straight line.

Before the semester ends, we will be able to remove all 3 of these assumptions. The next problem will show you how dot products help us remove the first assumption.

Recall the set up to problem 2.18. We want to lower a box down a ramp (which we will assume is frictionless). Gravity exerts a force of $\vec{F} = \langle 0, -200 \rangle$ N. If we apply no other forces to this system, then gravity will do work on the box through a displacement of $\langle 6, -3 \rangle$ m. The work done by gravity will transfer the potential energy of the box into kinetic energy (remember that work is a transfer of energy). How much energy is transferred?

Problem 2.21 Find the amount of work done by the force $\vec{F} = \langle 0, -200 \rangle$ through the displacement $\vec{d} = \langle 6, -3 \rangle$. Find this by doing the following: See 12.3: 24, 41-44.

1. Find the projection of $\vec{F}$ onto $\vec{d}$. This tells you how much force acts in the direction of the displacement. Find the magnitude of this projection.
2. Since work equals $W = Fd$, multiply your answer above by $|\vec{d}|$.
3. Now compute $\vec{F} \cdot \vec{d}$. You have just shown that $W = \vec{F} \cdot \vec{d}$ when $\vec{F}$ and $\vec{d}$ are not in the same direction.

The dot product gives us the work done by $\vec{F}$ through a displacement $\vec{d}$ when $\vec{F}$ and $\vec{d}$ are not in the same direction. Remember that the dot product is a number, which means it may be hard to visual. Connecting the dot product to work done by one vector in the direction of another can often lead to a good geometric description of the dot product.

Problem Answer each of the following, assuming that none of the vectors are the zero vector.

1. Suppose $\vec{u} \cdot \vec{v} = 0$. What do you know about the two vectors?
2. Suppose $\vec{u} \cdot \vec{v} > 0$. What do you know about the two vectors?
3. Suppose $\vec{u} \cdot \vec{v} < 0$. What do you know about the two vectors?

See [1] for a solution.

---

[1] When the dot product is zero, we know that the two vectors meet at a 90° angle. Thinking about this in terms of work, this means that the force has no portion in the direction of the displacement, hence there is no work done. If the dot product is positive, then the force has a portion acting in the direction of the displacement. This means that the angle between the two vectors is acute. Similarly if the dot product is negative then the angle must be obtuse (greater than 90°.)

2.3 The Cross Product and Planes

The dot product gave us a way of multiplying two vectors together, but the result was a number, not a vector. We now define the cross product, which will allow us to multiply two vectors together to give us another vector. We were able to define the dot product in all dimensions. The cross product is only defined in $\mathbb{R}^3$.

Definition 2.8: The Cross Product. The cross product of two vectors $\vec{u}$ and $\vec{v}$ is a new vector $\vec{u} \times \vec{v}$. This new vector is (1) orthogonal to both $\vec{u}$ and $\vec{v}$, (2) has a length equal to the area of the parallelogram whose sides are these two vectors, and (3) points in the direction your thumb points as you curl the base of your right hand from $\vec{u}$ to $\vec{v}$.

The definition of a the cross product tells us what kind of vector we need (orthogonal to both, magnitude equal to an area, and direction following the right hand rule), but doesn't give us a formula for computing it. The formula given here is nontrivial to develop from the definition. Wikipedia (see this link) gives a decent explanation, but does skip one difficult step in their computation. We will use the formula given here without proof. See 12.3: 9-14.

Problem 2.22 Consider $\mathbf{i} = (1, 0, 0)$, $2\mathbf{j} = (0, 2, 0)$, and $3\mathbf{k} = (0, 0, 3)$. Remember that the definition above defines the cross product in terms of areas of parallelograms.

1. Explain why $\mathbf{i} \times 2\mathbf{j} = 2\mathbf{k}$ and why $2\mathbf{j} \times \mathbf{i} = -2\mathbf{k}$. Make sure you draw the appropriate parallelogram whose area you need to compute, and be ready to explain why one points in the direction of $\mathbf{k}$ and the other points in the direction of $-\mathbf{k}$.
2. Compute $\mathbf{i} \times 3\mathbf{k}$ and $3\mathbf{k} \times \mathbf{i}$.
3. Compute $2\mathbf{j} \times 3\mathbf{k}$ and $3\mathbf{k} \times 2\mathbf{j}$.
4. Compute $\mathbf{i} \times \mathbf{i}$. In particular, what is $\vec{u} \times \vec{u}$ for any vector $\vec{u}$?

The problem above gives us the following facts for the cross product of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j},$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \mathbf{i} \times \mathbf{k} = -\mathbf{j},$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \vec{0}.$$

In the next problem, we'll develop a formula that works for the cross product of any two vectors. To do this, we'll need some more facts about the cross product. First, constant multiples of a vector can be applied either before or after computing the cross product, giving us

$$(a\vec{u}) \times (b\vec{v}) = ab(\vec{u} \times \vec{v}).$$

Second, the cross product satisfies the distributive laws

$$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w} \quad \text{and} \quad \vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}.$$

If you are interested, ask me in class to show you a proof of why the cross product satisfies these distributive laws.

Problem 2.23 Draw two non parallel vectors on your paper and label them $\vec{u}$ and $\vec{v}$. Then use the definition of the cross product, in terms of areas of parallelograms, to explain why $(2\vec{u}) \times (3\vec{v}) = (2 \cdot 3)(\vec{u} \times \vec{v})$. This is essentially the proof of why $(a\vec{u}) \times (b\vec{v}) = ab(\vec{u} \times \vec{v})$.

Then let $\vec{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Use the constant multiple rule, the distributive laws, and facts about the cross products of $\mathbf{i}$, $\mathbf{i}$, and $\mathbf{k}$, to compute and simplify the cross product $\vec{u} \times \vec{v}$. Simplify your work till you obtain the formula for the cross product that is given below.

Definition 2.9: Cross Product Formula. A formula for the cross product is

$$\vec{u}\times\vec{v}=\langle u_2v_3-u_3v_2,-(u_1v_3-u_3v_1),u_1v_2-u_2v_1\rangle=\det\begin{bmatrix}\vec{i}&\vec{j}&\vec{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{bmatrix}.$$

Problem 2.24 Let $\vec{u}=(1,-2,3)$ and $\vec{v}=(2,0,-1)$.

See 12.4: 1-8.

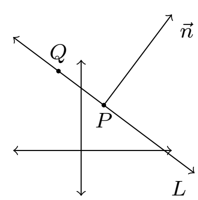
1. Compute $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$. How are they related?
2. Compute $\vec{u}\cdot(\vec{u}\times\vec{v})$ and $\vec{v}\cdot(\vec{u}\times\vec{v})$. Why did you get the answer you got?
3. Compute $\vec{u}\times(2\vec{u})$ using the formula for the cross product. Then explain, using an area argument, why you got $(0,0,0)$.

Problem 2.25 Let $P=(2,0,0)$, $Q=(0,3,0)$, and $R=(0,0,4)$. Find a vector that orthogonal to both $\vec{PQ}$ and $\vec{PR}$. Then find the area of the triangle PQR. Construct a 3D graph of this triangle.

See 12.4: 15-18. Remember, the magnitude of the cross product gives the area of the parallelogram formed using the two vectors as the edges.

We will now combine the dot product with the cross product to develop an equation of a plane in 3D. Before doing so, let's look at what information we need to obtain a line in 2D, and a plane in 3D. To obtain a line in 2D, one way is to have 2 points. The next problem introduces the new idea by showing you how to find an equation of a line in 2D.

Problem 2.26 Suppose the point $P=(1,2)$ lies on line L. Suppose that the angle between the line and the vector $\vec{n}=\langle 3,4\rangle$ is 90° (whenever this happens we say the vector $\vec{n}$ is normal to the line). Let $Q=(x,y)$ be another point on the line L. Use the fact that $\vec{n}$ is orthogonal to $\vec{PQ}$, together with the dot product, to obtain an equation of the line L.



Problem 2.27 Let $P=(a,b,c)$ be a point on a plane in 3D. Let $\vec{n}=(A,B,C)$ be a normal vector to the plane (so the angle between the plane and and $\vec{n}$ is 90°). Let $Q=(x,y,z)$ be another point on the plane. Show that an equation of the plane through point P with normal vector $\vec{n}$ is

See page 709.

$$A(x-a)+B(y-b)+C(z-c)=0.$$

Problem 2.28 Consider the three points $P=(1,0,0),Q=(2,0,-1),R=(0,1,3)$. Find an equation of the plane which passes through these three points. [Hint: first find a normal vector to the plane.]

See 12.5: 21-28.

Problem 2.29 Consider the two planes $x+2y+3z=4$ and $2x-y+z=0$. These planes meet in a line. Find a vector that is parallel to this line. Then find a vector equation of the line.

See 12.5: 57-60.

Problem 2.30 Find an equation of the plane containing the lines $\vec{r}_1(t)=(1,3,0)t+(1,0,2)$ and $\vec{r}_2(t)=(2,0,-1)t+(2,3,2)$.

Problem 2.31 Consider the points $P = (2, -1, 0)$, $Q = (0, 2, 3)$, and $R = (-1, 2, -4)$.

1. Give a vector equation $(x, y, z) = (?, ?, ?)$ of the line through P and Q.
2. Give a vector equation of the line through P and R.
3. Give an equation of the plane through P, Q, and R.

Problem 2.32 Consider $P = (2, 4, 5)$, $Q = (1, 5, 7)$, and $R = (-1, 6, 8)$.

1. What is the area of the triangle PQR.
2. Give a normal vector to the plane through these three points.
3. What is the distance from the point $A = (1, 2, 3)$ to the plane PQR. [Hint: What does the projection of $\vec{PA}$ onto $\vec{n}$ have to do with this problem?]

2.4 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 3

Curves

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Be able to graph and give equations of parabolas, ellipses, and hyperbolas.
2. Use a change-of-coordinates involving translation and stretching to give an equation of and graph a curve.
3. Model motion in the plane using parametric equations.
4. Find derivatives and tangent lines for parametric equations. Explain how to find velocity, speed, and acceleration from parametric equations.
5. Use integrals to find the length of a parametric curve and related quantities.

You'll have a chance to teach your examples to your peers prior to the exam.

3.1 Creating Good Graphs in the Plane

Before we jump fully into $\mathbb{R}^3$, we need some good examples of planar curves (curves in $\mathbb{R}^2$) that we'll extend to objects in 3D. For now, we'll focus on parabolas, circles, ellipses, and hyperbolas. We need to become comfortable drawing these graphs, as well as translating, streching (rescaling), and reflecting them about lines.

Given a graph of a function $y = f(x)$, how do we modify the equation $y = f(x)$ to obtain a new function that has been shifted? You might recall several rules that allow you to translate functions left and right, up and down, or even rescale (stretch) the functions vertically and horizontally. For example, if we start with the parabola $y = x^2$, then the equation $y = (x - 2)^2 + 3$, or equivalently $y - 3 = (x - 2)^2$, is the same parabola except we have shifted it right 2 and up 3.

In this section, we'll revisit the concepts of translating and stretching functions. All of these ideas are part of a bigger picture which we'll refer to as changing coordinates. In the example above we had two curves, namely $y = x^2$ and the translated $y - 3 = (x - 2)^2$. To simplify our work, let's use the variables u and v for the starting equation and x and y for the translated equation. Notice then that we have $v = u^2$ and $y - 3 = (x - 2)^2$. If we just let $v = y - 3$ and $u = x - 2$, or equivalently $x = u + 2$ and $y = v + 3$, then we have equations that allow us to change between uv and xy coordinatse. We call each pair of equations a change-of-coordinates. We'll often write our changes of coordinates

In practice, we generally don't use new variables but might instead write the change-of-coordinates as $x_n = x_o + 2$ and $y_n = y_o + 3$ where n stands for "new" and o stands for "old". After making the change, we just drop subscripts.

by solving for x and y, as the equations $x = u + 2$ and $y = v + 3$ clearly show us that the x-values should be the old u-values shifted 2 units right and the y-values should be the old v-values shifted 3 units up.

Problem 3.1 Consider the circle $u^2 + v^2 = 1$ and the change-of-coordinates given by $x = 2u + 1$ and $y = 3v + 4$. If you didn't read the paragraphs above this problem, please do so before you start working on this problem.

1. Draw the curve $u^2 + v^2 = 1$ in the uv plane.

2. The change of coordinates give above allows us to construct a graph of this curve in the xy plane. Once simple way to do this is make a u, v, x, y table. We know the circle above passes through the points $(\pm 1, 0)$ and $(0, \pm 1)$, so we can use the change of coordinate equations $x = 2u + 1$ and $y = 3v + 4$ to find the corresponding points in the xy plane, as seen on the right. Use this table to construct a graph of the curve in the xy plane.

(u, v)	(x, y)
$(1, 0)$	$(3, 4)$
$(-1, 0)$	$(-1, 4)$
$(0, 1)$	$(?, ?)$
$(0, -1)$	$(?, ?)$

3. Solve the change-of-coordinate equations for u and v and use substitution to give an equation of the curve using x and y coordinates.

4. Use the same change-of-coordinates with the curve $v = u^2$ to graph the curve in both the uv and xy plane. Then state an equation of the curve in the x and y coordinates. You may find the table to the right helpful.

(u, v)	(x, y)
$(-2, 4)$	$(-3, 16)$
$(-1, 1)$	$(?, ?)$
$(0, 0)$	$(?, ?)$
$(1, 1)$	$(3, 7)$
$(2, 4)$	$(?, ?)$

5. How would you describe the connection between the graphs you made in the uv plane and their corresponding graph in the xy plane?

In the previous problem you were given a curve using uv coordinates, and then asked to use a change-of-coordinates to construct a graph in the xy plane. The next problem has you do this in reverse, namely gives you curve in the xy plane and asks you to state the change of coordinates that would reduce the curve to a simple object in the uv plane.

Problem 3.2 Start by graphing the parabola $y = 3(x - 1)^2 + 2$.

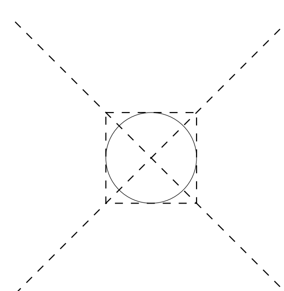
1. Give a change-of-coordinates of the form $x = ?u + ?$, $y = ?v + ?$ that will transform the curve $v = u^2$ in the uv plane to the parabola $y = 3(x-1)^2+2$.

2. Which of $y = 3(x - 1)^2 + 2$ or $\frac{y-2}{3} = (x - 1)^2$ makes it easier to see the change of coordinates?

3. Construct a graph of the parabola $\frac{y+1}{2} = \left(\frac{x-3}{4}\right)^2$. Optionally, state the change-of-coordinates you used.

Problem 3.3 Consider the curve $x^2 - y^2 = 1$, which we call a hyperbola.

1. Show that $y = \pm x\sqrt{1 - \frac{1}{x^2}}$, and then use this fact to explain why y approaches the lines $y = \pm x$ as x gets large. We call these two lines the asymptotes of the hyperbola, and any good graph of a hyperbola should include them.



2. We'll now construct a graph of the hyperbola. One simple way to draw the asymptotes is to start by constructing a rectangular box with corners at $(1, \pm 1)$ and $(-1, \pm 1)$. Connecting opposing corners of this box gives

the asymptotes $y = \pm x$. The circle $x^2 + y^2 = 1$ should fit nicely inside your box (see the picture on the right). Now use software to view a graph of the hyperbola $x^2 - y^2 = 1$ and add it to your picture, making sure the hyperbola follows the asymptotes as $|x|$ gets large. When you construct your graph on your paper, make sure your sketch includes the box, lines, and circle, as well as the hyperbola.

3. Now construct a graph of $\frac{(x-1)^2}{4} - \frac{(y-4)^2}{9} = 1$, including an appropriate box and asymptotes. If you want to find the box easily, start by drawing the ellipse $\frac{(x-1)^2}{4} + \frac{(y-4)^2}{9} = 1$, and then add the box, the asymptotes, and finally the hyperbola.

Problem 3.4 The equation $4x^2 + 4y^2 + 6x - 8y - 1 = 0$ represents a circle (though initially it does not look like it). Use the method of completing the square to rewrite the equation in the form $(x - a)^2 + (y - b)^2 = r^2$ (hence telling you the center and radius). Then generalize your work to find the center and radius of any circle written in the form $x^2 + y^2 + Dx + Ey + F = 0$.

Problem 3.5 Consider the parabola $v = u^2$ and the hyperbola $u^2 - v^2 = 1$. With each problem below, please make a u, v, x, y table before constructing your graph.

1. Using the change of coordinates $x = v$, $y = u$, draw the corresponding parabola and hyperbola in the xy-plane.

2. Using the change of coordinates $x = 2v + 1$, $y = 3u + 4$, draw the corresponding parabola in the xy-plane.

3. Draw both the ellipse $\frac{(y-4)^2}{9} + \frac{(x-1)^2}{4} = 1$ and hyperbola $\frac{(y-4)^2}{9} - \frac{(x-1)^2}{4} = 1$ in the xy-plane.

Problem 3.6 Consider the change of coordiantes $x = au + h$, $y = bv + k$.

1. Use this change of coordinates to rewrite the parabola $v = u^2$, the ellipse $u^2 + v^2 = 1$, and the hyperbola $u^2 - v^2 = 1$ using xy coordinates.

2. In your own words, how do each of the values of a, b, h, and k, change the graph of the curve in the uv plane when you draw the graph in the xy plane. Include pictures to accompany your words.

Problem 3.7 Graph each of the four ellipses below by hand. Be prepared to explain how you obtained the graph. See 11.6: 17-24.

1. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

2. $16x^2 + 25y^2 = 400$ [Hint: divide by 400.]

3. $\frac{(x-1)^2}{5} + \frac{(y-2)^2}{9} = 1.$

4. $x^2 + 2x + 2y^2 - 8y = 9$ (You'll need to complete the square.)

Problem 3.8 Graph each of the four hyperbolas below by hand. Make sure your graph shows the hyperbola's asymptotes.

See 11.6: 27-34.

1. $\frac{x^2}{25} - \frac{y^2}{9} = 1$ and $\frac{y^2}{9} - \frac{x^2}{25} = 1$
2. $25y^2 - 16x^2 = 400$ [Hint: divide by 400.]
3. $\frac{(x-1)^2}{5} - \frac{(y-2)^2}{9} = 1.$
4. $x^2 + 2x - 2y^2 + 8y = 9$ (You'll need to complete the square.)

Problem 3.9 Consider the hyperbola $\frac{(x-1)^2}{5} - \frac{(y-2)^2}{9} = 1$ from the previous problem. Use Mathematica and the ContourPlot[] command to produce a nice plot with reasonable bounds. Then add to your plot the asymptotes, using a different color. Your final plot should include both the hyperbola and the asymptotes in the same plot.

See 11.6: 27-34.

If you are struggling with getting the graphs to show up in the same plot, try using the Show[] command to combine several plots. Look up Show[] in the help menu, and you'll see several examples of how to combine several plots into one. Then you can make one plot for each curve, pick the color you want for that plot, and finish by combining all the plots with Show[].

If you need help changing the color, open the help menu for ContourPlot[]. Scroll to the bottom of the examples and expand the "Options" section. There are several options that have Color in the name, and Contour in the name. You want to change the style of the Contour, so expand the "ContourStyle" option. From there, look for an example that you like.

3.2 Parametric Equations

In middle school, you learned to write an equation of a line as $y = mx + b$. In the vector unit, we learned to write this in vector form as $(x, y) = (1, m)t + (0, b)$. The latter equation we call a vector equation. Equivalently we can write the two equations

$$x = 1t + 0, y = mt + b,$$

which we call parametric equations for the line. We were able to quickly develop equations of lines in space, by just adding a third equation for z.

Parametric equations provide us with a way of specifying the location (x, y, z) of an object by giving an equation for each coordinate. We will use these equations to model motion in the plane and in space. In this section we'll focus mostly on planar curves.

Definition 3.1. If each of f and g are continuous functions, then the curve in the plane defined by $x = f(t), y = g(t)$ is called a parametric curve, and the equations $x = f(t), y = g(t)$ are called parametric equations for the curve. You can generalize this definition to 3D and beyond by just adding more variables.

Problem 3.10 By plotting points, construct graphs of the three parametric curves given below (just make a t, x, y table, and then plot the (x, y) coordinates). Place an arrow on your graph to show the direction of motion.

See 11.1: 1-18. This is the same for all the problems below.

1. $x = \cos t, y = \sin t$, for $0 \le t \le 2\pi$.

2. $x = \sin t, y = \cos t$, for $0 \leq t \leq 2\pi$.
3. $x = \cos t, y = \sin t, z = t$, for $0 \leq t \leq 4\pi$. You'll need an x, y, z, t table. Plot your points (x, y, z) in 3D.
4. Now use Mathematica to plot these curves. Use the ParametricPlot[] command for the first two, and ParametricPlot3D[] for the last.

Problem 3.11 Plot the path traced out by the parametric curve $x = 1 + 2\cos t, y = 3 + 5\sin t$. Then use the trig identity $\cos^2 t + \sin^2 t = 1$ to give a Cartesian equation of the curve (an equation that only involves x and y).

Problem 3.12 Find parametric equations for a line that passes through the points $(0, 1, 2)$ and $(3, -2, 4)$.

What we did in the previous chapter should help here.

Problem 3.13 Plot the path traced out by the parametric curve $\vec{r}(t) = (t^2 + 1, 2t - 3)$. Give a Cartesian equation of the curve (eliminate the parameter t).

Problem 3.14 Consider the parametric curve given by $x = \tan t, y = \sec t$. Plot the curve for $-\pi/2 < t < \pi/2$. Give a Cartesian equation of the curve. (A trig identity will help - what identity involves both tangent and secant?) [Hint: this problem will probably be easier to draw if you first find the Cartesian equation, and then plot the curve.]

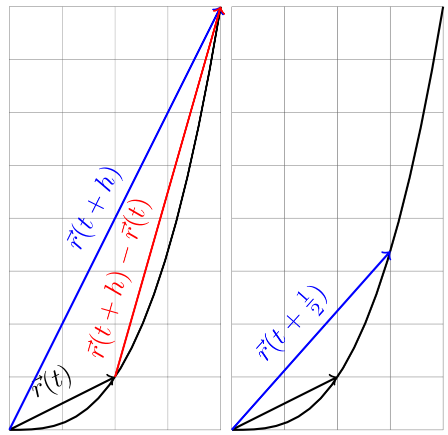
3.2.1 Derivatives and Tangent lines

We're now ready to discuss calculus on parametric curves. The derivative of a vector valued function is defined using the same definition as first semester calculus.

Definition 3.2. If $\vec{r}(t)$ is a vector equation of a curve (or in parametric form just $x = f(t), y = g(t)$), then we define the derivative to be

$$\frac{d\vec{r}}{dt} = \lim_{h \to 0} \frac{\vec{r}(t + h) - \vec{r}(t)}{h}.$$

Problem 3.15 Consider the curve $\vec{r}(t) = (2t, t^3)$. We'll analyze this curve at $t = 1$, where $\vec{r}(1) = (2, 1)$. When $h = 1$, we have $\vec{r}(t + h) = \vec{r}(2) = (4, 8)$ and the difference quotient $\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$ equals the difference $\vec{r}(2) - \vec{r}(1)$ and simply connects the heads of these two vectors, as shown below on the left.



1. The picture above on the right shows $\vec{r}(t)$ and $\vec{r}(t+h)$ when $t = 1$ and $h = 1/2$. Add to this picture the difference $\vec{r}(t+h)-\vec{r}(t)$ and the difference quotient $\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$.
2. Leaving $t = 1$ but changing h to $h = 1/4$ and then $h = 1/8$, construct a third and fourth picture that shows $\vec{r}(t), \vec{r}(t+h)$, the difference, and the difference quotient.
3. Letting $t = 1$, as $h \to 0$ what happens to $\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$? Draw this vector.

The previous problem gave a geometric intuition of the derivative, and emphasizes why the derivative is tangent to a curve. The following problem will provide a simple way to compute derivatives.

Problem 3.16 Let $\vec{r}(t) = (f(t), g(t))$. Show that $\frac{d\vec{r}}{dt} = (f'(t), g'(t))$. See page 728.

[The definition of the derivative is $\frac{d\vec{r}}{dt} = \lim_{h\to 0} \dfrac{\vec{r}(t+h)-\vec{r}(t)}{h}$. We were told $\vec{r}(t) = (f(t), g(t))$, so use this in the derivative definition. Perform the vector arithmetic componentwise, and you should obtain $\frac{d\vec{r}}{dt} = (f'(t), g'(t))$.]

The previous problem shows you can take the derivative of a vector valued function by just differentiating each component separately. The next problem shows you that velocity and acceleration are still connected to the first and second derivatives.

Problem 3.17 Consider the parametric curve given by $\vec{r}(t) = (3\cos t, 3\sin t)$. See 13.1:5-8 and 13.1:19-20

1. Graph the curve $\vec{r}$, and compute $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$.
2. On your graph, draw the vectors $\frac{d\vec{r}}{dt}\left(\frac{\pi}{4}\right)$ and $\frac{d^2\vec{r}}{dt^2}\left(\frac{\pi}{4}\right)$ with their tail placed on the curve at $\vec{r}\left(\frac{\pi}{4}\right)$. These vectors represent the velocity and acceleration vectors.
3. Give a vector equation of the tangent line to this curve at $t = \frac{\pi}{4}$. (You know a point and a direction vector.)

Definition 3.3. If an object moves along a path $\vec{r}(t)$, we can find the velocity and acceleration by just computing the first and second derivatives. The velocity is $\frac{d\vec{r}}{dt}$, and the acceleration is $\frac{d^2\vec{r}}{dt^2}$. Speed is a scalar, not a vector. The speed of an object is just the length of the velocity vector.

Problem 3.18 Consider the curve $\vec{r}(t) = (2t + 3, 4(2t - 1)^2)$.

1. Construct a graph of $\vec{r}$ for $0 \leq t \leq 2$.
2. If this curve represented the path of a horse running through a pasture, find the velocity of the horse at any time t, and then specifically at $t = 1$. What is the horse's speed at $t = 1$?
3. Give a vector equation of the tangent line to $\vec{r}$ at $t = 1$. Include this on your graph.
4. Explain how to obtain the slope of the tangent line, and then write an equation of the tangent line using point-slope form. [Hint: How can you turn the direction vector, which involves (dx/dt) and (dy/dt), into the number given by the slope (dy/dx)?]

Problem 3.19 Suppose an object travels along the path given by $\vec{r}(t) = (3t, -2t^2)$. The velocity is $\vec{v}(t) = (3, -4t)$ and the acceleration is $\vec{a}(t) = (0, -4)$. At time $t = 1$, these vectors are $\vec{v}(1) = (3, -4)$ and $\vec{a}(1) = (0, -4)$.

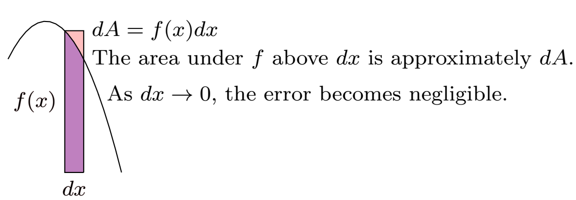
1. Why do we know that the acceleration and velocity vectors are not in the same direction?
2. What is the vector component of the acceleration vector that points in the same direction as the velocity vector? In other words, what is $\text{proj}_{\vec{v}}\vec{a}$. We'll call this vector $\vec{a}_{\parallel \vec{v}}$.
3. What is the vector component of the acceleration vector that is orthogonal to the velocity vector? We'll call this vector $\vec{a}_{\perp \vec{v}}$.
4. Draw a picture that shows the relationship among $\vec{v}$, $\vec{a}$, $\vec{a}_{\parallel \vec{v}}$, and $\vec{a}_{\perp \vec{v}}$.

3.2.2 Integration, Arc Length, and More

In this section, we will develop ways to integrate along paths. Everything in this section is a generalization of integration from first semester calculus. Try the following exercise whose solution is provided in the footnotes.

Exercise Consider a function $y = f(x)$ for $a \leq x \leq b$ and assume that $f(x) \geq 0$. Imagine cutting the x-axis up into many little bits, where we use dx to represent the length of each little bit. See [1] for a solution.

[1] The quantity $dA = f(x)dx$ is the area of a rectangle whose base is dx wide and whose height is $f(x)$. If dx is really small, then the function f is almost constant, so $f(x)$ and $f(x + dx)$ are really close. The little bit of area dA is extremely close the actual area under f that lies above the x axis between x and $x + dx$, off by the small amount of the rectangle that lies above the curve as shown below. This extra area becomes negligible as $dx \to 0$.



To find the total area under the curve, all we have to do is add up the little bits of area. In terms of Riemann sums, we would write $\sum dA$. The integral symbol just means that we're letting $dx \to 0$, and so the total area is found using $A = \int dA$. To obtain the total area, we just add up the little bits of area. When we replace dA with $f(x)dx$, we put the bounds $x = a$ to $x = b$ on the integral to obtain $A = \int dA = \int_a^b f(x)dx$.

1. If we pick one of the tiny bits of length dx whose left endpoint is located at x, what does the quantity $dA = f(x)dx$ give us? Construct a picture to illustrate this.
2. Why is the total area given by $A = \int_a^b f(x)dx$.

If an object moves at a constant speed, then the distance traveled is

$$\text{distance} = \text{speed} \times \text{time}.$$

This requires that the speed be constant. What if the speed is not constant? Over a really small time interval dt, the speed is almost constant, so we can still use the idea above. The following problem will help you develop the key formula for arc length.

Problem 3.20: Derivation of the arc length formula Suppose an object moves along the path given by $\vec{r}(t) = (x(t), y(t))$ for $a \leq t \leq b$. We know that the velocity is $\frac{d\vec{r}}{dt}$, and so the speed is just the magnitude of this vector.

1. Show that we can write the object's speed at any time t as $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.
2. If you move at constant speed $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ for a time length dt, what's the distance ds you have traveled.
3. Explain why the length of the path given by $\vec{r}(t)$ for $a \leq t \leq b$ is

$$s = \int ds = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

This is the arc length formula. Ask me in class for an alternate way to derive this formula.

Problem: Alternate derivation of arc length formula Suppose an object moves along the path given by $\vec{r}(t) = (x(t), y(t))$ for $a \leq t \leq b$. Imagine slicing the path up into hundreds of tiny slices. Let ds represent the length of each tiny slice.

1. Draw an appropriate diagram showing an arbitrary curve, a tiny chunk of the curve of length ds, and a triangle so that the Pythagorean theorem gives the approximation $ds = \sqrt{(dx)^2 + (dy)^2}$.
2. Use algebra to show that $\sqrt{(dx)^2 + (dy)^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$.
3. Explain why the length of the path given by $\vec{r}(t)$ for $a \leq t \leq b$ is

$$s = \int ds = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Now that we have a formula for computing arc length, let's practice using it with a few problems. First, we'll have you actually evaluate an integral. Next, we'll walk through setting up a block of code to do the same thing in Mathematica. Then, we'll have you set up several more integrals to find the arc length of several curves. You'll find that arc length problems can become quite messy and sometimes impossible to compute exactly because of the square root term in the integrand.

Problem 3.21 Find the length of the curve $\vec{r}(t) = \left(t^3, \dfrac{3t^2}{2}\right)$ for $t \in [1,3]$. The notation $t \in [1,3]$ means $1 \leq t \leq 3$. Be prepared to show us your integration steps in class (you'll need a substitution).

See 11.2: 25-30

Problem 3.22 Now let's use the parameterization from the previous problem to write a block of code in Mathematica to compute the arc length of a parameterized curve. We'll use the previous problem as a test problem.

1. First, define a vector function in Mathematica to represent the parameterized curve $\vec{r}(t) = \left(t^3, \dfrac{3t^2}{2}\right)$. In addition, define some variables to hold the upper and lower limits for the parameter t (i.e., $a = 1$ and $b = 3$).
2. Add a line to your block of code that uses ParametricPlot[] to create a graph of the function. This verifies that the function is defined correctly.
3. Using the vector function and limits you defined, add another line to your block of code to set up and evaluate an integral that will compute the path length of the curve. Use the derivative function in the integrand where necessary. *Hint: you may have to use a square root and a dot product to find the magnitude of a vector function.*
4. Copy the block of code that you created, then change the interval of integration to $2 \leq t \leq 5$.
5. Finally, copy your block of code one more time and use it to compute the length of the curve given by $x = \cos t + t \sin t, y = \sin t - t \cos t$, for $0 \leq t \leq 4\pi$. This curve, called the *involute* of the circle, is the path you would trace if you were skating around a barrel of radius 1 while holding taut a string that was initially wound around the barrel.

For more, visit http://mathworld.wolfram.com/Involute.html to see an animation of an involute of a circle, as well as more details.

Problem 3.23 For each curve below, set up an integral formula which would give the length, and sketch the curve. Do not worry about integrating them.

1. The parabola $\vec{r}(t) = (t, t^2)$ for $t \in [0,3]$.
2. The ellipse $\vec{r}(t) = (4\cos t, 5\sin t)$ for $t \in [0, 2\pi]$.
3. The hyperbola $\vec{r}(t) = (\tan t, \sec t)$ for $t \in [-\pi/4, \pi/4]$.

The reason I don't want you to actually compute the integrals is that they will get ugly really fast. Try doing one in Wolfram Alpha and see what the computer gives.

To actually compute the integrals above and find the lengths, we would use a numerical technique to approximate the integral (something akin to adding up the areas of lots and lots of rectangles as you did in first semester calculus).

Let's finish this chapter with some examples that illustrate how the arc length formula gives us much more than just length. This first problem comes from physics and asks you to find the total charge on a rod if you know the charge per length. The same type of problem shows up in engineering as finding the total mass of wire whose density (mass per length) is known. Since we know that density is mass per length, then all we have to do is times density by length to obtain the mass.

Problem 3.24 A wire lies along the curve $\vec{r}(t) = (7\cos t, 7\sin t)$ for $0 \leq t \leq \pi$. The wire contains charged particles where the charge per unit length at location (x, y) is given by $q(x, y) = y$. In this problem we'll compute the total charge on the wire.

If the wire were a conductor, then the charged particles (electrons) would not stay put, but rather flow freely along the wire until the repulsive forces are minimized. This wire is an insulator.

1. Why is the charge over a small distance ds approximately given by $dQ = q(x, y)ds$?

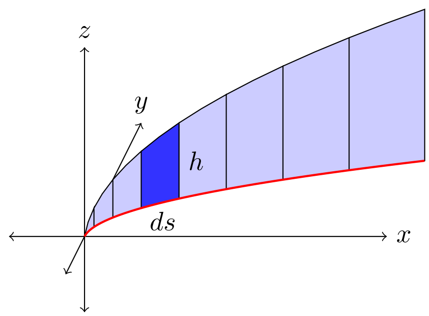
2. The total charge is the sum of the charges over all the little pieces on the rod. This gives us the total charge as

$$Q_{\text{total}} = \int_C dQ = \int_C q(x, y)ds = \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt.$$

   Replace x and y with what they are in terms of t and then finish by computing the integral above.

We can also use the arc length formula to find the surface area of some types of surfaces. Expanding this idea, we could use the formula developed in the next problem to compute the charge on a surface, the mass of a surface, and much more. For now, let's just compute the surface area.

Problem 3.25 A metal sheet lies above the parabola $\vec{r}(t) = (t^2, t)$ for $0 \leq t \leq 2$. Above the point (x, y), the height of the metal sheet is $h(x, y) = y$. The picture below shows the sheet, sliced into 8 bits.



1. If we slice the surface into many tiny vertical strips with base length ds, explain why the surface area of each vertical strip is approximately $d\sigma = h(x, y)ds$.

2. The total surface area is the sum of the surface areas over all the vertical strips. This gives us the total surface area as

$$\sigma = \int_C d\sigma = \int_C h(x, y)ds = \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt.$$

   Replace x and y with what they are in terms of t and then finish by computing the integral above.

3.3 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Exam 1 Review

At the end of each chapter, the following words appeared.

> Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam.

I've summarized the objectives from each chapter below. For our in class review, please come to class with examples to help illustrate each idea below. You'll get a chance to teach another member of class the examples you prepared. If you keep the examples simple, you'll have time to review each key idea.

Review

1. Give a summary of the ideas you learned in 112, including graphing, derivatives (product, quotient, power, chain, trig, exponential, and logarithm rules), and integration (u-sub and integration by parts).
2. Compute the differential dy of a function and use it to approximate the change in a function.
3. Explain how to perform matrix multiplication and compute determinants of square matrices.
4. Illustrate how to solve systems of linear equations, including how to express a solution parametrically (in terms of t) when there are infinitely solutions.

Vectors

1. Define, draw, and explain what a vector is in 2 and 3 dimensions.
2. Add, subtract, and multiply (scalar, dot product, cross product) vectors. Be able to illustrate each operation geometrically, where possible.
3. Use vector products to find angles, length, area, projections, and work.
4. Use vectors to give equations of lines and planes, and be able to draw lines and planes in 3D.

Curves

1. Be able to graph and give equations of parabolas, ellipses, and hyperbolas.
2. Use a change-of-coordinates involving translation and stretching to give an equation of and graph a curve.
3. Model motion in the plane using parametric equations.
4. Find derivatives and tangent lines for parametric equations. Explain how to find velocity, speed, and acceleration from parametric equations.
5. Use integrals to find the length of a parametric curve and related quantities.

Chapter 4

New Coordinates

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Use a change-of-coordinates to convert between rectangular and another coordinate system. In particular, be able to convert points and equations between rectangular and polar coordinates.
2. Graph polar functions $r = f(\theta)$ in the xy plane, and set up the arc length formula to find their length.
3. Given a change-of-coordinates, find the differentials dx and dy and write them in both vector and matrix form. Use these to compute tangent vectors, slope $\frac{dy}{dx}$, and equations of tangent lines.
4. Compute double integrals to find the area of regions in the xy plane, and use the determinant to explain how area between different coordinate systems is related.
5. Shade regions in the plane bounded by $\alpha \leq \theta \leq \beta$ and $r_1(\theta) \leq r \leq r_2(\theta)$, and use double integrals to compute their area.

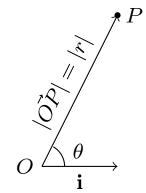
You'll have a chance to teach your examples to your peers prior to the exam.

4.1 Polar Coordinates

Up to now, we most often give the location of a point (or coordinates of a vector) by stating the (x, y) coordinates. These are called the Cartesian (or rectangular) coordinates. Some problems are much easier to work with if we know how far a point is from the origin, together with the angle between the x-axis and a ray from the origin to the point.

Problem 4.1 There are two parts to this problem. See 11.3:5-10.

1. Consider the point P with Cartesian (rectangular) coordinates $(2, 1)$. Find the distance r from P to the origin. Consider the ray $\vec{OP}$ from the origin through P. Find an angle between $\vec{OP}$ and the x-axis.
2. Given a generic point $P = (x, y)$ in the plane, write a formula to find the distance r from P to the origin (in terms of x and y) as well as a formula to find the angle θ between the vector $(1, 0)$ (the positive x-axis) and the vector from the origin to P. [Hint: A picture of a triangle will help here.]



Definition 4.1. Let P be be a point in the plane with Cartesian coordinates (x, y). Let $O = (0, 0)$ be the origin. We say that (r, θ) is a polar coordinates of P if (1) we have $|\vec{OP}| = |r|$, and (2) the angle between $\mathbf{i} = (1, 0)$ and $\vec{OP}$ is θ, or coterminal with θ.

Problem 4.2 The following points are given using polar coordinates. Plot the points in the Cartesian plane, and give the Cartesian (rectangular) coordinates of each point. The points are

See 11.3:5-10.

$$(r, \theta) = (1, \pi), \left(6, \frac{\pi}{4}\right), \left(-3, \frac{\pi}{4}\right), \left(3, \frac{5\pi}{4}\right), \text{ and } \left(-2, -\frac{\pi}{6}\right).$$

Finish by explaining why a general formula for x and y if we know a point has polar coordinates (r, θ) is $x = r\cos\theta$ and $y = r\sin\theta$.

See page 647.

The equations above, namely

$$x = r\cos\theta, \quad y = r\sin\theta$$

are a typical example of what we call a change-of-coordinates. We've seen that these equations allow us transfer points back and forth between Cartesian coordinates and polar coordinates. We can also use this change-of-coordinates to transfer equations back and forth between coordinate system. The next two problems have you do this.

Problem 4.3 Each of the following equations is written in the Cartesian (rectangular) coordinate system. Convert each to an equation in polar coordinates, and then solve for r so that the equation is in the form $r = f(\theta)$. You'll want to use the change-of-coordinates to replace any x and y you see so that it is in terms of r and θ.

See 11.3: 53-66.

1. $x^2 + y^2 = 7$
2. $2x + 3y = 5$
3. $x^2 = y$

Problem 4.4 Each of the following equations is written using polar coordinates. Convert each to an equation in using Cartesian coordinates (sometimes called rectangular coordinates). You'll want to use the change-of-coordinates to replace any r and θ you see so that it is in terms of x and y.

See 11.3: 27-52. I strongly suggest that you do many of these as practice.

1. $r = 9\cos\theta$
2. $r = \dfrac{4}{2\cos\theta + 3\sin\theta}$
3. $\theta = 3\pi/4$

We've been writing the change-of-coordinates by listing the two equations $x = r\cos\theta$, $y = r\sin\theta$. We can also write this in vector notation as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}.$$

This is a vector equation in which you input polar coordinates (r, θ) and get out Cartesian coordinates (x, y). So you input one thing to get out one thing, which means that we have a function. We could also write $\vec{T}(r, \theta) = (r\cos\theta, r\sin\theta)$, where we've used the letter T as the name for the function because the function is a $t$ransformation between coordinate systems.

To emphasize that the domain and range are both two dimensional systems, we could also write $\vec{T} : \mathbb{R}^2 \to \mathbb{R}^2$. In the next chapter, we'll spend more time with this notation. The following problem will show you one way to graph a change-of-coordinates, or coordinate transformation. When you're done, you should essentially have polar graph paper.

Problem 4.5 Consider the polar coordinate transformation

For this problem, you are just drawing many parametric curves. This is what we did in the previous chapter.

$$\vec{T}(r, \theta) = (r\cos\theta, r\sin\theta).$$

1. Let $r = 3$ and then graph $\vec{T}(3, \theta) = (3\cos\theta, 3\sin\theta)$ for $\theta \in [0, 2\pi]$ in the xy plane. Remember, the notation $\theta \in [0, 2\pi]$ just means $0 \le \theta \le 2\pi$. If you get a circle, you're doing this right.
2. Let $\theta = \frac{\pi}{4}$ and then, on the same axes as above, add the graph of $\vec{T}\left(r, \frac{\pi}{4}\right) = \left(r\frac{\sqrt{2}}{2}, r\frac{\sqrt{2}}{2}\right)$ for $r \in [0, 5]$.
3. To the same axes as above, add the graphs of $\vec{T}(1, \theta), \vec{T}(2, \theta), \vec{T}(4, \theta)$ for $\theta \in [0, 2\pi]$ and $\vec{T}(r, 0), \vec{T}(r, \pi/2), \vec{T}(r, 3\pi/4), \vec{T}(r, \pi)$ for $r \in [0, 5]$.

If you ended up circles and rays, then you're doing this correctly. Congrats, you just drew a four dimensional graph (we'll talk more about this in class).

Make sure you ask me in class to show you the corresponding graph in the $r\theta$ plane, or come to class with it drawn and ready to share.

Problem 4.6 We have two equations $x = r\cos\theta$ and $y = r\sin\theta$. Suppose that a point is moving through space and x, y, r, θ all depend on time t.

1. Explain why $\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}$. Obtain a similar equation for $\frac{dy}{dt}$.

Hint: Use implicit differentiation.

2. We can obtain the differential dx and dy in terms of r, θ, dr, and $d\theta$ if we multiply through by dt. This gives $dx = \cos\theta dr - r\sin\theta d\theta$ and $dy =?$. Write you answer as the vector equation
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta \\ ? \end{pmatrix} dr + \begin{pmatrix} -r\sin\theta \\ ? \end{pmatrix} d\theta.$$
3. Find a 2 by 2 matrix so that we can write the above vector equation as the matrix equation
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}.$$

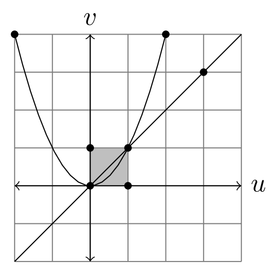
The vector equation above is the sum of vectors times scalars. Matrix multiplication was invented to abbreviate this type of sums. The vector are placed in the columns of the matrix, and the scalars are placed in a column vector to the right of the matrix.

Let's try the last two problems with a different change-of-coordinates, of the form $x = au + bv, y = cu + dv$. Any change of coordinates of this form we call a linear change-of-coordinates. You should see that lines map to lines in your work below.

Problem 4.7 Consider the change-of-coordinates $x = u-v$, $y = u+v$, which we could also write as the coordinate transformation $\vec{T}(u,v) = (u-v, u+v)$.

1. In the table below, you're given several (u,v) points. Find the corresponding (x,y) pair.

(u,v)	(x,y)
$(0,0)$	$(0,0)$
$(1,0)$	$(1-0,1+0)=(1,1)$
$(0,1)$	$(0-1,0+1)=(-1,1)$
$(1,1)$	
$(3,3)$	
$(2,4)$	
$(-2,4)$	



2. In the graph above is a plot of the points from the table, graphed in the uv plane. In addition, we see the parabola $v = u^2$, the line $v = u$, and the shaded box whose corners are the first few points. Construct a plot (please make a grid) in the xy planes that contains the points from above. Connect the points in your xy plot to show how the parabola, line, and shaded box transform because of this change-of-coordinates.

Problem 4.8 Consider the change-of-coordinates from the problem above, namely $x = u-v$, $y = u+v$, or equivalently $\vec{T}(u,v) = (u-v, u+v)$.

1. If we assume x, y, u, v are all functions of t, we can compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Do so and then multiply your equations on both sides by dt to obtain the differentials dx and dy. Write your answer as the vector equation
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} du + \begin{pmatrix} ? \\ ? \end{pmatrix} dv.$$

2. Find a 2 by 2 matrix so that we can write the above vector equation as the matrix equation
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

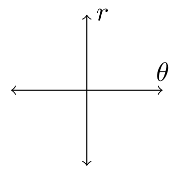
3. If we use the change-of-coordinates $x = 2u + 3v$, $y = 4u + 5v$, then find the differential dx and dy and write your answer as both
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} du + \begin{pmatrix} ? \\ ? \end{pmatrix} dv \quad \text{and} \quad \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

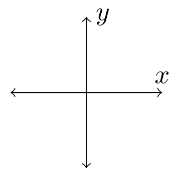
4.2 Graphing Transformed Equations

You've spent a lot of time in your past graphing equations of the form $y = f(x)$. Let's now graph equations of the form $r = f(\theta)$ in the xy plane.

Problem 4.9 In the θr plane, graph the curve $r = \sin\theta$ for $\theta \in [0, 2\pi]$ (make a table where you pick several values for θ and then compute r). Then graph the curve $r = \sin\theta$ for $\theta \in [0, 2\pi]$ in the xy plane (add to your table the corresponding x and y values). The graphs should look very different. If one looks like a sine wave, and the other looks like a circle, you're on the right track. Here's the start of a table to help you, as well as the axes you'll need to put your graphs on.

θ	r	$x = r\cos\theta$	$y = r\sin\theta$
0	$\sin(0) = 0$	0	0
$\frac{\pi}{6}$	$\sin\frac{\pi}{6} = \frac{1}{2}$	$\frac{1}{2}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{4}$	$\frac{1}{2}\sin\frac{\pi}{6} = \frac{1}{4}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}\cos\frac{\pi}{4} = \frac{1}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$			
$\frac{\pi}{2}$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$





In general, to construct a graph of a polar curve in the xy plane, we create an r, θ table. We choose values for θ that will make it easy to compute any trig functions involved. If you need to, add x and y to your table before plotting the location of the polar point in the xy plane. Then connect the points in a smooth manner, making sure that your radius grows or shrinks appropriately as your angle increases. Ask me in class to show you some animations of this, or you can see these animations before class if you open up the Mathematica Technology Introduction.

Problem 4.10 Graph the polar curve $r = 2 + 2\cos\theta$ in the xy plane. See 11.4: 1-20.

Problem 4.11 Graph the polar curve $r = 2\sin 3\theta$ in the xy plane. [Hint: You'll want to chose values for theta so that 3θ hits all multiples of ninety degrees, the places where r attains it maximums and minimums.]

Problem 4.12: Mathematica Problem In this problem we'll use Mathematica to plot the polar curve $r = a\cos(n\theta)$ for various values of a and n.

1. Use the command PolarPlot[] to plot the curve $r = 3\cos 2\theta$ for $0 \leq \theta \leq 2\pi$.
2. Use the command ParametricPlot[] to plot the curve $r = 3\cos 2\theta$ for the same bounds. We know that $x = r\cos\theta$ and $y = r\sin\theta$, so you just need to plot $\vec{r}(t) = \langle(3\cos 2\theta)\cos\theta, (3\cos 2\theta)\sin\theta\rangle$.
3. Use your code above to graph $r = 3\cos(n\theta)$ for $0 \leq \theta \leq 2\pi$ for each integer n from 2 to 8. What patterns do you see? Make a conjecture and then plug in higher values for n to see if you are correct.
4. With software you can quickly change parts of a function to see how they affect behavior. In the function $r = a\cos(n\theta)$, how does the graph change if instead of having $a = 3$ you pick a to be another number? What happens if you pick n to be something other than an integer? What happens if you change cos to sin?

4.3 Calculus with Change-of-Coordinates

Problem 4.13 We saw in some previous problems that we can express the differential dx and dy as the matrix product

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}.$$

1. Use the matrix equation above to compute $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of r and $\frac{dr}{d\theta}$, if we assume that r is a function of θ.

Hint: Just multiply everything out and divide by $d\theta$.

2. Explain why the slope of a tangent line in the xy plane to the curve $r = f(\theta)$ is
$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

For parametric curves $\vec{r}(t) = (x(t), y(t))$, to find the slope of the curve we just compute
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$
A polar curve of the form $r = f(\theta)$ is just the parametric curve $(x, y) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. The previous problem showed us that we can find the slope by computing
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Problem 4.14 Consider the polar curve $r = 1 + 2\cos\theta$, graphed in the xy plane. (It wouldn't hurt to provide a quick sketch of the curve.)

See 11.2: 1-14.

1. Compute both $dx/d\theta$ and $dy/d\theta$.
2. Find the slope dy/dx of the curve at $\theta = \pi/2$.
3. Give both a vector equation of the tangent line, and a Cartesian equation of the tangent line at $\theta = \pi/2$.

Problem 4.15 Consider the parabola $v = u^2$ and the change-of-coordinates $x = 2u + v$, $y = u - 2v$.

1. Construct a graph of the parabola in the xy plane.
2. Compute both dx/du and dy/du. Then find the slope dy/dx of the parabola at $u = 1$.
3. Give both a vector equation of the tangent line, and a Cartesian equation of the tangent line at $u = 1$.

We showed in the curves section that you can find the arc length for a parametric curve by using the formula
$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$
If we replace t with θ, this becomes a formula for the arc length of a curve given in polar coordinates.

Problem 4.16 Set up (do not evaluate) an integral formula to compute each of the following (draw the curve to be sure your bounds are correct - getting the right bounds is perhaps the toughest part of this problem.):

See 11.5: 21-28.

1. The length of one petal of the rose $r = 3\cos 2\theta$.
2. The length of the entire rose $r = 2\sin 3\theta$.

We've now seen one example of how we can use a change-of-coordinates to compute an integral, namely to find arc length. You've actually been using a change-of-coordinates since first semester calculus, every time you performed a substitution to complete an integral. The next problem has you revisit this, and notice something crucial about differentials.

Problem 4.17 Consider the integral $\displaystyle\int_{-1}^{2} e^{-3x}\,dx$.

1. To complete this integral we use the substitution $u = -3x$. Solve for x and compute the differential dx.
2. Now perform the substitution, filling in the missing parts of
$$\int_{x=-1}^{x=2} e^{-3x}\,dx = \int_{u=?}^{u=?} e^{u}?du.$$
To find the u bounds, just ask, "If $x = -1$, then $u =$?" Don't spend any time completing the integral, rather just focus on completing the substitution above.
3. The x values range from -1 to 2. This is an interval whose width is 3 units along the x-axis. Our substitution $u = -3x$ gives us an interval along the u-axis. How long is this interval, and what does your differential equation $dx = -\frac{1}{3}du$ have to do with this?
4. The substitution $u = -3x$ is a one-dimensional change-of-coordinates. We can write the differential $dx = -\frac{1}{3}du$ in the matrix form
$$\begin{pmatrix} dx \end{pmatrix} = \begin{bmatrix} -\frac{1}{3} \end{bmatrix} \begin{pmatrix} du \end{pmatrix}.$$
We have not defined the determinant of a 1 by 1 matrix. What would you define the determinant of a 1 by 1 matrix to be, and why?

Note: When a definite integral ends with du, the bounds should be in terms of u. Many of you have always ignored this step, and instead would first compute $\int e^u?du$ without bounds, replace u with $-3x$, and then finish. We need the approach on the left in high dimensions.

We've now seen that the differential equation $dx = \frac{dx}{du}du$ tells us how to relate lengths along the u-axis to lengths along the x-axis. The next two problems have you focus on how a two dimensional change-of-coordinates helps us connect areas in the uv plane to areas in the xy plane.

Problem 4.18 Consider the change-of-coordinates $x = 2u$, $y = 3v$.

1. The lines $u = 0, u = 1, u = 2$ and $v = 0, v = 1, v = 2$ correspond to lines in the xy plane. Draw these lines in the xy-plane. [Hint: One option is to find the xy coordinates of the (u, v) points $(0, 0)$, $(0, 1)$, $(0, 2)$ and connect the dots to make a line. Then repeat with the (u, v) coordinates $(1, 0)$, $(1, 1)$, $(1, 2)$ and draw another line. Eventually you'll have a grid.]
2. The box in the uv plane with $0 \le u \le 1$ and $1 \le v \le 2$ should correspond to a box in the xy plane. Draw and shade this box in the xy plane and find its area.

3. Compute the differentials dx and dy. State these differentials using both the vector and matrix forms

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} du + \begin{pmatrix} ? \\ ? \end{pmatrix} dv \quad \text{and} \quad \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

4. What's the determinant of the matrix above, and what does the determinant have to do with your picture?

5. Consider the box given by $-1 \leq u \leq 1$ and $-1 \leq v \leq 1$. State the area of this box in both the uv plane and the xy plane.

6. Consider the circle $u^2 + v^2 = 1$. The area inside this circle in the uv plane is $A = \pi$. Guess the area inside the corresponding ellipse in the xy plane.

Problem 4.19 Consider the change-of-coordinates $x = 2u + v$, $y = u - 2v$.

1. The lines $u = 0, u = 1, u = 2$ and $v = 0, v = 1, v = 2$ correspond to lines in the xy plane. Draw these lines in the xy-plane. [Hint: One option is to find the xy coordinates of the (u, v) points $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, etc., and then just connect the dots to make a rotated grid.]

2. The box in the uv plane with $0 \leq u \leq 1$ and $1 \leq v \leq 2$ should correspond to a parallelogram in the xy plane. Shade this parallelogram in your picture above and find the area of the parallelogram.

3. Compute the differentials dx and dy. State these differentials using the vector form

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} du + \begin{pmatrix} ? \\ ? \end{pmatrix} dv$$

What do the two vectors above have to do with your picture?

4. Write the differentials above in the matrix form

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

What's the determinant of the matrix above, and what does the determinant have to do with your picture?

The next problems have you analyze the integrals $\int_C dx$ and $\int_C dy$, and from them develop a way to compute area using double integrals.

Problem 4.20 Consider the ellipse given by the vector equation $\vec{r}(t) = (3\cos t, 4\sin t)$ or the parametric equations $x = 3\cos t$ and $y = 4\sin t$.

1. Start by drawing the curve and computing the differentials dx and dy.

2. The integral $\int_C dx$ adds up little changes in x. Adding up lots of little changes in x gives a total change in x. Verify this by computing $\displaystyle\int_{t=0}^{t=\pi/2} dx$ and comparing your result the physical change in x from $t = 0$ to $t = \pi/2$.

3. Compute $\displaystyle\int_{t=0}^{t=\pi/2} dy$. Explain how you could obtain this answer without doing any integration.

4. For each interval $[a,b]$ given below, give the value of both $\displaystyle\int_{t=a}^{t=b} dx$ and $\displaystyle\int_{t=a}^{t=b} dy$ by connecting the integral to your graph in part a.

$[a,b]$	$\displaystyle\int_{t=a}^{t=b} dx$	$\displaystyle\int_{t=a}^{t=b} dy$
$[0,\pi]$		
$[\pi/2,\pi]$		
$[0,3\pi/2]$		
$[\pi,2\pi]$		

Problem 4.21 Consider the region R between the functions $y=x^2$ and $y=-x$ for $0\leq x\leq 3$. Draw both functions and shade the region R. Your goal in this problem is to explain why the iterated integral $\displaystyle\int_{x=0}^{x=3}\left(\int_{y=-x}^{y=x^2} dy\right)dx$ gives the area of the region R.

1. Pick a value of x, such as $x=2$. The inner integral $\displaystyle\int_{y=-x}^{y=x^2} dy$ adds up little changes in y for that specific x value. Compute this integral when $x=2$ (so $\int_{y=-2}^{y=4} dy$) to verify that you get a total change in y of 6 units, the vertical distance between the two points $(2,-2)$ and $(2,4)$. Draw a vertical line segment inside your region that connects these two points. Then repeat this for various other values of x, adding appropriate segments.

2. Compute the integral $\displaystyle\int_{y=-x}^{y=x^2} dy$ for arbitrary x. Then explain what physical quantity this integral measures.

3. Recall that dx is a small width. When we multiply the previous integral by this width dx, we will obtain the area of a small region. Construct a new picture that includes the original region R together with the small region whose area is given by the product $\displaystyle\left(\int_{y=-x}^{y=x^2} dy\right)dx$.

4. Explain why $\displaystyle\int_{x=0}^{x=2}\left(\int_{y=-x}^{y=x^2} dy\right)dx$ gives the area of the region R.

Problem 4.22 Consider the double integral

$$\int_{y=-1}^{y=2}\left(\int_{x=y^2}^{x=y+2} dx\right)dy.$$

1. The bounds in the integral above describe a region in xy plane where $-1\leq y\leq 2$ and $y^2\leq x\leq y+2$. Sketch this region.

2. Consider the inner integral $\int_{x=y^2}^{x=y+2} dx$. What physical quantity does this integral compute? Add to your sketch several line segments whose widths are given by this integral.

3. When we multiply a width $\int_{x=y^2}^{x=y+2} dx$ by a small height dy, we get a little bit of area dA. Pick a value y between -1 and 2, and then at that height draw a small rectangle whose area is given by $dA = \left(\int_{x=y^2}^{x=y+2} dx\right) dy$.

4. Adding up little bits of area gives total area, so the double integral at the start of this problem gives an area. Compute the integral.

Problem 4.23 The double integral $\displaystyle\int_{x=a}^{x=b}\left(\int_{y=g(x)}^{y=f(x)} dy\right) dx$ computes the area of a region in the xy plane that you should be quite familiar with. Compute the inner integral $\int_{y=g(x)}^{y=f(x)} dy$ to obtain the single variable formula you should be more familiar with. Provide a sketch of the region, using some specific functions to illustrate this abstract idea.

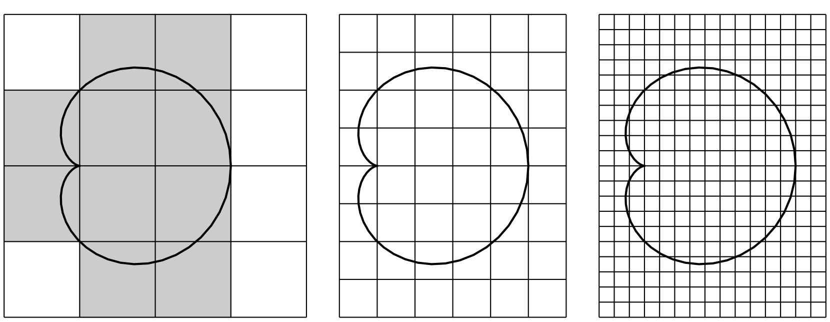
Since area is a two dimensional quantity, a double integral provides a natural way to compute the area. The above problems have shown that the area A of a region R can be found by adding up little bits of area using any of

$$A = \int_{x=a}^{x=b}\left(\int_{y=g(x)}^{y=f(x)} dy\right) dx = \int_{y=c}^{y=d}\left(\int_{x=a(y)}^{x=b(y)} dy\right) dx.$$

We call these iterated integrals, as we iteratively give the bounds for each variable. Notice that in each of the integrals above, we took a slice of the region, thickened it up to get a thin rectangle whose area was dA, and then found the area by adding up these thin rectangles.

Another way to compute the area of a region R is overlay the region with a rectangular grid, where dx and dy are the distances between the vertical and horizontal lines of the grid. To find the area of the region, we first determine which of the rectangles contains a portion of the region R, and then add up the areas of of all such rectangles. This will overestimate the area, but we then use limits to shrink both dx and dy to zero to obtain the area.

Problem 4.24 Consider the polar curve $r = 1 + \cos\theta$. We will use the approach described above this problem to estimate the area or region R that is inside this polar curve. The bounds for each graph below are $-1 \le x \le 2$ and $-2 \le y \le 2$. To present this problem in class, please print this page so you can appropriate shade things as asked below.



$dx = dy = 1$ $\quad$ $dx = dy = .5$ $\quad$ $dx = dy = .2$

1. For the first picture above, there are 10 rectangles (shaded) that contain a portion of the region R. Each of these rectangles has area $dA = dxdy = (1)(1) =$, which means an overestimate for the area of R is $A \approx 10\, dA = 10$. Describe a way to use these same rectangles to get an underestimate for the area of R.

2. Now use the middle picture above (where $dx = dy = .5$) to shade and then count the number of rectangles that contain a portion of R. What is the area dA of each little rectangle. Finish by giving an estimate for A.

3. Now use the last picture with $dx = dy = .2$ to estimate the area of R.

4. How can we obtain the exact value for the area of R?

We've been considering double integrals of the form

$$\int_{x=a}^{x=b}\left(\int_{y=g(x)}^{y=f(x)} dy\right) dx \quad \text{and} \quad \int_{y=c}^{y=d}\left(\int_{x=a(y)}^{x=b(y)} dy\right) dx.$$

These integrals give us the area of a region R in the (x, y) plane. Setting up the bounds for these integrals requires being able to describe the bounds of the region using inequalities of the form $a \leq x \leq b$ and $g(x) \leq y \leq f(x)$, or of the form $c \leq y \leq d$ and $a(y) \leq y \leq b(y)$. This can become a problem if the region is not easily described using rectangular coordinates.

Problem 4.25 Shade the region in the xy plane described by each set of inequalities.

1. $0 \leq \theta \leq \pi/4$ and $0 \leq r \leq 4$
2. $0 \leq x \leq 3$ and $0 \leq y \leq \sqrt{9 - x^2}$
3. $-\pi/6 \leq \theta \leq \pi/6$ and $0 \leq r \leq 2\cos 3\theta$
4. $0 \leq \theta \leq 2\pi$ and $2 \leq r \leq 5 + 2\cos\theta$

Our goal now is to learn how to use double integrals to compute area if the region is easily described using polar coordinates instead of rectangular coordinates. Basically, we need to perform a substitution from (x, y) to (r, θ) coordinates. Earlier we saw that for the change-of-coordinates $x = 2u + v$, $y = u - 2v$, we can write the differentials dx and dy in the matrix form

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

The determinant of the matrix above, namely 5, gave us the scale factor that connected areas in the xy plane to areas in the uv plane. A rectangle with width du and height dv in the uv plane would have an aread 5 times larger when transformed to the xy plane. We can write this as $dA_{xy} = 5dudv$. The next problem has you repeat this process with polar coordinates.

Problem 4.26 Consider the change-of-coordinates $x = r\cos\theta$, $y = r\sin\theta$.

1. The lines $r = 1, r = 2, r = 3$ and $\theta = 0, \theta = \frac{\pi}{6}, \theta = \frac{\pi}{3}$ correspond to circles and lines in the xy plane. Draw these circles and lines in the xy-plane. The box in the $r\theta$ plane with $2 \leq r \leq 3$ and $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$ corresponds to a region in the xy plane. Shade this region in the xy plane.

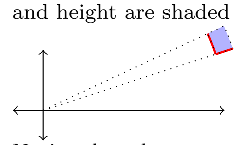
2. Compute the differentials dx and dy. State these differentials using the vector form
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} dr + \begin{pmatrix} ? \\ ? \end{pmatrix} d\theta.$$

3. Write the differentials above (at arbitrary r and θ) in the matrix form
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}.$$
Compute the determinant of the matrix above. Make a guess about what the determinant has to do with your picture.

Did you obtain r as the determinant of the matrix in the last step above? This means that a little rectange in the $r\theta$ plane will have its area increased by a scale factor of r when transforming the region to the xy plane. We can express this as $dA_{xy} = rdrd\theta$, or just $dA = rdrd\theta$. The next problem has you give a geometric proof of the same fact.

Problem 4.27 Let (r, θ) be an arbitray point. Our goal is to develop a formula for the area of the region R in the xy plane where the radius ranges from r to $r + dr$ and the angle ranges from θ to $\theta + d\theta$, shown in the diagram to the right. Copy a similar diagram on to your paper and then do the following.

A small polar rectancle, when transformed into the xy plane, looks like a rectangle whose width and height are shaded red below.

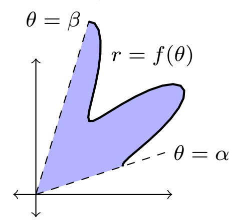


Notice that the rectangle's area will increase as r increases.

1. Add the labels r, θ, dr, $d\theta$, $r + dr$, and $\theta + d\theta$ to appropriate places in your diagram.
2. The shaded region is approximately a rectangle. Explain why the area of this rectangle is $dA = rdrd\theta$ by first finding the width and height.

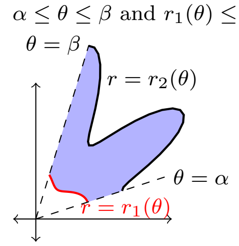
Problem 4.28 Consider the region R in the xy plane bounded by and $\alpha \leq \theta \leq \beta$ and $0 \leq r \leq f(\theta)$.

Here's a typical region with $\alpha \leq \theta \leq \beta$ and $0 \leq r \leq f(\theta)$.



1. The area of a region R in the xy plane can be found using the double integral $A = \int\int_R dA$. If it's easy to describe the bounds using rectangular coordinates, then we can use either $A = \int_a^b \left(\int_{g(x)}^{f(x)} dy\right) dx$ or $A = \int_c^d \left(\int_{a(y)}^{b(y)} dy\right) dx$. However, if the bounds for a region in the xy plane are given by the polar inequalities $\alpha \leq \theta \leq \beta$ and $0 \leq r \leq f(\theta)$, explain why the area of the region in the xy plane is
$$A = \int_\alpha^\beta \int_0^{f(\theta)} rdrd\theta.$$
2. Now consider the region R bounded by $\alpha \leq \theta \leq \beta$ and $r_1(\theta) \leq r \leq r_2(\theta)$, as show in the diagram to the right. Set up a double integral that would give the area of this region R.

Here's a typical region with $\alpha \leq \theta \leq \beta$ and $r_1(\theta) \leq r \leq r_2(\theta)$.



Let's use what we have just developed to examine several polar integrals.

Problem 4.29 Complete both parts below.

1. Draw the region in the xy plane whose area is given by the polar integral
$$\int_0^{3\pi/2} \int_1^{3+3\cos\theta} rdrd\theta.$$
2. Set up a double integral to find the area in the xy plane that is inside one petal of the curve $r = 3\sin 2\theta$.

Problem 4.30 Find the area of the region enclosed by the positive x-axis and the spiral $r = 4\theta/3$ for $0 \leq \theta \leq 2\pi$. The region looks like a snail shell.

Problem 4.31 Find the area enclosed by one leaf of the rose $r = 5\cos 3\theta$ (a sketch may help you define limits for θ). Compute the integral by hand.

You may need the power reduction formula $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$.

For the remainder of the semester, any time an integral involves a power reduction formula, you may use software to finish the integral.

Problem 4.32 For each region R described below, start by drawing the region. Then set up a formula involving interated integral to find the area of R.

1. R is inside the cardioid $r = 1 + \cos\theta$ but outside the circle $r = 1$.
2. R is inside both the circles $r = \cos\theta$ and $r = \sin\theta$.
3. R is inside the circle $r = 5\cos\theta$ but to the right of the line $r = 3\sec\theta$.

4.4 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 5

Functions

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Describe uses for, and construct graphs of, space curves and parametric surfaces. Find derivatives of space curves, and use this to find velocity, acceleration, and find equations of tangent lines.

2. Describe uses for, and construct graphs of, functions of several variables. For functions of the form $z = f(x, y)$, this includes both 3D surface plots and 2D level curve plots. For functions of the form $w = f(x, y, z)$, construct plots of level surfaces.

3. Describe uses for, and construct graphs of, vector fields and transformations. Develop the formulas for cylindrical and spherical coordinates.

4. If you are given a description of a vector field, curve, or surface (instead of a function or parametrization), explain how to obtain a function for the vector field, or a parametrization for the curve or surface.

You'll have a chance to teach your examples to your peers prior to the exam.

Most of the work in this chapter requires graphing. You'll find many links throughout this chapter that point to SageMath and/or WolframAlpha plots (see sagemath.org for more information about SageMath). Alternatively, you can use this Mathematica Technology introduction to have technology create plots for you. Please use technology to check how you are doing.

5.1 Function Terminology

A function is a set of instructions involving two sets (called the domain and codomain). A function assigns to each element of the domain D exactly one element in the codomain R. We'll often refer to the codomain R as the target space. We'll write

$$f : D \to R$$

when we want to remind ourselves of the domain and target space. In this class, we will study what happens when the domain and target space are subsets of $\mathbb{R}^n$ (Euclidean n-space). In particular, we will study functions of the form

$$f : \mathbb{R}^n \to \mathbb{R}^m,$$

when m and n are 3 or less. The value of n is the dimension of the input vector (or number of inputs). The number m is the dimension of the output vector (or number of outputs). Our goal is to understand uses for each type of function, and be able to construct graphs to represent the function.

We will focus most of our time this semester on two- and three-dimensional problems. However, many problems in the real world require a higher number of dimensions. When you hear the word "dimension", it does not always represent a physical dimension, such as length, width, or height. If a quantity depends on 30 different measurements, then the problem involves 30 dimensions. As a quick illustration, the formula for the distance between two points depends on 6 numbers, so distance is really a 6-dimensional problem. As another example, if a piece of equipment has a color, temperature, age, and cost, we can think of that piece of equipment being represented by a point in four-dimensional space (where the coordinate axes represent color, temperature, age, and cost).

Problem 5.1 A pebble falls from a 64 ft tall building. Its height (in ft) above the ground t seconds after it drops is given by the function $y = f(t) = 64 - 16t^2$. What are n and m when we write this function in the form $f\colon \mathbb{R}^n \to \mathbb{R}^m$? Construct a graph of this function. How many dimensions do you need to graph this function?

See Sage or Wolfram Alpha. Follow the links to Sage or Wolfram Alpha in all the problems below to see how to get the computer to graph the function.

5.2 Parametric Curves: $\vec{f}\colon \mathbb{R} \to \mathbb{R}^m$

Problem 5.2 A horse runs around an elliptical track. Its position at time t is given by the function $\vec{r}(t) = (2\cos t, 3\sin t)$. We could alternatively write this as $x = 2\cos t, y = 3\sin t$.

See Sage or Wolfram Alpha. See also Chapter 3 of this problem set. There's a lot more practice of this idea in 11.1. You'll also find more practice in 13.1: 1-8.

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. Construct a graph of this function.
3. Next to a few points on your graph, include the time t at which the horse is at this point on the graph. Include an arrow for the horse's direction.
4. How many dimensions do you need to graph this function?

Notice in the problem above that we placed a vector symbol above the function name, as in $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$. When the target space (codomain) is 2-dimensional or larger, we place a vector above the function name to remind us that the output is more than just a number.

Problem 5.3 Consider the pebble from problem 5.1. The pebble's height was given by $y = 64 - 16t^2$. The pebble also has some horizontal velocity (it's moving at 3 ft/s to the right). If we let the origin be the base of the 64 ft building, then the position of the pebble at time t is given by $\vec{r}(t) = (3t, 64 - 16t^2)$.

See Sage or Wolfram Alpha. The text has more practice in 13.1: 1-8.

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. At what time does the pebble hit the ground (the height reaches zero)? Construct a graph of the pebble's path from when it leaves the top of the building till when it hits the ground.
3. Find the pebble's velocity and acceleration vectors at $t = 1$? Draw these vectors on your graph with their base at the pebble's position at $t = 1$.

See Section 3.2.1 and Definition 3.3.

4. At what speed is the pebble moving when it hits the ground?

In the next problem, we keep the input as just a single number t, but the output is now a vector in $\mathbb{R}^3$.

Problem 5.4 A jet begins spiraling upwards to gain height. The position of the jet after t seconds is modeled by the equation $\vec{r}(t) = (2\cos t, 2\sin t, t)$. We could alternatively write this as $x = 2\cos t$, $y = 2\sin t$, $z = t$.

See Sage or Wolfram Alpha. The text has more practice in 13.1: 9-14.

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. Construct a graph of this function by picking several values of t and plotting the resulting points $(2\cos t, 2\sin t, t)$.
3. Next to a few points on your graph, include the time t at which the jet is at this point on the graph. Include an arrow for the jet's direction.
4. How many dimensions do you need to graph this function?

In all the problems above, you should have noticed that in order to draw a function (provided you include arrows for direction, or use an animation to represent "time"), you can determine how many dimensions you need to graph a function by just summing the dimensions of the domain and codomain. This is true in general.

Problem 5.5 Use the same set up as problem 5.4, namely

See Section 3.2.1 and Definition 3.3.
The text has more practice in 13.1: 19-22.

$$\vec{r}(t) = (2\cos t, 2\sin t, t).$$

You'll need a graph of this function to complete this problem.

1. Find the first and second derivative of $\vec{r}(t)$.
2. Compute the velocity and acceleration vectors at $t = \pi/2$. Place these vectors on your graph with their tails at the point corresponding to $t = \pi/2$.
3. Give an equation of the tangent line to this curve at $t = \pi/2$.

5.3 Parametric Surfaces: $\vec{f}\colon \mathbb{R}^2 \to \mathbb{R}^3$

We now increase the number of inputs from 1 to 2. This will allow us to graph many space curves at the same time.

Problem 5.6 The jet from problem 5.4 is actually accompanied by several jets flying side by side. As all the jets fly, they leave a smoke trail behind them (it's an air show). The smoke from each jet spreads outwards to mix together, so that it looks like the jets are leaving wide sheet of smoke behind them as they spiral upwards. The position of two of the many other jets is given by $\vec{r}_3(t) = (3\cos t, 3\sin t, t)$ and $\vec{r}_4(t) = (4\cos t, 4\sin t, t)$. A function which represents the smoke stream from these jets is $\vec{r}(a,t) = (a\cos t, a\sin t, t)$ for $0 \le t \le 4\pi$ and $2 \le a \le 4$.

More practice in 16.5: 1-16.

1. What are n and m when we write the function $\vec{r}(a,t) = (a\cos t, a\sin t, t)$ in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?

2. Start by graphing the position of the three jets $\vec{r}(2,t) = (2\cos t, 2\sin t, t)$, $\vec{r}(3,t) = (3\cos t, 3\sin t, t)$ and $\vec{r}(4,t) = (4\cos t, 4\sin t, t)$.
3. Let $t = 0$ and graph the curve $r(a,0) = (a,0,0)$ for $a \in [2,4]$, which represents the segment along which the smoke has spread. Then repeat this for $t = \pi/2, \pi, 3\pi/2$.
4. Describe the resulting surface, and make sure you check your answer with technology (use the links to the side).

See Sage or Wolfram Alpha.

We call the surface you drew above a parametric surface. The vector equation describing the smoke screen is a parametrization of this surface.

Definition 5.1: Parametric Surface, Parametrization of a surface. A parametrization of a surface is a collection of three equations to tell us the position

$$x = x(u,v), y = y(u,v), z = z(u,v)$$

of a point (x,y,z) on the surface. We call u and v parameters, and these parameters give us a two dimensional pair (u,v), the input, needed to obtain a specific location (x,y,z), the output, on the surface. We can also write a parametrization in vector form as

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v)).$$

We'll often give bounds on the parameters u and v, which help us describe specific portions of the surface. A parametric surface is a surface together with a parametrization.

We draw parametric surfaces by joining together many parametric space curves, as done in the previous problem. Just pick one variable, hold it constant, and draw the resulting space curve. Repeat this several times, and you'll have a 3D surface plot. Most of 3D computer animation is done using parametric surfaces. Woody's entire body in *Toy Story* is a collection of parametric surfaces. Car companies create computer models of vehicles using parametric surfaces, and then use those parametric surfaces to study collisions. Often the mathematics behind these models is hidden in the software program, but parametric surfaces are at the heart of just about every 3D computer model.

Problem 5.7 Consider the parametric surface $\vec{r}(u,v) = (u\cos v, u\sin v, u^2)$ for $0 \le u \le 3$ and $0 \le v \le 2\pi$. Construct a graph of this function. Remember, to do so we just let u equal a constant (such as 1, 2, 3) and then graph the resulting space curve where we let v vary. After doing this for several values of u, swap and let v equal a constant (such as 0, $\pi/2$, etc.) and graph the resulting space curve as u varies. [Hint: Did you get a satellite dish? Use the software links to the right to make sure you did this right.]

See Sage or Wolfram Alpha.

5.4 Functions of Several Variables: $f\colon \mathbb{R}^n \to \mathbb{R}$

In this section we'll focus on functions where the output is a single number. These functions take the form $f\colon \mathbb{R}^2 \to \mathbb{R}^1$ and $f\colon \mathbb{R}^3 \to \mathbb{R}^1$. In the next problem, you should notice that the input is a vector (x,y) and the output is a number z. There are two common ways we graph functions of this type. The next two problems show you how.

Problem 5.8 A computer chip has been disconnected from electricity and sitting in cold storage for quite some time. The chip is connected to power, and a few moments later the temperature (in Celsius) at various points (x, y) on the chip is measured. From these measurements, statistics is used to create a temperature function $z = f(x, y)$ to model the temperature at any point on the chip. Suppose that this chip's temperature function is given by the equation $z = f(x, y) = 9 - x^2 - y^2$. We'll be creating a 3D model of this function in this problem, so you'll want to place all your graphs on the same x, y, z axes.

See Sage or Wolfram Alpha.

1. What is the temperature at $(0, 0)$, $(1, 2)$, and $(-4, 3)$?

   See 14.1: 1-4.

2. If you let $y = 0$, construct a graph of the temperature $z = f(x, 0) = 9 - x^2 - 0^2$, or just $z = 9 - x^2$. In the xz plane (where $y = 0$) draw this upside down parabola.
3. Now let $x = 0$. Draw the resulting parabola in the yz plane.
4. Now let $z = 0$. Draw the resulting curve in the xy plane.
5. Once you've drawn a curve in each of the three coordinate planes, it's useful to pick an input variable (either x or y) and let it equal various constants. So now let $x = 1$ and draw the resulting parabola in the plane $x = 1$. Then repeat this for $x = 2$.
6. Describe the shape. Add any extra features to your graph to convey the 3D image you are constructing.

   See 14.1: 37-48.

Problem 5.9 We'll be using the same function $z = f(x, y) = 9 - x^2 - y^2$ as the previous problem. However, this time we'll construct a graph of the function by only studying places where the temperature is constant. We'll create a graph in 2D of the surface (similar to a topographical map).

See Sage or Wolfram Alpha.

1. Which points in the plane have zero temperature? Just let $z = 0$ in $z = 9 - x^2 - y^2$. Plot the curve corresponding to these points in the xy-plane with the same temperature, and write $z = 0$ next to this curve. We call this curve a level curve. As long as you stay on this curve, your temperature will remain level; it will not increase nor decrease.

   See 14.1: 13-16 and 31-36, 37-48.

   Because the function here represents temperature, we can also call this curve an isotherm. If the function represented pressure, we'd call it an isobar. There are many names given to level curves. We'll use the words "level curve" throughout the semester rather than isotherm, isobar, isocline, etc.

2. Which points in the plane have temperature $z = 5$? Add this level curve to your 2D plot and write $z = 5$ next to it.
3. Repeat the above for $z = 8$, $z = 9$, and $z = 1$. What's wrong with letting $z = 10$?
4. Using your 2D plot, construct a 3D image of the function by lifting each level curve to its corresponding height.

Definition 5.2. A level curve of a function $z = f(x, y)$ is a curve in the xy-plane found by setting the output z equal to a constant. Symbolically, a level curve of $f(x, y)$ is the curve $c = f(x, y)$ for some constant c. A 2D plot consisting of several level curves is called a contour plot of $z = f(x, y)$.

Problem 5.10 Consider the function $f(x, y) = x - y^2$.

See Sage or Wolfram Alpha.More practice is in 14.1: 37-48.

1. Construct a 3D surface plot of f. [So just graph in 3D the curves given by $x = 0$ and $y = 0$ and then try setting x or y equal to some other constants, like $x = 1$, $x = 2$, $y = 1$, $y = 2$, etc.]

2. Construct a contour plot of f. [So just graph in 2D the curves given by setting z equal to a few constants, like $z = 0$, $z = 1$, $z = -4$, etc.]

3. Which level curve passes through the point $(2, 2)$? Draw this level curve on your contour plot.

See 14.1: 49-52.

Notice that when we graphed the previous two functions (of the form $z = f(x, y)$) we could either construct a 3D surface plot, or we could reduce the dimension by 1 and construct a 2D contour plot by letting the output z equal various constants. The next function is of the form $w = f(x, y, z)$, so it has 3 inputs and 1 output. We could write $f: \mathbb{R}^3 \to \mathbb{R}^1$. We would need 4 dimensions to graph this function, but graphing in 4D is not an easy task. Instead, we'll reduce the dimension and create plots in 3D to describe the level surfaces of the function.

Problem 5.11 Suppose that an explosion occurs at the origin $(0, 0, 0)$. Heat from the explosion starts to radiate outwards. Suppose that a few moments after the explosion, the temperature at any point in space is given by $w = T(x, y, z) = 100 - x^2 - y^2 - z^2$.

See Sage. Wolfram Alpha currently does not support drawing level surfaces. You could also use Mathematica or Wolfram Demonstrations.
You can access more problems on drawing level surfaces in 12.6:1-44 or 14.1:53-60.

1. Which points in space have a temperature of 99? To answer this, replace $T(x, y, z)$ by 99 to get $99 = 100 - x^2 - y^2 - z^2$. Use algebra to simplify this to $x^2 + y^2 + z^2 = 1$. Draw this object.

2. Which points in space have a temperature of 96? of 84? Draw the surfaces.

3. What is your temperature at $(3, 0, -4)$? Draw the level surface that passes through $(3, 0, -4)$.

4. The 4 surfaces you drew above are called level surfaces. If you walk along a level surface, what happens to your temperature?

5. As you move outwards, away from the origin, what happens to your temperature?

Problem 5.12 Consider the function $w = f(x, y, z) = x^2 + z^2$. This function has an input y, but notice that changing the input y does not change the output of the function.

See Sage.

1. Draw a graph of the level surface $w = 4$. [When $y = 0$ you can draw one curve. When $y = 1$, you should draw the same curve. When $y = 2$, again you draw the same curve. This kind of graph is called a cylinder, and is important in manufacturing where you extrude an object through a hole.]

2. Graph the surface $9 = x^2 + z^2$ (so the level surface $w = 9$).

3. Graph the surface $16 = x^2 + z^2$.

The examples I give you for functions of the form $w = f(x, y, z)$ we can draw by using our knowledge about conic sections. We can graph ellipses and hyperbolas if there are only two variables. So the key idea is to set one of the variables equal to a constant and then graph the resulting curve. Repeat this with a few variables and a few constants, and you'll know what the surface is. Sometimes when you set a specific variable equal to a constant, you'll get an ellipse. If this occurs, try setting that variable equal to other constants, as ellipses are generally the easiest curves to draw. If setting a variable equal to a contant gives you a hyperbola, try picking a different variable to set equal to a constant. It gets really messy to graph several hyperbolas on the same 3D axes by hand.

Problem 5.13 Consider the function $w = f(x, y, z) = x^2 - y^2 + z^2$.

See Sage. Remember you can find more practice in 12.6:1-44 or 14.1: 53-64.
We'll have a few people present this problem.

1. Draw a graph of the level surface $w = 1$. [You need to graph $1 = x^2 - y^2 + z^2$. Let $x = 0$ and draw the resulting curve. Then let $y = 0$ and draw the resulting curve. Let either x or y equal some more constants (whichever gave you an ellipse), and then draw the resulting ellipses.]

2. Graph the level surface $w = 4$. [Divide both sides by 4 (to get a 1 on the left) and then repeat the previous part.]

3. Graph the level surface $w = -1$. [Try dividing both sides by a number to get a 1 on the left. If $y = 0$ doesn't help, try $y = 1$ or $y = 2$.]

4. Graph the level surface that passes through the point $(3, 5, 4)$. [Hint: what is $f(3, 5, 4)$?]

5.4.1 Vector Fields and Transformations: $\vec{f}\colon \mathbb{R}^n \to \mathbb{R}^n$

We've covered the following types of functions in the problems above.

- $y = f(x)$ or $f\colon \mathbb{R} \to \mathbb{R}$ (functions of a single variable)
- $\vec{r}(t) = (x, y)$ or $f\colon \mathbb{R} \to \mathbb{R}^2$ (parametric curves)
- $\vec{r}(t) = (x, y, z)$ or $f\colon \mathbb{R} \to \mathbb{R}^3$ (space curves)
- $\vec{r}(u, v) = (x, y, z)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ (parametric surfaces)
- $z = f(x, y)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}$ (functions of two variables)
- $z = f(x, y, z)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}$ (functions of three variables)

In this class, we will ignore functions of the form $\vec{f}\colon \mathbb{R}^3 \to \mathbb{R}^2$, though one way to view these is to just create functions of the form $f_i\colon \mathbb{R}^3 \to \mathbb{R}$.

The only examples that remain are functions where the dimension of the input matches the dimension of the output. In our previous chapters we've look at two examples of this form, namely vector fields and coordinate transformations (change-of-coordinates). Let's finish this section by revisiting these two types of functions, namely

- $\vec{F}(x) = (M) = M\mathbf{i}$ or $f\colon \mathbb{R} \to \mathbb{R}$ (vector fields along a line)
- $\vec{F}(x, y) = (M, N)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (vector fields in the plane)
- $\vec{F}(x, y, z) = (M, N, P)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ (vector fields in space)
- $T(u) = x$ or $f\colon \mathbb{R} \to \mathbb{R}$ (1D change-of-coordinates)
- $\vec{T}(u, v) = (x, y)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (2D change-of-coordinates)
- $\vec{T}(u, v, w) = (x, y, z)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ (3D change-of-coordinates)

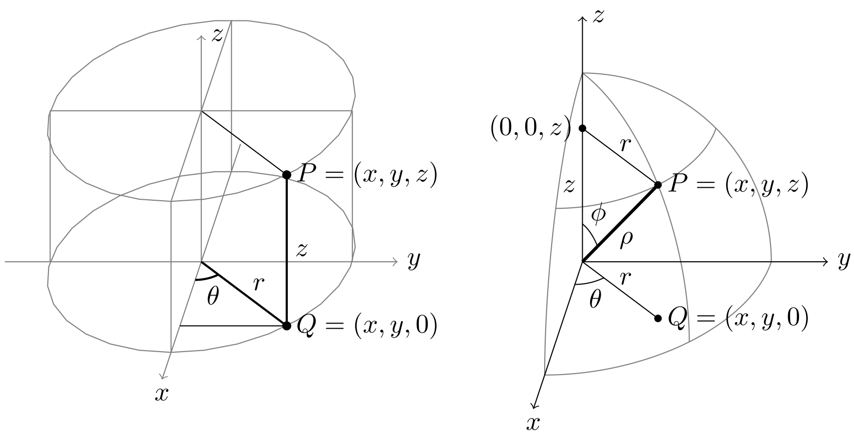


Figure 5.1: Cylindrical and spherical coordinates.

The difference between vector fields and transformations has to do with how we apply the function. Let's examine this difference first by considering change of coordinates in three dimensions.

The previous chapter focused quite a bit on how to work with a two-dimensional change-of-coordinates. In particular, we've already seen examples of coordinate transformations with polar coordinates. In three dimensions, some common coordinate systems are cylindrical and spherical coordinates. The equations for these coordinate systems are shown in the table below.

Cylindrical Coordinates	Spherical Coordinates
$x = r\cos\theta$ $y = r\sin\theta$ $z = z$	$x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$

The next two problems have you develop these equations, similar to the first few problems in the previous chapter.

Problem 5.14 Let $P = (x, y, z)$ be a point in space. This point lies on a cylinder of radius r, where the cylinder has the z axis as its axis of symmetry. The height of the point is z units up from the xy plane. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the ray $\vec{OQ}$ and the x-axis is θ. See Figure 5.1 for a picture. Use the graph and the information above to explain why the equations for cylindrical coordinates are See page 893.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Problem 5.15 Let $P = (x, y, z)$ be a point in space. This point lies on a sphere of radius ρ ("rho"), where the sphere's center is at the origin $O = (0, 0, 0)$. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the ray $\vec{OQ}$ and the x-axis is θ, and we call the azimuth angle. The angle between the ray $\vec{OP}$ and the z axis is ϕ ("phi"), and we call call the inclination angle, polar angle, or zenith angle. See Figure 5.1 for a picture. Use this information to develop the equations for spherical coordinates, in other words explain why See page 897.

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.$$

There is some disagreement between different scientific fields about the notation for spherical coordinates. In some fields (like physics), ϕ represents the azimuth angle and θ represents the inclination angle, swapped from what we see here. In some fields, like geography, instead of the inclination angle, the *elevation* angle is given — the angle from the xy-plane (lines of latitude are from the elevation angle). Additionally, sometimes the coordinates are written in a different order. You should always check the notation for spherical coordinates before communicating to others with them. As long as you have an agreed upon convention, it doesn't really matter how you denote them.

See the Wikipedia or MathWorld for a discussion of conventions in different disciplines.

Problem 5.16 Consider the spherical coordinates transformation

$$\vec{T}(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi).$$

Graphing this transformation requires $3+3 = 6$ dimensions. In this problem we'll construct parts of this graph by graphing various surfaces. We did something similar for the polar coordinate transformation in problem 4.5.

1. Let $\rho = 2$ and graph the resulting surface. What do you get if $\rho = 3$? See Sage or Wolfram Alpha.
2. Let $\phi = \pi/4$ and graph the resulting surface. What do you get if $\phi = \pi/2$? See Sage or Wolfram Alpha.
3. Let $\theta = \pi/4$ and graph the resulting surface. What do you get if $\theta = \pi/2$?

Let's now turn our focus to vector fields.

Problem 5.17 Consider the vector field $\vec{F}(x, y) = (2x + y, x + 2y)$. In this problem, you will construct a graph of this vector field by hand. We did something quite similar in Problem 2.10 on page 13.

See Sage or Wolfram Alpha. The computer will shrink the largest vector down in size so it does not overlap any of the others, and then reduce the size of all the vectors accordingly. See 16.2: 39-44 for more practice.

1. Compute $\vec{F}(1, 0)$. Then draw the vector $\vec{F}(1, 0)$ with its base at $(1, 0)$.
2. Compute $\vec{F}(1, 1)$. Then draw the vector $\vec{F}(1, 1)$ with its base at $(1, 1)$.
3. Repeat the above process for the points $(0, 1)$, $(-1, 1)$, $(-1, 0)$, $(-1, -1)$, $(0, -1)$, and $(1, -1)$. Remember, at each point draw a vector. When you finish, check your answer with software.

Problem 5.18: Spin field Consider the vector field $\vec{F}(x, y) = (-y, x)$. Construct a graph of this vector field. Remember, the key to plotting a vector field is "at the point (x, y), draw the vector $\vec{F}(x, y)$ with its base at (x, y)." Plot at least 8 vectors (a few in each quadrant), so we can see what this field is doing.

Use the links above to see the computer plot this. See 16.2: 39-44 for more practice.

Drawing 3D vector fields by hand can be tough, luckily Sage and Mathematica can help us visualize 3D vector fields. The sage example show a 3D visualization of the vector field $\vec{F}(x, y, z) = (y, z, x)$.

5.5 Constructing Functions

We now know how to draw a vector field provided someone tells us the equation. What we really need is to do the reverse. If we see vectors (forces, velocities, etc.) acting on something, how do we obtain an equation of the vector field? The spin field from the previous problem is directly related to the field you would need to understand the forces at play on a merry-go-round or carousel. The following problem will help you develop the gravitational vector field.

Problem 5.19: Radial fields Do the following:

Use Sage to plot your vector fields. See 16.2: 39-44 for more practice.

1. Let $P = (x, y, z)$ be a point in space. At the point P, let $\vec{F}(x, y, z)$ be the vector which points from P to the origin. Give a formula for $\vec{F}(x, y, z)$.
2. Give an equation of the vector field where at each point P in the plane, the vector $\vec{F}_2(P)$ is a unit vector that points towards the origin.
3. Give an equation of the vector field where at each point P in the plane, the vector $\vec{F}_3(P)$ is a vector of length 7 that points towards the origin.
4. Give an equation of the vector field where at each point P in the plane, the vector $\vec{G}(P)$ points towards the origin, and has a magnitude equal to $1/d^2$ where d is the distance to the origin.

If someone gives us parametric equations for a curve in the plane, we know how to draw the curve. Again, what we really need is the ability to go backwards. How do we obtain parametric equations of a curve that we can see? In problem 5.2, we were given the parametric equation for the path of a horse, namely $x = 2\cos t, y = 3\sin t$ or $\vec{r}(t) = (2\cos t, 3\sin t)$. From those equations, we drew the path of the horse, and could have written a Cartesian equation for the path. How do we work this in reverse, namely if we had only been given the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, could we have obtained parametric equations $\vec{r}(t) = (x(t), y(t))$ for the curve?

Problem 5.20 Give a parametrization of the top half (so $y \geq 0$) of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. You can write your parametrization in the vector form $\vec{r}(t) = (?, ?)$, or in the parametric form $x = ?,\ y = ?$, and you'll need to give bounds for t of the form $? \leq t \leq ?$ so that we only obtain the top half. [Hint: Read the paragraph above, and/or review Problem 5.2.]

Use Sage or Wolfram Alpha to visualize your parameterizations.

Problem 5.21 Give a parametrization of the straight line from $(a, 0)$ to $(0, b)$. You can write your parametrization in the vector form $\vec{r}(t) = (?, ?)$, or in the parametric form $x = ?,\ y = ?$. Remember to include bounds for t. [Hint: Review 2.9 and 3.12.]

We often use t as the parameter when writing equations for planar and space curves, because we'll often use the curve to describe the motion of an object as time elapses. You are welcome to use whatever variable you want for your parameter, such as x, y, z, θ, r, etc.

Problem 5.22 Give a parametrization ($\vec{r}(?) = (?, ?)$) of the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$. Remember the bounds for your parameter.

Problem 5.23 Give a parametrization of the function $y = f(x)$ for $x \in [a, b]$. You can write your parametrization in the vector form $\vec{r}(?) = (?, ?)$, or in the parametric form $x = ?,\ y = ?$. Include bounds for your parameter.

If someone gives us parametric equations for a surface, we can draw the surface. This is what we did in problems 5.6 and 5.7. How do we work backwards and obtain parametric equations for a given surface? This requires that we write an equation for x, y, and z in terms of two parameters, i.e. input variables (see 5.6 and 5.7 for examples). Using function notation, we need a function of the form $\vec{r}\colon \mathbb{R}^2 \to \mathbb{R}^3$.

Problem 5.24 Consider the surface $z = 9 - x^2 - y^2$ plotted in problem 5.8.

Use Sage or Wolfram Alpha to plot your parametrization. See 16.5: 1-16 for more practice.

1. Give a parametrization $\vec{r}\colon \mathbb{R}^2 \to \mathbb{R}^3$ of the surface. In other words, you'll need to give equations

$$x = ?, y = ?, z = ? \qquad \text{or} \qquad \vec{r}(?, ?) = (?, ?, ?).$$

   [Hint: You can use the parameters x and y to help you out. Then you just have $x = x$, $y = y$, and $z = ?$. This should be quite fast.]

2. What bounds must you place on x and y to obtain the portion of the surface above the plane $z = 0$?

3. If $z = f(x, y)$ is any surface, give a parametrization of the surface (i.e., $x = ?, y = ?, z = ?$ or $\vec{r}(?, ?) = (?, ?, ?)$.)

When a surface has a lot of symmetry, we can often use an appropriate coordinate transformation $\vec{T}\colon \mathbb{R}^3 \to \mathbb{R}^3$ to obtain a parametrization of a surface. Note that the coordinate transformation has three inputs and three outputs, whereas the parametric surface has only two inputs. All we have to do is remove one input variables by expressing it in terms of the others, and the function instantly describes a surface. We did this already in problem 5.16, where we obtained a 6 dimensional graph to represent spherical coordinates.

Problem 5.25 Again consider the surface $z = 9 - x^2 - y^2$.

1. Using cylindrical coordinates, so $\vec{T}(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$, obtain a parametrization $\vec{r}(r, \theta) = (?, ?, ?)$ of the surface using the two parameters r and θ. So you'll need to give equations

$$x = ?, y = ?, z = ? \qquad \text{or} \qquad \vec{r}(r, \theta) = (?, ?, ?).$$

   [Hint: We already know $x = r\cos\theta$ and $y = ?$ from cylindrical coordinates. The equation $z = 9 - x^2 - y^2$, when written in terms of r and θ, should give you the last equation for your parametrization.

2. What bounds must you place on r and θ to obtain the portion of the surface above the plane $z = 0$? Make sure you use technology to graph your parametric equations and verify that your bounds are correct.

Use Sage or Wolfram Alpha to plot your parametrization with your bounds (see 5.24 for examples). See 16.5: 1-16 for more practice.

Problem 5.26 Recall the spherical coordinate transformation

We did very similar things in problem 5.16. See 16.5: 1-16 for more practice.

$$\vec{T}(\rho, \phi, \theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi).$$

This is a function of the form $\vec{T}\colon \mathbb{R}^3 \to \mathbb{R}^3$. If we hold one of the three inputs constant, then we have a function of the form $\vec{r}\colon \mathbb{R}^2 \to \mathbb{R}^3$, which is a parametric surface.

1. Give a parametrization of the sphere of radius 2, using ϕ and θ as your parameters.

Use Sage or Wolfram Alpha to plot each parametrization (see 5.24 for examples).

2. What bounds should you place on ϕ and θ if you want to hit each point on the sphere exactly once?

3. What bounds should you place on ϕ and θ if you only want the portion of the sphere above the plane $z = 1$?

Sometimes you'll have to invent your own coordinate system when constructing parametric equations for a surface. If you notice that there are lots of circles parallel to one of the coordinate planes, try using a modified version of cylindrical coordinates. Instead of circles in the xy plane ($x = r\cos\theta, y = r\sin\theta, z = z$), maybe you need circles in the yz-plane ($x = x, y = r\cos\theta, z = r\sin\theta$) or the xz plane. Just look for lots of symmetry, and then construct your parametrization accordingly.

Problem 5.27 Find parametric equations for the surface $x^2 + z^2 = 9$. [Hint: read the paragraph above.]

1. What bounds should you use to obtain the portion of the surface between $y = -2$ and $y = 3$?
2. What bounds should you use to obtain the portion of the surface above $z = 0$?
3. What bounds should you use to obtain the portion of the surface with $x \geq 0$ and $y \in [2, 5]$?

Use Sage or Wolfram Alpha to plot each parametrization (see 5.24 for examples).

We'll finish the chapter with a few review problems.

Problem 5.28 Construct a graph of the surface $z = x^2 - y^2$. Do so in 2 ways. (1) Construct a 3D surface plot. (2) Construct a contour plot, which is a graph with several level curves. Which level curve passes through the point $(3, 4)$? Use Wolfram Alpha to know if you're right.

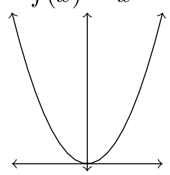
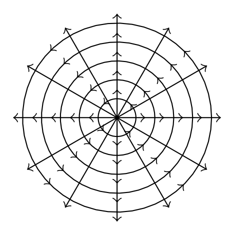
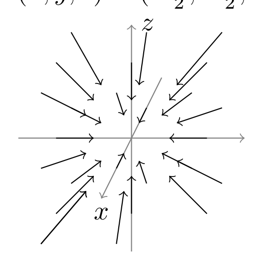
Problem 5.29 Construct a plot of the vector field

$$\vec{F}(x, y) = (x + y, -x + 1)$$

by graphing the field at many integer points around the origin (I generally like to get the 8 integer points around the origin, and then a few more). Then explain how to modify your graph to obtain a plot of the vector field

$$\hat{F}(x, y) = \frac{(x + y, -x + 1)}{\sqrt{(x + y)^2 + (1 - x)^2}}.$$

Problem 5.30 For this problem, print the grid below which contains some examples of the different types of functions and how we graph them. Once we know the dimensions of the domain and codomain, there are specific ways we graph the function. In each cell, I've given you the function form. Your job is to select a function that fits this form, and then appropriately graph it. I've filled in a few for you. Feel free to use examples from earlier in this chapter.

Single-valued functions of a single variable $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$ 	Single-valued functions of two variables $f : \mathbb{R}^2 \to \mathbb{R}$ $f(x, y) =$ Surface Plot | Contour Plot	Single-valued functions of three variables $f : \mathbb{R}^3 \to \mathbb{R}$ $f(x, y, z) =$ 4D Plot (Skip) | 3D Contour Plot
Parametric Curve (Planar Curve) $\vec{r} : \mathbb{R} \to \mathbb{R}^2$	Coordinate Change $\vec{T} : \mathbb{R}^2 \to \mathbb{R}^2$ $T(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$  Vector Field $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ $F(x, y) =$	$\vec{f} : \mathbb{R}^3 \to \mathbb{R}^2$ Skip (Math 316)
Parametric Curve (Space Curve) $\vec{r} : \mathbb{R} \to \mathbb{R}^3$	Parametric Surface $\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$	Coordinate Change $\vec{T} : \mathbb{R}^3 \to \mathbb{R}^3$ Vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ $\vec{F}(x, y, z) = (-\frac{x}{2}, -\frac{y}{2}, -\frac{z}{2})$ 

5.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 6

Differentials and the Derivative

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Compute differentials and partial derivatives, and explain how they are connected.
2. Explain how to obtain the total derivative from the partial derivatives (using a matrix).
3. Find equations of tangent lines and tangent planes to surfaces.
4. Find derivatives of composite functions, using the chain rule (substitution and matrix multiplication).
5. Give general chain rule formulas in terms of partial derivatives.

You'll have a chance to teach your examples to your peers prior to the exam.

6.1 Differentials and Partial Derivatives

Let's recall the definition of a differential. If $y = f(x)$ is a function, then we say the differential dy is the expression $dy = f'(x)dx$. We can also write this in the form $dy = \frac{dy}{dx}dx$.

Observation 6.1. Here's the key. Think of differential notation $dy = f'(x)dx$ in the following way:

> A small change in the output y equals the derivative multiplied by a small change in the input x. To get dy, we just need the derivative times dx.

To get the derivative in all dimensions, we just substitute in vectors to obtain the differential notation $d\vec{y} = f'(\vec{x})d\vec{x}$. The derivative is precisely the thing that tells us how to get $d\vec{y}$ from $d\vec{x}$. We'll quickly see that the derivative is a matrix, and the columns of that matrix we'll call partial derivatives. We'll start using the notation Df instead of f'.

Let's examine some problems you have seen before.

Problem 6.1 The volume of a right circular cylinder is $V(r,h)=\pi r^2 h$. Imagine that each of V, r, and h depends on t (we might be collecting rain water in a can, or crushing a cylindrical concentrated juice can, etc.).

See 3.10 for more practice.

1. Compute $\frac{dV}{dt}$ in terms of both $\frac{dr}{dt}$ and $\frac{dh}{dt}$. Then multiply both sides by dt to obtain the differential dV in terms of the differentials dr and dh. Write your answer in the form
$$dV=(?)dr+(?)dh.$$
2. Show that we can write dV as the matrix product
$$dV=\begin{bmatrix}2\pi rh & ?\end{bmatrix}\begin{bmatrix}dr\\ dh\end{bmatrix}.$$
3. If h is constant, what is $\frac{dV}{dr}$? Similarly, if r is constant, what is $\frac{dV}{dh}$?

The matrix $\begin{bmatrix}2\pi rh & ?\end{bmatrix}$ is the derivative of V. The columns we call the partial derivatives. The partial derivatives make up the whole.

When h was held constant, you should have gotten $\frac{dV}{dr}=2\pi rh$. We call this the partial derivative of V with respect to r and we write $\frac{\partial V}{\partial r}=2\pi rh$. This is the part of the differential that is multiplied by dr. Similarly, the partial derivative of V with respect to h, which we write as $\frac{\partial V}{\partial h}$, is the part of the differential that we times by dh. Using the partial derivative notation, we have
$$dV=\frac{\partial V}{\partial r}dr+\frac{\partial V}{\partial h}dh.$$

Definition 6.2: Partial Derivative. Given a function $f(x,y)$, we can write the differential df in the form $df=Mdx+Ndy$. The partial derivative of f with respect to x, written $\frac{\partial f}{\partial x}$ is the portion of this differential that we multiply by dx, so $\frac{\partial f}{\partial x}=M$. Similarly the partial derivative of f with respect to y, written $\frac{\partial f}{\partial y}$ is the portion of this differential that we multiply by dy, $\frac{\partial f}{\partial y}=N$. Symbolically, we can write the differential as
$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy.$$

We can extend this definition to functions with any number of inputs and/or outputs. For example, for the vector field $\vec{F}(x,y,z)$, we have
$$d\vec{F}=\frac{\partial \vec{F}}{\partial x}dx+\frac{\partial \vec{F}}{\partial y}dy+\frac{\partial \vec{F}}{\partial z}dz.$$

Different disciplines use different notations for the partial derivative. Four common uses are $\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}f=f_x=D_xf$.

Problem 6.2 The volume of a box is $V(x,y,z)=xyz$.

1. Compute the differential dV and write it in the form
$$dV=(?)dx+(?)dy+(?)dz.$$
2. Show that we can write dV as the matrix product (fill in the blanks)
$$dV=\begin{bmatrix}yz & ? & ?\end{bmatrix}\begin{bmatrix}dx\\ dy\\ dz\end{bmatrix}.$$

The matrix $\begin{bmatrix}yz & ? & ?\end{bmatrix}$ is the derivative. The columns we call the partial derivatives. The partial derivatives make up the whole.

3. Compute $\frac{\partial V}{\partial x}$. Then also state $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$. Read the paragraph before this problem if you need help with the notation.

Problem 6.3 For each function below, you'll first find the differential, then find the corresponding partial derivatives, and finish by organizing your work into a matrix product.

1. Let $f(x, y) = 3x^2 + 2xy$. State df, and then $\frac{\partial f}{\partial x}$ and f_y. Then fill in the blanks in the matrix product

$$df = \begin{bmatrix} ? & ? \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}.$$

2. Let $g(r, s, t) = r^2 s^3 + 4rt^2$. State dg, and then give the partials $D_r g$, g_s, and $\frac{\partial g}{\partial t}$. Then fill in the blanks in the matrix product

$$dg = \begin{bmatrix} ? & ? & ? \end{bmatrix} \begin{bmatrix} dr \\ ds \\ dt \end{bmatrix}.$$

3. Let $\vec{r}(u, v) = (u \cos v, u \sin v, v)$. State $d\vec{r}$, and then give the partials $\vec{r}_u$ and $\frac{\partial \vec{r}}{\partial v}$. Then fill in the blanks in the matrix product

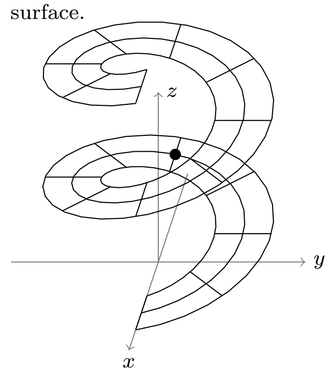
$$d\vec{r} = \begin{bmatrix} ? & ? \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}.$$

Note, if you find the matrix has more than one row in this example, then you are doing it correctly.

Review If you know that a line passes through the point $(1, 2, 3)$ and is parallel to the vector $(4, 5, 6)$, give a vector equation, and parametric equations, of the line. See [1] for an answer.

Problem 6.4 Consider the parametric surface $\vec{r}(a, t) = (a \cos t, a \sin t, t)$ for $2 \le a \le 4$ and $0 \le t \le 4\pi$. We encountered this parametric surface in chapter 5 when we considered a smoke screen left by multiple jets.

Here's a rough sketch of the surface.



1. Compute the differential $d\vec{r}$ which is the same as finding dx, dy, and dz. Write your answer in both vector and matrix forms

$$d\vec{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} da + \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} dt \quad \text{and} \quad d\vec{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} da \\ dt \end{pmatrix}.$$

2. Suppose an object is on this surface at the point $\vec{r}(3, \pi) = (-3, 0, \pi)$ (the dot on the graph to the right). Evaluate the matrix above at this point. Each column of the matrix above is an important vector, called a partial derivative. Draw both vectors with their tail at the point $\vec{r}(3, \pi)$.

---

[1] A vector equation is $\vec{r}(t) = (4, 5, 6)t + (1, 2, 3)$ or $\vec{r}(t) = (4t + 1, 5t + 2, 6t + 3)$. Parametric equations for this line are $x = 4t + 1$, $y = 5t + 2$, and $z = 6t + 3$.

3. Give vector equations for two tangent lines to the surface at $\vec{r}(3,\pi)$.

   [Hint: You've got the point as $\vec{r}(3,\pi)$, and you've got two different direction vectors as the columns of the matrix. Use the ideas from chapter 2 to get an equation of a line, or see the review problem above.]

In the previous problem, you should have noticed that the columns of your matrix are tangent vectors to the surface. Because we have two tangent vectors to the surface, we should be able to use them to construct a normal vector to the surface, and from that we can get the equation of a tangent plane.

Review If you know that a plane passes through the point $(1,2,3)$ and has normal vector $(4,5,6)$, then give an equation of the plane. See [2] for an answer.

Problem 6.5 Consider again the parametric surface

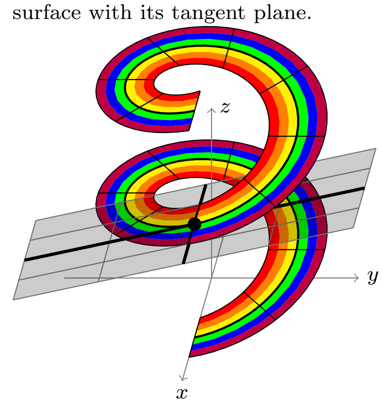
$$\vec{r}(a,t) = (a\cos t, a\sin t, t)$$

for $2 \leq a \leq 4$ and $0 \leq t \leq 4\pi$. We'd like to obtain an equation of the tangent plane to this surface at the point $\vec{r}(3,2\pi)$. Once you have a point on the plane, and a normal vector to the surface, we can use the concepts in chapter 2 to get an equation of the plane. Give an equation of the tangent plane.

[Hint: To get the point, what is $\vec{r}(3,2\pi)$? The columns of the matrix we obtain, when computing the differential $d\vec{r}$, give us two tangent vectors. How do we obtain a vector orthogonal to both these vectors?]

[Here's an alternate version of this problem, for Mario Kart fans. Mario and Luigi are booking it up rainbow road. About half way up, there is a glitch in the computer game and the road temporarily disappears. Instead of following the road, they instead are stuck on an infinite plane that meets the road tangentially where the glitch occurred. Give an equation of this plane.]

Here's a rough sketch of the surface with its tangent plane.



Problem 6.6 We can use differential notation to approximate tolerances.

1. We showed that a change in the volume of a cylinder is approximately

$$dV = \begin{bmatrix} 2\pi rh & \pi r^2 \end{bmatrix} \begin{bmatrix} dr \\ dh \end{bmatrix}.$$

   If we know that $r=3$ and $h=4$, and we know that r could increase by about .1 and h could increase by about .2, then by about how much will V increase by?

2. The volume of a box is given by $V = xyz$. We know the differential of the volume is $dV = \begin{bmatrix} yz & xz & xy \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$. If the current measurements are $x=2$, $y=3$, and $z=5$, and we know that $dx=.01$, $dy=.02$, and $dz=.03$, then by about how much will the volume increase.

Make sure you ask me in class to show you physically exactly how you can see these differential formulas.

[2] An equation of the plane is $4(x-1)+5(y-2)+6(y-3)=0$. If (x,y,z) is any point in the plane, then the vector $(x-1,y-2,z-3)$ is a vector in the plane, and hence orthogonal to $(4,5,6)$. The dot product of these two vectors should be equal to zero, which is why the plane's equation is $(4,5,6)\cdot(x-1,y-2,z-3)=0$.

We have been finding partial derivatives by first finding the differential, and then stating the partial derivatives as the parts of the differential that are multiplied by each corresponding variable. Can we find the partial derivatives first before finding the differential?

Problem 6.7 Consider the function $f(x, y) = x^2y + 3x + 4\sin(5y)$.

1. Suppose that y is a constant, so $f(x) = x^2y+3x+4\sin(5y)$. Compute $\frac{df}{dx}$.
2. Suppose that x is a constant, so $f(y) = x^2y+3x+4\sin(5y)$. Compute $\frac{df}{dy}$.
3. Now compute the differential df, and write your answer in both vector and matrix forms as
$$df = (?)dx + (?)dy \quad \text{and} \quad df = \begin{bmatrix} ? & x^2 + 20\cos(5y) \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}.$$
4. How can you obtain the partial derivatives of a function without first computing the entire differential?

If your answer above was something along the lines of "The partial derivative of f with respect to x is the regular derivative of f, provided we hold every every input variable constant except x," then you nailed it. We'll finish this section with one final problem, which will require you to grab your textbook and do some practice.

Problem 6.8 Open your calculus textbook to the section on partial derivatives (14.3 in Thomas's Calculus). The first 40 problems ask you to compute partial derivatives (with no mention of a differential). Do lots of the odd numbered problems (so you can check your work) until you are getting them correct each time. Then neatly organize your work from a few problems to illustrate the key ideas you learned as you practiced computing partial derivatives. You'll get to share this with the class.

6.2 The Derivative

In the previous section, we found differentials and partial derivatives. Most of the problems had your express your answer in both vector form, and matrix form. The derivative is precisely the matrix you obtained. You should have noticed that the columns of this matrix are the partial derivatives.

Definition 6.3: Derivatives and Partial Derivatives. Let f be a function. The derivative of f is a matrix. The columns of the derivative are the partial derivatives of f. When there's more than one input variable, we'll use Df rather than f' to talk about the derivative. The order of the columns matches the order you list the variables in the function. For example, if the function is $f(x, y)$, then the derivative is $Df(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$. If the function is $V(x, y, z)$, then the derivative is $DV(x, y, z) = \begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{bmatrix}$.

We've added one new definition, so let's practice.

Problem 6.9 Compute the partial derivatives, the derivative, and then the differential, as requested below.

If you haven't yet, then please go back and see 14.3: 1-40 in Thomas's Calculus for more practice. I strongly suggest you practice until you can compute partial derivatives with ease.

1. For $f(x,y) = x^2+2xy+3y^2$, compute $\dfrac{\partial f}{\partial x}$ and f_y. Then state the derivative $Df(x,y)$ and then finally the differential df.
2. For $f(x,y,z) = x^2y^3z^4$, compute all three of f_x, $\dfrac{\partial f}{\partial y}$, and $D_z f$. Then state the derivative $Df(x,y,z)$ and then the differential df.

Remember, the partial derivative of a function with respect to x is just the regular derivative with respect to x, provided you hold all other variables constant. We put the partials into the columns of a matrix to obtain the (total) derivative.

Your textbook has lots of examples to help you with partial derivatives in section 14.3. However, the textbook leaves out the actual derivative (putting the parts into a single matrix). The exercise below has 6 problems, with solutions, that you can use as extra practice for total derivatives. Complete the exercise below before moving on.

Exercise For each function below, compute the total derivative.

1. $f(x,y) = 9 - x^2 + 3y^2$
2. $\vec{r}(t) = (t, \cos t, \sin t)$
3. $f(x,y,z) = xy^2z^3$
4. $\vec{r}(u,v) = (u^2, v^2, u - v)$
5. $\vec{F}(x,y) = (-y + 3x, x + 4y)$
6. $\vec{T}(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$.

See [3] for answers.

Problem 6.10 Compute the requested partial and total derivatives.

1. Consider the parametric surface $\vec{r}(u,v) = (u, v, v\cos(uv))$. Compute both $\dfrac{\partial \vec{r}}{\partial u}$ and $\dfrac{\partial \vec{r}}{\partial v}$. Then state $D\vec{r}(u,v)$ and the differential $d\vec{r}$. If you end up with a 3 by 2 matrix for the derivative, you did this correctly.
2. Consider the vector field $\vec{F}(x,y) = (-y, xe^{3y})$. Compute both $\dfrac{\partial \vec{F}}{\partial x}$ and $\dfrac{\partial \vec{F}}{\partial y}$. Then state $D\vec{F}(x,y)$ and the differential $d\vec{F}$.

As you completed the problems above, did you notice any connections between the size of the matrix and the size of the input and output vectors? Make sure you ask in class about this. We'll make a connection.

We've now seen that the derivative of $z = f(x,y)$ is a matrix $Df(x,y) = \begin{bmatrix} f_x & f_y \end{bmatrix}$. This is a function itself that has inputs x and y, and outputs f_x and f_y. This means it has 2 inputs and 2 outputs, so it's a vector field. What does the vector field tell us about the original function?

[3] The derivatives of each function are shown below.

1. $Df(x,y) = \begin{bmatrix} -2x & 6y \end{bmatrix}$
2. $D\vec{r}(t) = \begin{bmatrix} 1 \\ -\sin t \\ \cos t \end{bmatrix}$
3. $Df(x,y,z) = \begin{bmatrix} y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{bmatrix}$
4. $D\vec{r}(u,v) = \begin{bmatrix} 2u & 0 \\ 0 & 2v \\ 1 & -1 \end{bmatrix}$
5. $D\vec{F}(x,y) = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}$
6. $D\vec{T}(r,\theta,z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 6.11 Consider the function $f(x,y)=x^2-y$, whose derivate is the vector field $Df(x,y)=\begin{bmatrix}2x & -1\end{bmatrix}$.

1. In the xy plane, carefully draw several level curves of f (maybe $z=0$, $z=1$, $z=-4$, etc.) Write the height on each curve (so you're making a topographical map). You should end up with several parabolas. You need this graph to be fairly close to scale, or you will completely miss the point of this problem. So if your graph is sloppy, then draw it again neatly.
2. On the same graph, we'll now draw the vector field. To do this, pick several points in the xy plane that lie on the level curves you already drew (such as $(0,0)$, $(1,1)$, $(2,4)$, and more). At these points, add the vector given by the derivative. (So at (0,0), you'll need to draw the vector $(0,-1)$. At (1,1), you'll need to draw the vector $(2,-1)$.) Add at least 8 vectors to your picture.
3. How are the vectors you drew related to the curves? There are lots of right answers here, just give an observation.

We'll come back to this problem more in chapter 7 as we discuss optimization. There are lots of connections between the derivative and level curves.

Since a partial derivative is a function, we can take partial derivatives of that function as well. This gives us second-order partial derivatives.

Definition 6.4: Second-Order Partial Derivatives. The second-order partial derivative of f is a partial derivative of one of the partial derivatives of f. The the second-order partial of f with respect to x and then y is the quantity $\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$, so we first compute the partial of f with respect to x, and then compute the partial of the result with respect to y. Alternate notations exist, for example the same second-order partial above we can write as

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]=f_{xy}=\frac{\partial}{\partial y}\frac{\partial}{\partial x}f=\frac{\partial}{\partial y}\frac{\partial f}{\partial x}=\frac{\partial^2 f}{\partial y\partial x}.$$

Did you notice the swap in order between the fractional notation and the subscript notation?

The subscript notation f_{xy} is easiest to write. In upper-level courses, we will use subscripts to mean other things. At that point, we'll have to use the fractional partial notation to avoid confusion.

Problem 6.12 Consider the functions $f(x,y,z)=xy^2z^3$ and $g(x,y)=x\cos(xy)$.

1. Compute f_x, f_{xy}, f_{xyy}, and $\frac{\partial^2 f}{\partial z^2}$.
2. Compute g_x and g_{xy}, and then compute g_y and g_{yx}.

Problem 6.13: Mixed Partials Agree Complete the following:

1. Let $f(x,y)=3xy^3+e^x$. Compute the four second partials
$$\frac{\partial^2 f}{\partial x^2},\quad \frac{\partial^2 f}{\partial y\partial x},\quad \frac{\partial^2 f}{\partial y^2},\quad \text{and}\quad \frac{\partial^2 f}{\partial x\partial y}.$$
2. For $f(x,y)=x^2\sin(y)+y^3$, compute both f_{xy} and f_{yx}.
3. Make a conjecture about a relationship between f_{xy} and f_{yx}. Then use your conjecture to quickly compute f_{xy} if
$$f(x,y)=3xy^2+\tan^2(\cos(x))(x^{49}+x)^{1000}.$$

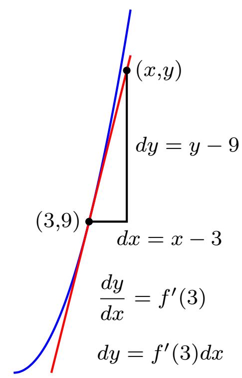
6.3 Tangent Planes

We can obtain most of the results in multivariate calculus by replacing the x and y in $dy = f'dx$ with $\vec{x}$ and $\vec{y}$. As an example, we can use differential notation to find an equation of the tangent plane to a function of the form $z = f(x, y)$. Let's first review how to do it for functions of the form $y = f(x)$, and then generalize.

Example 6.5: Tangent Lines. Consider the function $y = f(x) = x^2$.

1. The derivative is $f'(x) = 2x$. When $x = 3$ this means the derivative is $f'(3) = 6$ and the output y is $y = f(3) = 9$.

2. We know the tangent line passes through the point $P = (3, 9)$. We let $Q = (x, y)$ be any other point on the tangent line, and then a vector between these points is $\vec{PQ} = (x, y) - (3, 9) = (x - 3, y - 9)$. This vector tells us that when our change in x is $dx = x - 3$, then the change in y is $dy = y - 9$.



3. Differential notation states that a change in the output dy equals the derivative times a change in the input dx. In symbols, we have the equation $dy = f'(3)dx$. We then replace dx, dy, and $f'(3)$ with what we know they equal from the parts above to obtain

$$\underbrace{y - 9}_{dy} = \underbrace{6}_{f'(3)}\underbrace{(x - 3)}_{dx}.$$

This is an equation of the tangent line.

In first semester calculus, differential notation is $dy = f'dx$. At $x = c$, the line passes through the point $P = (c, f(c))$. If $Q = (x, y)$ is any other point on the line, then the vector $\vec{PQ} = (x - c, y - f(c))$ tells us that when $dx = x - c$ we have $dy = y - f(c)$. Substitution give us an equation for the tangent line tangent line as

$$\underbrace{y - f(c)}_{dy} = f'(c)\underbrace{(x - c)}_{dx}.$$

This equation tells us that a change in the output $(y - f(c))$ equals the derivative times a change in the input $(x - c)$. We now repeat this for the next problem, where the output is z and input is (x, y), which means differential notation says

$$dz = Df\begin{bmatrix} dx \\ dy \end{bmatrix}.$$

Problem 6.14 Consider the function $z = f(x, y) = 9 - x^2 - y^2$. If you haven't yet, read the example above.

See Sage for a picture.

See 14.6: 9-12 for more practice.

1. Compute the derivative $Df(x, y)$ and differential df. Then at $(x, y) = (2, 1)$, evaluate the derivative $Df(2, 1)$ and the output $z = f(2, 1)$.

2. One point on the tangent plane to the surface at $(2, 1)$ is the point $P = (2, 1, f(2, 1))$. Let $Q = (x, y, z)$ be another point on this plane. Use the vector $\vec{PQ}$ obtain dz when $dx = x - 2$ and $dy = y - 1$.

3. We'd like an equation of the tangent plane to $f(x, y)$ when $x = 2$ and $y = 1$. Differential notation tells us that

$$dz = Df(2, 1)\begin{bmatrix} dx \\ dy \end{bmatrix} \quad \text{or} \quad z - ? = \begin{bmatrix} -4 & ? \end{bmatrix}\begin{bmatrix} x - 2 \\ ? \end{bmatrix}$$

Fill in the blanks and compute this matrix product. When you are done you should have an equation of a plane.

4. Rewrite the equation you got in the form $A(x-a)+B(y-b)+C(z-c)=0$ and state a normal vector to the plane.

The first semester calculus tangent line equation, with differential notation, generalizes immediately to the tangent plane equation for functions of the form $z = f(x, y)$. Let's try this on another problem.

Problem 6.15 Let $f(x, y) = x^2 + 4xy + y^2$. Give an equation of the tangent plane at $(3, -1)$, and then state a normal vector to this plane. [Hint: find $Df(x, y)$, $Df(3, -1)$, df, dx, dy, and dz. Then substitute, as done in the previous problem.]

See Sage.
See 14.6: 9-12 for more practice.

Let's now return to the function $z = 9 - x^2 - y^2$, and show how parametric surfaces can add more light to unlocking the derivative and its geometric meaning. With a parametrization, partial derivatives are vectors, instead of just numbers. Once we have vectors, we can describe motion. This makes it easier to visualize.

Problem 6.16 Let $z = f(x, y) = 9 - x^2 - y^2$. We can parameterize this function by writing $x = x, y = y, z = 9 - x^2 - y^2$, or in vector notation

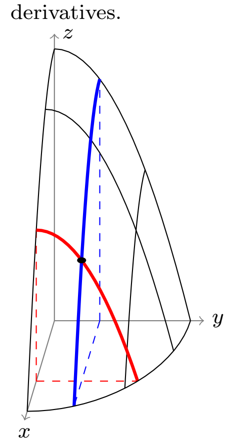
$$\vec{r}(x, y) = (x, y, f(x, y)).$$

1. Compute $\dfrac{\partial \vec{r}}{\partial x}$ and $\dfrac{\partial \vec{r}}{\partial y}$ and then evaluate these partials at $(x, y) = (2, 1)$. The surface is drawn to the right, where $x = 2$ is highlighted in red and $y = 1$ is highlighted in blue. Based at the point $(2, 1, 4)$, draw both of these partial derivatives (they are vectors).

2. You should see that the partial derivatives above are tangent vectors to the surface. Cross them to obtain a normal vector to the tangent plane.

3. Give an equation of the tangent plane to the surface at $(2, 1, 4)$.

See 16.5: 27-30 for more practice. Here's a picture of the surface on which you can draw your partial derivatives.



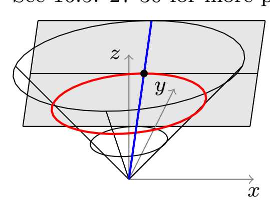
The next problem generalizes the tangent plane and normal vector calculations above to work for any parametric surface $\vec{r}(u, v)$.

Problem 6.17 Let $\vec{r}(u, v) = (u \cos v, u \sin v, u)$, a parametrization of a cone.

1. Give vector equations of two tangent lines to the surface at $\vec{r}(2, \pi/2)$ (so $u = 2$ and $v = \pi/2$).

2. Give a normal vector to the surface at $\vec{r}(2, \pi/2)$ and an equation of the tangent plane at $\vec{r}(2, \pi/2)$.

See 16.5: 27-30 for more practice.



We now have two different ways to compute tangent planes. One way generalizes differential notation $dy = f'dx$ to $dz = Df \begin{bmatrix} dx \\ dy \end{bmatrix}$ and then uses matrix multiplication. This way will extend to tangent objects in EVERY dimension. It's the key idea needed to work on really large problems. The other way requires that we parametrize the surface $z = f(x, y)$ as $\vec{r}(x, y) = (x, y, f(x, y))$ and then use the cross product on the partial derivatives to obtain a normal vector. The next problem has you give a general formula for a tangent plane. To tackle this problem, you'll need to make sure you can use symbolic notation. The review problem should help with this.

Review Joe wants to to find the tangent line to $y = x^3$ at $x = 2$. He knows the derivative is $y = 3x^2$, and when $x = 2$ the curve passes through 8. So he writes an equation of the tangent line as $y - 8 = 3x^2(x - 2)$. What's wrong? What part of the general formula $y - f(c) = f'(c)(x - c)$ did Joe forget? See [4] for an answer.

Problem 6.18: Tangent Plane General Formula First read the review problem above, and its solution. Now, consider the function $z = f(x, y)$. Prove that an equation of the tangent plane to f at $(x, y) = (a, b)$ is given by

$$z - f(a, b) = \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$

Then give an equation of the tangent plane to $f(x, y) = x^2 + 3xy$ at $(3, -1)$. [Hint: Use either differential notation or a parametrization. Try both ways.]

6.4 The Chain Rule (or just Substitution)

Suppose we know that the temperature at points in the plane is given by some function $T = f(x, y)$. We also know that an object is traveling around the plane following the curve $\vec{r}(t) = (x(t), y(t))$. As the object moves around, it encounters different temperatures. One function f tells us the temperature based on position. The other function $\vec{r}$ tells position based on time. Combining these two functions together (function composition $f(\vec{r}(t))$ we can compute the temperature based on time. These functions are like a chain of events. Changing t causes position to change, which in turn causes the temperature to change. This might cause something else to change. The chain rule helps us see how to compute the derivative of a function that is composed of several smaller pieces.

Problem 6.19 Consider $f(x, y) = 9 - x^2 - y^2$ and $\vec{r}(t) = (2\cos t, 3\sin t)$. Imagine the following scenario. A horse runs around outside in the cold. The horse's position at time t is given parametrically by the elliptical path $\vec{r}(t)$. The function $T = f(x, y)$ gives the temperature of the air at any point (x, y).

1. At time $t = 0$, what is the horse's position $\vec{r}(0)$, and what is the temperature $f(\vec{r}(0))$ at that position? Find the temperatures at $t = \pi/2$, $t = \pi$, and $t = 3\pi/2$ as well.

2. In the plane, draw the path of the horse for $t \in [0, 2\pi]$. Then, on the same 2D graph, include a contour plot of the temperature function f. Make sure you include the level curves that pass through the points in part 1, and write the temperature on each level curve you draw.

   If you end up with an ellipse and several concentric circles, then you've done this right.

3. As the horse runs around, the temperature of the air around the horse is constantly changing. At which t does the temperature around the horse reach a maximum? A minimum? Explain, using your graph.

   This idea leads to an optimization technique, Lagrange multipliers, later in the semester.

4. As the horse moves past the point at $t = \pi/4$, is the temperature of the surrounding air increasing or decreasing? In other words, is $\frac{df}{dt}$ positive or negative? Use your graph to explain.

[4] Joe forgot to replace x with 2 in the derivative. The equation should be $y - 8 = 12(x - 2)$. The notation $f'(c)$ is the part he forgot. He used $f'(x) = 3x^2$ instead of $f'(2) = 12$.

Notice above that we wanted $\frac{df}{dt}$, the rate of change of temperature with respect to time, even though the function $f(x, y)$ does not explicitly have t as an input. The proper notation would be $\frac{d(f\circ r)}{dt}$, but this is so cumbersome that it's generally avoided.

Problem 6.20 Consider again $f(x, y) = 9-x^2-y^2$ and $\vec{r}(t) = (2\cos t, 3\sin t)$, which means $x = 2\cos t$ and $y = 3\sin t$.

1. At the point $\vec{r}(t)$, we'd like a formula for the temperature $f(\vec{r}(t))$. What is the temperature of the horse at any time t? [In $f(x, y)$, replace x and y with what they are in terms of t.]
2. Compute df/dt (the derivative as you did in first-semester calculus).
3. Construct a graph of $f(t)$ (use software to draw this if you like). From your graph, at what time values do the maxima and minima occur?
4. What is $\frac{df}{dt}$ at $t = \pi/4$?

Let's now look at the same problem, but first compute differentials before we do any substitution.

Problem 6.21 Consider again $f(x, y) = 9-x^2-y^2$ and $\vec{r}(t) = (2\cos t, 3\sin t)$.

1. Compute both $Df(x, y)$ and $D\vec{r}(t)$ as matrices. One should have two columns. The other should have one column (but two rows).
2. State the differential df in terms of x, y, dx, and dy. Then state the differential $d\vec{r}$ in terms of t and dt.
3. Remember that $\vec{r} = (x, y)$, so clearly state dx and dy in terms of t and dt.
4. Use substitution to write df in terms of t and dt, and then state $\frac{df}{dt}$ (it should match the previous problem).

Problem 6.22 A horse walks on a curved path given by $(x, y)(t) = (5\cos t+ 3, 2\sin t)$. The temperature of the air at points along the path is given by $f(x, y) = x^2e^y + y^3$.

1. Compute the composite function $f(\vec{r}(t))$ which gives the temperature encountered by the horse at time t. Then compute the derivative of the temperature with respect to time, namely $\frac{df}{dt}$.
2. Compute the differential df in terms of x, y, dx, dy. Then compute the differential $d(x, y)$ in terms of t and dt. Then use substitution to obtain df in terms of t and dt, and finally state $\frac{df}{dt}$.
3. Verify that the two computations above both yield the same result.

Did you notice in the previous problems that you can get the same answer by either first substituting and then differentiating, or instead you can first differentiate and then substitute. The order does not matter, as the result will be the same. Let's look at a problem where the order does matter.

Problem 6.23 Suppose a horse walks along the parabolic path $(x, y) = (3t, t^2)$. The temperature encountered, $f(x, y)$, is currently unknown (another team is currently trying to figure out this function). We need to obtain a formula for the rate of the change of the horse's temperature as t increases.

1. Why is computing the composite function $f(\vec{r}(t))$ and simplifying not possible?
2. Even though we do not know a formula currently for f, why do we know $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$?
3. Compute both dx and dy in terms of t and dt. Then use substitution to obtain df in terms of t and dt.
4. Use your work on the previous part to show that $\frac{df}{dt} = 3\frac{\partial f}{\partial x} + 2t\frac{\partial f}{\partial y}$.

When a formula for a function is currently unknown, we can still compute derivatives of a composite function by first differentiating and then substituting. Wait. How often in life do you look at something and see a formula written on it to describe the function involved? We know that the temperature of points on the earth at a specific time are given by some function $f(x, y, z)$, but do we know the formula for that function? To find oil or natural gas under the earth, we would love to know before we start digging what the density of material under the earth is (so $f(x, y, z)$), but this function is now known either. In fact, the world before us NEVER has a function provided, rather we have to construct these functions. The class after this, differential equations, uses precisely the ideas we are learning now to work backwards and find formulas for $f(x, y, z)$ from knowledge about velocities (derivatives) and forces (mass time acceleration).

What we need to do now is learn how to compute derivatives of composite functions when some (or all) of the functions involved are currently not fully known. Let's start with a problem related to notation.

Problem 6.24 Consider the two functions $f(x, y)$ and $g(x, y, z)$.

1. Because of the definition of partial derivatives, we know that
$$df = f_x dx + f_y dy.$$
Give a formula for dg in terms of its partial derivatives.
2. Use the formula above to explain why $\frac{df}{dx} = f_x + f_y\frac{dy}{dx}$. Then give a similar formula for $\frac{dg}{dz}$.
3. Consider now the specific function $f(x, y) = x^2y + xy^3$. The regular derivative $\frac{df}{dx}$ assumes that all other variables depend on x when computing a derivative, whereas the partial derivative f_x assumes that all other variables are constants when computing a derivative. Compute both $\frac{df}{dx}$ and $\frac{\partial f}{\partial x}$, and then compare the results.

Problem 6.25 Consider the two functions $\vec{r}(t) = (x, y)(t)$ (a parametrization of a curve) and $\vec{r}(u, v) = (x, y, z)(u, v)$ (a parametrization of a surface).

1. Using the function $\vec{r}(t) = (x, y)(t)$ together with the fact that $dx = \frac{dx}{dt}dt$ and a similar fact for dy, we obtain the differential

$$d\vec{r} = \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt}dt \\ \frac{dy}{dt}dt \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} dt.$$

   Obtain a formula for the differential of $(x, y, z)(u, v)$. [Hint: your answer can be written in the form below.]

$$d\vec{r} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} du + \begin{bmatrix} \frac{\partial x}{\partial v} \\ ? \\ ? \end{bmatrix} dv.$$

2. For the first function, compare and contrast $\frac{d\vec{r}}{dt}$ and $\frac{\partial \vec{r}}{\partial t}$?

3. For the second function, compare and contrast $\frac{d\vec{r}}{du}$ and $\frac{\partial \vec{r}}{\partial u}$?

Let's now apply what we've learned above to develop some formulas for the composition of two functions. These formulas are called "chain rule" formulas.

Problem 6.26 Suppose w is a function of x and y (so $w = f(x, y)$) and suppose x and y are functions of t (so $(x, y) = \vec{r}(t)$). The previous two problems showed that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \quad \text{and} \quad d\vec{r} = \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt}dt \\ \frac{dy}{dt}dt \end{bmatrix}.$$

Use substitution to show that

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Then obtain a similar function if w is a function of x, y, and z (so $w = f(x, y, z)$) which are all functions of t (so $(x, y, z) = \vec{r}(t)$).

Problem 6.27 Suppose w is a function of x and y (so $w = f(x, y)$) and suppose x and y are functions of u and v (so $(x, y) = \vec{r}(u, v)$). Your work on previous problems showed that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \quad \text{and} \quad d\vec{r} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \\ \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \\ \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \end{bmatrix}.$$

Use substitution and some rearranging to show that (fill in the blank)

$$df = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\right) du + (?)\, dv.$$

What is $\frac{\partial f}{\partial u}$? What is f_v?

6.5 Matrices and the Chain Rule

This section focuses on using matrices to develop the chain rule. We'll find that the first semester calculus chain rule will generalize to all dimensions, if we replace f' with the matrix Df. First. let's recall the chain rule from first-semester calculus.

Theorem (The Chain Rule)**.** *Let x be a real number and f and g be functions of a single real variable. Suppose f is differentiable at $g(x)$ and g is differentiable at x. The derivative of $f \circ g$ at x is*

$$(f \circ g)'(x) = \frac{d}{dx}(f \circ g)(x) = f'(g(x)) \cdot g'(x).$$

Please complete this review problem to make sure you are comfortable with the notation of the chain rule.

Exercise Suppose we know that $f'(x) = \dfrac{\sin(x)}{2x^2+3}$ and $g(x) = \sqrt{x^2+1}$. Notice we don't know $f(x)$. See [5] for an answer.

Not knowing a function f is actually quite common in real life. We can often measure how something changes (a derivative) without knowing the function itself.

1. State $f'(x)$ and $g'(x)$.
2. State $f'(g(x))$, and explain the difference between $f'(x)$ and $f'(g(x))$.
3. Use the chain rule to compute $(f \circ g)'(x)$.

In the previous section, we focused on using differentials to compute derivatives, as the following example shows.

Example 6.6. Suppose that $y = f(u)$ and that $u = g(x)$. This means we have the differentials

$$dy = f'(u)du \quad \text{and} \quad du = g'(x)dx.$$

Simple substitution tells us that

$$dy = f'(\underbrace{g(x)}_{u})\underbrace{g'(x)dx}_{du}.$$

This means instantly that the derivative of y with respect to x must be the product $f'(g(x))g'(x)$.

Problem 6.28 Suppose that $f(x, y) = 3xy^2 + \sin x$ and $\vec{r}(t) = (e^{2t}, t^3)$, which is just shorthand for saying $x(t) = e^{2t}$ and $y(t) = t^3$.

1. State the composite function $f(\vec{r}(t))$. Then compute the derivative $\frac{df}{dt}$ directly.
2. Find the differential df in terms of x, y, dx, and dy. Then find dx and dy in terms of t and dt. Then use substitution to obtain the derivative $\frac{df}{dt}$.
3. Compute the derivative $Df(x, y)$ and the derivative $D\vec{r}(t)$. How can you combine these two matrices to obtain the derivative $\frac{df}{dt}$?

---

[5] We have $f'(x) = \frac{\sin(x)}{2x^2+3}$ and $g'(x) = \frac{1}{2}(x^2+1)^{-1/2}(2x)$. We have $f'(g(x)) = \frac{\sin((\sqrt{x^2+1}))}{2(\sqrt{x^2+1})^2+3}$. The difference between $f'(x)$ and $f'(g(x))$ is whether we've replaced x in f with $g(x)$ or not. The final derivative is $(f \circ g)'(x) = f'(g(x))g'(x) = \frac{\sin((\sqrt{x^2+1}))}{2(\sqrt{x^2+1})^2+3}\frac{1}{2}(x^2+1)^{-1/2}(2x)$.

Did you see that multiplying together the two matrices above gives you the derivative? Matrix multiplication was invented precisely so that we can generalize the chain rule to higher dimensions. Since the chain rule in first semester calculus states $(f(g(x))' = f'(g(x))g'(x)$, then in high dimensions, with matrices, it states $D(\vec{f}(\vec{g}(\vec{x})) = D\vec{f}(\vec{g}(\vec{x}))D\vec{g}(\vec{x})$, the product of two matrices.

Problem 6.29 In problem 6.1, we showed that for a circular cylinder with volume $V = \pi r^2 h$, the derivative is

$$DV(r,h) = \begin{bmatrix} 2\pi rh & \pi r^2 \end{bmatrix}.$$

Suppose that the radius and height are both changing with respect to time, where $r = 3t$ and $h = t^2$. We'll write this parametrically as $(r,h)(t) = (3t, t^2)$.

1. In $V = \pi r^2 h$, replace r and h with what they are in terms of t. Then compute $\frac{dV}{dt}$.
2. Now instead, compute dV in terms of r, h, dr, and dh. Then compute (dr, dh) in terms of t and dt. Finish by using substitution to obtain dV in terms of t and dt.
3. We know $DV(r,h) = \begin{bmatrix} 2\pi rh & \pi r^2 \end{bmatrix}$ and $D(r,h)(t) = \begin{bmatrix} 3 \\ 2t \end{bmatrix}$. In first semester calculus, the chain rule was the product of derivatives. Compute the matrix product
$$DV((r,h)(t)) \cdot D(r,h)(t)$$
and verify that you get $\frac{dV}{dt}$.
4. To get the correct answer to the previous part, you had to replace r and h with what they equaled in terms of t. What part of the notation $\frac{dV}{dt} = DV((r,h)(t)) \cdot D(r,h)(t)$ tells you to replace r and h with what they equal in terms of t?

Theorem 6.7 (The Chain Rule). *Let $\vec{x}$ be a vector and $\vec{f}$ and $\vec{g}$ be functions so that the composition $\vec{f}(\vec{g}(\vec{x}))$ makes sense (we can use the output of g as an input to f). Suppose $\vec{f}$ is differentiable at $\vec{g}(\vec{x})$ and that $\vec{g}$ is differentiable at $\vec{x}$. Then the derivative of $\vec{f} \circ \vec{g}$ at $\vec{x}$ is the matrix product*

$$D(\vec{f} \circ \vec{g})(\vec{x}) = D\vec{f}(\vec{g}(\vec{x})) \cdot D\vec{g}(\vec{x}).$$

This is exactly the same as the chain rule in first-semester calculus. The only difference is that now we have vectors above every variable and function, and we replaced the one-by-one matrices f' and g' with potentially larger matrices Df and Dg. If we write everything in vector notation, the chain rule in all dimensions is the EXACT same as the chain rule in one dimension.

Problem 6.30 Suppose that $f(x,y) = x^2 + xy$ and that $x = 2t + 3$ and $y = 3t^2 + 4$.

1. Use substitution to express f in terms of t. Then compute $\frac{df}{dt}$.
2. Compute the differential df in terms of x, y, dx and dy. Then compute dx and dy in terms of t and dt. Then use substitution to obtain $\frac{df}{dt}$.

See 14.4: 1-6 for more practice. Don't use the formulas in the chapter, rather practice using matrix multiplication. The formulas are just a way of writing matrix multiplication without writing down the matrices, and only work for functions from $\mathbb{R}^n \to \mathbb{R}$. Our matrix multiplication method works for any function from $\mathbb{R}^n \to \mathbb{R}^m$.

3. Rewrite the parametric equations $x = 2t + 3$ and $y = 3t^2 + 4$ in vector form, so we can apply the chain rule. This means you need to create a function $\vec{r}(t) = (\_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_)$. Then compute the derivatives $Df(x, y)$ and $D\vec{r}(t)$, and multiply the matrices together to obtain $\dfrac{df}{dt}$. How can you make your answer only depend on t (not x or y)?
4. The chain rule states that $D(f \circ \vec{r})(t) = Df(\vec{r}(t))D\vec{r}(t)$. Explain why we write $Df(\vec{r}(t))$ instead of $Df(x, y)$.

Problem 6.31 Suppose $f(x, y, z) = x + 2y + 3z^2$ and $x = u + v$, $y = 2u - 3v$, and $z = uv$. Our goal is to find how much f changes if we were to change u (so $\partial f/\partial u$) or if we were to change v (so $\partial f/\partial v$).

See 14.4: 7-12 for more practice.

1. Compute $\partial f/\partial u$ and $\partial f/\partial v$ by first substituting to get f written in terms of u and v, and then computing partial derivatives.
2. Compute $\partial f/\partial u$ and $\partial f/\partial v$ by first obtaining differentials and then using substitution.
3. Compute $\partial f/\partial u$ and $\partial f/\partial v$ by first obtaining the derivatives of $f(x, y, z)$ and $\vec{r}(u, v)$, and then multiplying the two matrices together.

Review Suppose $f(x, y) = x^2 + 3xy$ and $(x, y) = \vec{r}(t) = (3t, t^2)$. Compute both $Df(x, y)$ and $D\vec{r}(t)$. Then explain how you got your answer by writing what you did in terms of partial derivatives and regular derivatives. See [6] for an answer.

Problem 6.32: General Chain Rule Formulas Complete the following:

See 14.4: 13-24 for more practice. Practice these problems by using matrix multiplication. The examples problems in the text use a "branch diagram," which is just a way to express matrix multiplication without having to introduce matrices.

1. Suppose that $w = f(x, y, z)$ and that x, y, z are all function of one variable t (so $x = g(t), y = h(t), z = k(t)$). Use the chain rule with matrix multiplication to explain why
$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt}.$$
which is equivalent to writing
$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$
[Hint: Rewrite the parametric equations for x, y, and z in vector form $\vec{r}(t) = (x, y, z)$ and compute $Dw(x, y, z)$ and $D\vec{r}(t)$.]
2. Suppose that $R = f(V, T, n, P)$, and that V, T, n, P are all functions of x. Give a formula (similar to the above) for $\dfrac{dR}{dx}$.

Problem 6.33 Suppose $z = f(s, t)$ and s and t are functions of u, v and w. Use the chain rule to give a general formula for $\partial z/\partial u$, $\partial z/\partial v$, and $\partial z/\partial w$.

Make sure you practice problems 14.4: 13-24. Use matrix multiplication, rather than the "branch diagram" referenced in the text.

[6] We have $Df(x, y) = \begin{bmatrix} 2x + 3y & 3y \end{bmatrix}$ and $D\vec{r}(t) = \begin{bmatrix} 3 \\ 2t \end{bmatrix}$. We just computed f_x and f_y, and dx/dt and dy/dt, which gave us $Df(x, y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \end{bmatrix}$ and $D\vec{r}(t) = \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix}$.

Review If $w = f(x, y, z)$ and x, y, z are functions of u and v, obtain formulas for $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$. See [7] for an answer.

You've now got the key ideas needed to use the chain rule in all dimensions. The chain rule shows up many places in upper-level math, physics, and engineering courses as the key tool needed to develop new formulas. The following problem will show you one such use, namely how you can use the general chain rule to get an extremely quick way to perform implicit differentiation from first-semester calculus.

Problem 6.34 Suppose $z = f(x, y)$. If z is held constant, this produces a level curve. As an example, if $f(x, y) = x^2 + 3xy - y^3$ then $5 = x^2 + 3xy - y^3$ is a level curve. Our goal in this problem is to find dy/dx in terms of partial derivatives of f.

See 14.4: 25-32 to practice using the formula you developed. To practice the idea developed in this problem, show that if $w = F(x, y, z)$ is held constant at $w = c$ and we assume that $z = f(x, y)$ depends on x and y, then $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$. This is done on page 798 at the bottom.

1. Suppose $x = x$ and $y = y(x)$, so y is a function of x. We can write this in parametric form as $\vec{r}(x) = (x, y(x))$. We now have $z = f(x, y)$ and $\vec{r}(x) = (x, y(x))$. Compute both $Df(x, y)$ and $D\vec{r}(x)$ symbolically. Don't use the function $f(x, y) = x^2 + 3xy - y^3$ until you get to part 4 below.
2. Use the chain rule to compute $D(f(\vec{r}(x)))$. What is dz/dx (i.e., df/dx)?
3. Since z is held constant, we know that $dz/dx = 0$. Use this fact, together with part 2 to explain why $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$.
4. For the curve $5 = x^2 + 3xy - y^3$, use this formula to compute dy/dx.

Problem: Optional Suppose $\vec{F}(u, v) = (3u - v, u + 2v, 3v)$, $\vec{G}(x, y, z) = (x^2 + z, 4y - x)$, and $\vec{r}(t) = (t^3, 2t + 1, 1 - t)$. We want to examine $\vec{F}(\vec{G}(\vec{r}(t)))$. This means that $\vec{F} \circ \vec{G} \circ \vec{r}$ is a function from $\mathbb{R}^n \to \mathbb{R}^m$ for what n and m? Similar to first-semester calculus, since we have several functions nested inside of each other, we'll just need to apply the chain rule twice. Our goal is to find $d\vec{F}/dt$. Try to do this problem without looking at the steps below.

1. Compute $D\vec{F}(u, v)$, $D\vec{G}(x, y, z)$, and $D\vec{r}(t)$.
2. Use the chain rule (matrix multiplication) to find the derivative of $\vec{F}$ with respect to t. What size of matrix should we expect for the derivative? See [8] for an answer.

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[7] We have $Df(x, y, z) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}$. The parametrization $\vec{r}(u, v) = (x, y, z)$ has derivative $D\vec{r} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$. The product is $D(f(\vec{r}(u, v))) = \begin{bmatrix} \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} & \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v} \end{bmatrix}$. The first column is $\frac{\partial f}{\partial u}$, and the second column is $\frac{\partial f}{\partial v}$.

[8] The requested derivatives are

$$D\vec{F}(u, v) = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}, D\vec{G}(x, y, z) = \begin{bmatrix} 2x & 0 & 1 \\ -1 & 4 & 0 \end{bmatrix}, D\vec{r}(t) = \begin{bmatrix} 3t^2 \\ 2 \\ -1 \end{bmatrix}.$$

6.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

The product of these matrices is

$$\begin{aligned}\frac{d\vec{F}}{dt} &= D(\vec{F}(\vec{G}(\vec{r}(t)))) = D\vec{F}(\vec{G}(\vec{r}(t)))D\vec{G}(\vec{r}(t))D\vec{r}(t)\\ &= \begin{bmatrix}3 & -1\\ 1 & 2\\ 0 & 3\end{bmatrix}\begin{bmatrix}2x & 0 & 1\\ -1 & 4 & 0\end{bmatrix}\begin{bmatrix}3t^2\\ 2\\ -1\end{bmatrix} = \begin{bmatrix}18xt^2+3t^2-11\\ 6xt^2-6t^2+15\\ 24-9t^2\end{bmatrix}\\ &= \begin{bmatrix}18(t^3)t^2+3t^2-11\\ 6(t^3)t^2-6t^2+15\\ 24-9t^2\end{bmatrix}.\end{aligned}$$

The final step comes from noting that $x = t^3$, $y = 2t + 1$, and $z = 1 - t$, so we replace x with t^3 so that all variables are in terms of t.

Exam 2 Review

At the end of each chapter, the following words appeared.

> Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam.

I've summarized the objectives from each chapter below. For our in class review, please come to class with examples to help illustrate each idea below. You'll get a chance to teach another member of class the examples you prepared. If you keep the examples simple, you'll have time to review each key idea.

New Coordinates

1. Use a change-of-coordinates to convert between rectangular and another coordinate system. In particular, be able to convert points and equations between rectangular and polar coordinates.
2. Graph polar functions $r = f(\theta)$ in the xy plane, and set up the arc length formula to find their length.
3. Given a change-of-coordinates, find the differentials dx and dy and write them in both vector and matrix form. Use these to compute tangent vectors, slope $\frac{dy}{dx}$, and equations of tangent lines.
4. Compute double integrals to find the area of regions in the xy plane, and use the determinant to explain how area between different coordinate systems is related.
5. Shade regions in the plane bounded by $\alpha \leq \theta \leq \beta$ and $r_1(\theta) \leq r \leq r_2(\theta)$, and use double integrals to compute their area.

Functions

1. Describe uses for, and construct graphs of, space curves and parametric surfaces. Find derivatives of space curves, and use this to find velocity, acceleration, and find equations of tangent lines.
2. Describe uses for, and construct graphs of, functions of several variables. For functions of the form $z = f(x, y)$, this includes both 3D surface plots and 2D level curve plots. For functions of the form $w = f(x, y, z)$, construct plots of level surfaces.
3. Describe uses for, and construct graphs of, vector fields and transformations. Develop the formulas for cylindrical and spherical coordinates.
4. If you are given a description of a vector field, curve, or surface (instead of a function or parametrization), explain how to obtain a function for the vector field, or a parametrization for the curve or surface.

Differentials and The Derivative

1. Compute differentials and partial derivatives, and explain how they are connected.
2. Explain how to obtain the total derivative from the partial derivatives (using a matrix).
3. Find equations of tangent lines and tangent planes to surfaces.
4. Find derivatives of composite functions, using the chain rule (substitution and matrix multiplication).
5. Give general chain rule formulas in terms of partial derivatives.

Chapter 7

Optimization

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Explain the properties of the gradient, its relation to level curves and level surfaces, and how it can be used to find directional derivatives.

2. Find equations of tangent planes using the gradient and level surfaces. Use the derivative (tangent planes) to approximate functions, and use this in real world application problems.

3. Explain the second derivative test in terms of eigenvalues. Use the second derivative test to optimize functions of several variables.

4. Use Lagrange multipliers to optimize a function subject to constraints.

You'll have a chance to teach your examples to your peers prior to the exam.

7.1 The Gradient

When we take the derivative of single valued function, the derivative is a matrix with just a single row. The derivative of $f(x, y, z)$ is the matrix $Df(x, y, z) = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix}$. Because the number of input variables matches the number of entries in the matrix, we can think of such a derivative as a vector field. When we want to use the derivative as a vector field, we call it the gradient of f.

Definition 7.1: Gradient. Let $\vec{f} : \mathbb{R}^n \to \mathbb{R}$. When we want to emphasize that the derivative of f is a vector field, we call Df the gradient of f and write $Df = \vec{\nabla} f$.

The symbol $\vec{\nabla} f$ is read "the gradient of f" or "del f."

Problem 7.1 Consider the functions $f(x, y) = 9 - x^2 - y^2$, $g(x, y) = 2x - y$, and $h(x, y) = \sin x \cos y$.

1. Compute $\vec{\nabla} f(x, y)$. Then draw both $\vec{\nabla} f$ and several level curves of f on the same axes.

2. Compute $\vec{\nabla} g(x, y)$. Then draw both $\vec{\nabla} g$ and several level curves of g on the same axes.

3. Compute $\vec{\nabla} h(x, y)$. Then draw both $\vec{\nabla} h$ and several level curves of h on the same axes.

You'll want a computer to help you construct the graphs, particularly h. Please use the Mathematica introduction or Sage. You could use Wolfram Alpha (use the links in the function chapter if you forgot how to graph vector fields and/or contour plots).

See Sage. You can modify these commands to help in the plots below too.

4. What relationships do you see between the gradient vector field and level curves?

When you present in class, be prepared to provide rough sketches of the level curves and gradients of each function.

The next few problems will focus on explaining why the relationships you saw are always true.

Problem 7.2 Suppose $\vec{r}(t)$ is a level curve of $f(x, y)$.

1. Suppose you know that at $t = 0$, the value of f at $\vec{r}(0)$ is 7. What is the value of f at $\vec{r}(1)$? [What does it mean to be on a level curve?]
2. As you move along the level curve $\vec{r}$, how much does f change? Use this to tell the class what $\frac{df}{dt}$ must equal.
3. At points along the level curve $\vec{r}$, we have the composite function $f(\vec{r}(t))$. Compute the derivative $\frac{df}{dt}$ using the chain rule.
4. Use your work from the previous parts to explain why the gradient always meets the level curve at a 90° angle. We say that the gradient is *normal* to level curves (i.e., a gradient vector is orthogonal to the tangent vector of the curve).

In the derivative chapter, we extended differential notation from $dy = f'dx$ to $d\vec{y} = D\vec{f}d\vec{x}$. The key idea is that a small change in the output variables is approximated by the product of the derivative and a small change in the input variables. As a quick refresher, if we have the function $z = f(x, y)$, then differential notation states that

$$dz = \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

where the derivative is $Df(x, y) = \begin{bmatrix} f_x & f_y \end{bmatrix}$.

Problem 7.3 Suppose the temperature at a point in the plane is given by the function $T(x, y) = x^2 - xy - y^2$ degrees Fahrenheit. A particle is at $P = (2, 3)$.

1. Use differentials to estimate the change in temperature if the particle moves 1 unit in the direction of $\vec{u} = (3, 4)$. [Hint: In the formula $df = Df d\vec{x}$, the differential $d\vec{x}$ should be a unit vector in the direction of $\vec{u}$.]
2. What is the actual change in temperature if the particle moves 1 unit in the direction of $\vec{u} = (3, 4)$?
3. Use differentials to estimate the change in temperature if the particle moves about .2 units in the direction of $\vec{u} = (3, 4)$.

We can define partial derivatives solely in terms of differential notation. We can define derivatives in any direction in terms of differential notation.

Problem 7.4 Suppose that $z = f(x, y)$ is a differentiable function (so the derivative is the matrix $\begin{bmatrix} f_x & f_y \end{bmatrix}$). Remember to use differential notation in this problem.

1. If $(dx, dy) = (1, 0)$, which means we've moved one unit in the x direction while holding y constant, what is dz?

2. If $(dx, dy) = (0, 1)$, which means we've moved one unit in the y direction while holding x constant, what is dz?

3. Consider the direction $\vec{u} = (2, 3)$. Find a unit vector in the direction of $\vec{u}$. If we move one unit in the direction of $\vec{u}$, what is dz? [It's all right to leave you answer as a dot product.]

Definition 7.2. The directional derivative of f in the direction of the unit vector $\vec{u}$ at a point P is defined to be

$$D_{\vec{u}}f(P) = Df(P)\vec{u} = \vec{\nabla} f \cdot \vec{u}.$$

We dot the gradient of f with the direction vector $\vec{u}$. The partial derivative of f with respect to x is precisely the directional derivative of f in the $(1, 0)$ direction. Similarly, the partial derivative of f with respect to y is precisely the directional derivative of f in the $(0, 1)$ direction. This definition extends to higher dimensions.

Note that in the definition above, we require the vector $\vec{u}$ to be a unit vector. If you are asked to find a directional derivative in some direction, make sure you start by finding a unit vector in that direction. We want to deal with unit vectors because when we say something has a slope of m units, we want to say "The function rises m units if we run 1 unit."

Problem 7.5 Consider the function $f(x, y) = 9 - x^2 - y^2$.

1. Draw several level curves of f.

2. At the point $P = (2, 1)$, place a dot on your graph. Then draw a unit vector based at P that points in the direction $\vec{u} = (3, 4)$ [not to the point $(3, 4)$, but in the direction $\vec{u} = (3, 4)$]. If you were to move in the direction $(3, 4)$, starting from the point $(2, 1)$, would the value of f increase or decrease?

3. Find the slope of f at $P = (2, 1)$ in the direction $\vec{u} = (3, 4)$ by finding the directional derivative. This should agree with your previous answer.

4. If you stand at $Q = (-2, 3)$ and move in the direction $\vec{v} = (1, -1)$, will f increase or decrease? Find the directional derivative of f in the direction $\vec{v} = (1, -1)$ at the point $Q = (-2, 3)$.

Problem 7.6 Recall that the directional derivative of $\vec{f}$ in the direction $\vec{u}$ is the dot product $\vec{\nabla} f \cdot \vec{u} = |\vec{\nabla} f||\vec{u}| \cos\theta$.

1. Why is the directional derivative of $\vec{f}$ the largest when $\vec{u}$ points in the exact same direction as $\vec{\nabla} f$? [Hint: What angle maximizes $\cos\theta$?]

2. When $\vec{u}$ points in the same direction as $\vec{\nabla} f$, show that $D_{\vec{u}}f = |\vec{\nabla} f|$. In other words, explain why the length of the gradient is precisely the slope of f in the direction of greatest increase (the slope in the steepest direction).

3. Which direction points in the direction of greatest decrease?

Problem 7.7 Suppose you are looking at a topographical map (see Wikipedia for an example). On this topographical map, each contour line represents 100 ft in elevation. You notice in one section of the map that the contour lines are really close together, and they start to form circles around a spot on the graph. You notice in another section of the map that the contour lines are spaced quite far apart. Let $f(x, y)$ be the elevation of the land, so that the topographical map is just a contour plot of f.

1. Where is the slope of the terrain larger, in the section with closely packed contour lines, or the section with contour lines that are spread out. In which section will the gradient be a longer vector?

2. At the very top of a mountain, or the very bottom of a valley, will the gradient be a long vector or a small vector? How do you locate a peak in a topographical map?

3. Create you own topographical map to illustrate the ideas above. Just make sure your map has a section with some contours that are closely packed together, and some that are far apart, as well as a contour that intersects itself. Then on your topographical map, please add a few gradient vectors, where you emphasize which ones are long, and which ones are short. Show us how to find a peak, as well as what the gradient vector would be at the peak.

If you're stuck, look at a contour plot of $f(x, y) = (x+1)^3 - 3(x+1)^2 - y^2 + 2$ in Sage. Then make your own example.

Theorem 7.3. *Let f be a continuously differentiable function, with $\vec{r}$ a level curve of the function.*

- *The gradient is always normal to level curves, meaning* $\vec{\nabla} f \cdot \dfrac{d\vec{r}}{dt} = 0$.
- *The gradient points in the direction of greatest increase.*
- *The directional derivative of f in the direction of the gradient is the length of the gradient. Symbolically, we write $D_{\vec{\nabla} f} f = |\vec{\nabla} f|$.*
- *At a maximum or minimum, the gradient is the zero vector.*

The next few problems have you practice using differentials, and then obtain tangent lines and planes to curves and surfaces using differentials.

Problem 7.8 The volume of a cylindrical can is $V(r, h) = \pi r^2 h$. Any manufacturing process has imperfections, and so building a cylindrical can with designed dimensions (r, h) will result in a can with dimensions $(r + dr, h + dh)$.

1. Compute both DV (the derivative of V) and dV (the differential of V).

2. If the can is tall and slender (h is big, r is small), which will cause a larger change in volume: an error in r or an error in h? Use dV to explain your answer.

3. If the can is short and wide (like a tuna can), which will cause a larger change in volume: an error in r or an error in h? Use dV to explain your answer.

Problem 7.9 Consider the function $f(x,y)=x^2+y^2$. Consider the level curve C given by $f(x,y)=25$. Our goal is to find an equation of the tangent line to C at $P=(3,-4)$.

1. Draw C. Compute $\vec{\nabla} f$ and add to your graph the vector $\vec{\nabla} f(P)$.
2. We know the point $P=(3,-4)$ is on the tangent line. Let $Q=(x,y)$ represent another point on the tangent line. Add to your graph the point Q and the vector $\vec{PQ}=(x-3,y+4)$.
3. Why are $\vec{\nabla} f(P)$ and $\vec{PQ}$ orthogonal? Use this fact to write an equation of the tangent line.
4. What is a normal vector to the line?

The previous problem had you give an equation of the tangent line to a level curve, by using differential notation. The next problems asks you to repeat this idea and give an equation of a tangent plane to a level surface.

Problem 7.10 Consider the function $f(x,y,z)=x^2+y^2+z^2$. Consider the level surface S given by $f(x,y,z)=9$. Our goal is to find an equation of the tangent plane to S at $P=(1,2,-2)$.

1. Draw S.
2. Compute $\vec{\nabla} f$. Add to your graph the vector $\vec{\nabla} f(P)$, with its base at P.
3. We know the point $P=(1,2,-2)$ is on the tangent plane. Let $Q=(x,y,z)$ be any other point on the tangent plane. What is the component form of the vector $\vec{PQ}$?
4. Why are $\vec{\nabla} f(P)$ and $\vec{PQ}$ orthogonal? Use this fact to write an equation of the tangent plane.
5. What is a normal vector to the plane?

Problem 7.11 Find an equation of the tangent plane to the hyperboloid of one sheet $1=x^2-y^2+z^2$ at the point $(-3,3,1)$.

Problem 7.12 The two surfaces $x^2+y^2+z^2=14$ and $3x+4y-z=-1$ intersect in a curve C. Draw both surfaces, and show us the curve C. Then, at the point $(2,-1,3)$, find an equation of the tangent line to this curve. [Hint: The line is in both tangent planes, so it is orthogonal to both normal vectors. The cross product gets you a vector that is orthogonal to two vectors.]

7.2 The Second Derivative Test

We start with a review problem from first-semester calculus.

Problem 7.13 Let $f(x)=x^3-3x^2$. Find the critical values of f by solving $f'(x)=0$. Then use the second derivative test to determine if each critical value leads to a local maximum or local minimum (look up the second derivative test if you don't remember it). State the local maxima/minima of f. End by sketching the function using the information you discovered.

We now generalize the second derivative test to all dimensions. We've already seen that the second derivative of a function such as $z = f(x, y)$ is a square matrix. The second derivative test relied on understanding if a function was concave up or concave down. We need a way to examine the concavity of f as we approach a point (x, y) from any of the infinitely many directions. Such a method exists, and leads to an eigenvalue/eigenvector problem. I'm assuming that most of you have never heard the word "eigenvalue." We could spend an entire semester just studying eigenvectors. We'd need a few weeks to discover what they are from a problem-based approach. Instead, here is an example of how to find eigenvalues and eigenvectors.

Definition 7.4. Let A be a square matrix, so in 2D we have $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The identity matrix I is a square matrix with 1's on the diagonal and zeros everywhere else, so in 2D we have $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The eigenvalues of A are the solutions λ to the equation $|A - \lambda I| = 0$. Remember that $|A|$ means, "Compute the determinant of A." So in 2D, we need to find the value λ so that

$$\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0.$$

This definition extends to any square matrix. In 3D, the eigenvalues are the solutions to the equation

$$\left| \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} a - \lambda & b & c \\ d & e - \lambda & f \\ g & h & i - \lambda \end{vmatrix} = 0.$$

An eigenvector of A corresponding to λ is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda x$.

As you continue taking more upper level science courses (in physics, engineering, mathematics, chemistry, and more) you'll soon see that eigenvalues and eigenvectors play a huge role. You'll start to see them in most of your classes. For now, we'll use them without proof to apply the second derivative test. In class, make sure you ask me to show you pictures with each problem we do, so we can see how eigenvalues and eigenvectors appear in surfaces.

Theorem 7.5 (The Second Derivative Test)**.** *Let $f(x, y)$ be a function so that all the second partial derivatives exist and are continuous. The second derivative of f, written $D^2 f$ and sometimes called the Hessian of f, is a square matrix. Let λ_1 be the largest eigenvalue of $D^2 f$, and λ_2 be the smallest eigenvalue. Then λ_1 is the largest possible second derivative obtained in any direction. Similarly, the smallest possible second derivative obtained in any direction is λ_2. The eigenvectors give the directions in which these extreme second derivatives are obtained. The second derivative test states the following.*

Suppose (a, b) is a critical point of f, meaning $Df(a, b) = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

- *If all the eigenvalues of $D^2 f(a, b)$ are positive, then in every direction the function is concave upwards at (a, b) which means the function has a local minimum at (a, b).*
- *If all the eigenvalues of $D^2 f(a, b)$ are negative, then in every direction the function is concave downwards at (a, b). This means the function has a local maximum at (a, b).*

- *If the smallest eigenvalue of $D^2f(a,b)$ is negative, and the largest eigenvalue of $D^2f(a,b)$ is positive, then in one direction the function is concave upwards, and in another the function is concave downwards. The point (a,b) is called a saddle point.*
- *If the largest or smallest eigenvalue of f equals 0, then the second derivative tests yields no information.*

Example 7.6. Consider the function $f(x,y) = x^2 - 2x + xy + y^2$. The first and second derivatives are

$$Df(x,y) = \begin{bmatrix} 2x - 2 + y, x + 2y \end{bmatrix} \quad \text{and} \quad D^2f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The first derivative is zero (the zero matrix) when both $2x - 2 + y = 0$ and $x + 2y = 0$. We need to solve the system of equations $2x + y = 2$ and $x + 2y = 0$. Double the second equation, and then subtract it from the first to obtain $0x - 3y = 2$, or $y = -2/3$. The second equation says that $x = -2y$, or that $x = 4/3$. So the only critical point is $(4/3, -2/3)$.

We find the eigenvalues of $D^2f(4/3, -2/3)$ by solving the equation

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 1 = 0.$$

In this example, the second derivative is constant, so the point $(4/3, -2/3)$ did not change the matrix. In general, the point will affect your matrix. See Sage to see a graph which shows the eigenvectors in which the largest and smallest second derivatives occur.

Expanding the left hand side gives us $4 - 4\lambda + \lambda^2 - 1 = 0$. Simplifying and factoring gives us $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$. This means the eigenvalues are $\lambda = 1$ and $\lambda = 3$. Since both numbers are positive, the function is concave upwards in every direction. The critical point $(4/3, -2/3)$ corresponds to a local minimum of the function. The local minimum is the output $f(4/3, -2/3) = (4/3)^2 - 2(4/3) + (4/3)(-2/3) + (-2/3)^2$.

Problem 7.14 Consider the function $f(x,y) = x^2 + 4xy + y^2$.

See 14.7 for more practice.

1. Find the critical points of f by finding when $Df(x,y)$ is the zero matrix.
2. Find the eigenvalues of D^2f at any critical points.
3. Label each critical point as a local maximum, local minimum, or saddle point, and state the value of f at the critical point.

Problem 7.15 Consider the function $f(x,y) = x^3 - 3x + y^2 - 4y$.

1. Find the critical points of f by finding when $Df(x,y)$ is the zero matrix.
2. Find the eigenvalues of D^2f at any critical points. [Hint: First compute D^2f. Since there are two critical points, evaluate the second derivative at each point to obtain 2 different matrices. Then find the eigenvalues of each matrix.]
3. Label each critical point as a local maximum, local minimum, or saddle point, and state the value of f at the critical point.

Problem 7.16 Consider the function $f(x,y) = x^3 + 3xy + y^3$.

1. Find the critical points of f by finding when $Df(x,y)$ is the zero matrix.
2. Find the eigenvalues of D^2f at any critical points.

3. Label each critical point as a local maximum, local minimum, or saddle point, and state the value of f at the critical point.

You now have the tools needed to find optimal solutions to problems in any dimension. Here's a silly problem that demonstrates how we can use what we've just learned.

Problem 7.17: Optional For my daughter's birthday, she has asked for a Barbie princess cake. I purchased a metal pan that's roughly in the shape of a paraboloid $z = f(x, y) = 9 - x^2 - y^2$ for $z \geq 0$. To surprise her, I want to hide a present inside the cake. The present is a bunch of small candy that can pretty much fill a box of any size. I'd like to know how large (biggest volume) of a rectangular box I can fit under the cake, so that when we start cutting the cake, she'll find her surprise present. The box will start at $z = 0$ and the corners of the box (located at $(x, \pm y)$ and $(-x, \pm y)$) will touch the surface of the cake $z = 9 - x^2 - y^2$.

1. What is the function $V(x, y)$ that we are trying to maximize?
2. If you find all the critical points of V, you'll discover there are 9. However, only one of these critical points makes sense in the context of this problem. Find that critical point.
3. Use the second derivative test to prove that the critical point yields a maximum volume.
4. What are the dimensions of the box? What's the volume of the box?

The only thing left for me is to now determine how much candy I should buy to fill the box. I'll take care of that.

In this problem, we'll derive the version of the second derivative test that is found in most multivariate calculus texts. The test given below only works for functions of the form $f : \mathbb{R}^2 \to \mathbb{R}$. The eigenvalue test you have been practicing will work with a function of the form $f : \mathbb{R}^n \to \mathbb{R}$, for any natural number n.

Problem 7.18: Optional Suppose that $f(x, y)$ has a critical point at (a, b).

1. Find a general formula for the eigenvalues of $D^2 f(a, b)$. Your answer will be in terms of the second partials of f.
2. Let $D = f_{xx} f_{yy} - f_{xy}^2$.
   - If $D < 0$, explain why f has a saddle point at (a, b).
   - If $D = 0$, explain why the second derivative test fails.
   - If $D > 0$, explain why f has either a maximum or minimum at (a, b).
   - If $D > 0$, and $f_{xx}(a, b) > 0$, does f have a local max or local min at (a, b). Explain.
3. The only critical point of $f(x, y) = x^2 + 3xy + 2y^2$ is at $(0, 0)$. Does this point correspond to a local maximum, local minimum, or saddle point? Give the eigenvalues (which should come instantly out of part 1). Find D, from part 2, to answer the question.

7.3 Lagrange Multipliers

The last problem was an example of an optimization problem where we wish to optimize a function (the volume of a box) subject to a constraint (the box has to fit inside a cake). If you are economics student, this section may be the key reason why you were asked to take multivariate calculus. In the business world, we often want to optimize something (profit, revenue, cost, utility, etc.) subject to some constraint (a limited budget, a demand curve, warehouse space, employee hours, etc.). An aerospace engineer will build the best wing that can withstand given forces. Everywhere in the engineering world, we often seek to create the "best" thing possible, subject to some outside constraints. Lagrange discovered an extremely useful method for answering this question, and today we call it "Lagrange Multipliers."

Rather than introduce Cobb-Douglass production functions (from economics) or sheer-stress calculations (from engineering), we'll work with simple examples that illustrate the key points. Sometimes silly examples carry the message across just as well.

Problem 7.19 Suppose an ant walks around the circle $g(x, y) = x^2 + y^2 = 1$. As the ant walks around the circle, the temperature is $f(x, y) = x^2 + y + 4$. Our goal is to find the maximum and minimum temperatures reached by the ant as it walks around the circle. We want to optimize $f(x, y)$ subject to the constraint $g(x, y) = 1$.

1. Draw the circle $g(x, y) = 1$. Then, on the same set of axes, draw several level curves of f. The level curves $f = 3, 4, 5, 6$ are a good start. Then add more (maybe at each 1/4th). If you make a careful, accurate graph, it will help a lot below.

2. Based solely on your graph, where does the minimum temperature occur? What is the minimum temperature?

3. If the ant is at the point $(0, 1)$, and it moves left, will the temperature rise or fall? What if the ant moves right?

4. On your graph, place a dot(s) where you believe the ant reaches a maximum temperature (it may occur at more than one spot). Explain why you believe this is the spot where the maximum temperature occurs. What about the level curves tells you that these spots should be a maximum.

5. Draw the gradient of f at the places where the minimum and maximum temperatures occur. Also draw the gradient of g at these spots. How are the gradients of f and g related at these spots?

Theorem 7.7 (Lagrange Multipliers)**.** *Suppose f and g are continuously differentiable functions. Suppose that we want to find the maximum and minimum values of f subject to the constraint $g(x, y) = c$ (where c is some constant). Then if a maximum or minimum occurs, it must occur at a spot where the gradient of f and the gradient of g point in the same, or opposite, directions. So the gradient of g must be a multiple of the gradient of f. To find the maximum and minimum values (if they exist), we just solve the system of equations that result from*

$$\vec{\nabla} f = \lambda \vec{\nabla} g, \quad \text{and} \quad g(x, y) = c$$

where λ is the proportionality constant. The maximum and minimum values will be among the solutions of this system of equations.

Problem 7.20 Suppose an ant walks around the circle $x^2 + y^2 = 1$. As the ant walks around the circle, the temperature is $T(x, y) = x^2 + y + 4$. Our goal is to find the maximum and minimum temperatures T reached by the ant as it walks around the circle.

1. What function $f(x, y)$ do we wish to optimize? What is the constraint $g(x, y) = c$?
2. Find the gradient of f and the gradient of g. Then solve the system of equations that you get from the equations

$$\vec{\nabla} f = \lambda \vec{\nabla} g, \qquad x^2 + y^2 = 1.$$

You should obtain 4 ordered pairs (x, y).

The most common error on this problem is to divide both sides of an equation by x, which could be zero. If you do this, you'll only get 2 ordered pairs.

3. At each ordered pair, find the temperature. What is the maximum temperature obtained? What is the minimum temperature obtained.

Problem 7.21 Consider the curve $xy^2 = 54$ (draw it). The distance from each point on this curve to the origin is a function that must have a minimum value. Find a point (a, b) on the curve that is closest to the origin.

See 14.8 for more practice.

[The distance to the origin is $d(x, y) = \sqrt{x^2 + y^2}$. This distance is minimized when $f(x, y) = x^2 + y^2$ is minimized. So use $f(x, y) = x^2 + y^2$ as the function you wish to minimize. What's the constraint $g(x, y) = c$?]

Problem 7.22 Find the dimensions of the rectangular box with maximum volume that can be inscribed inside the ellipsoid

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{5^2} = 1.$$

[What is the function f you wish to optimize? What is the constraint $g = c$? Try solving each equation for λ so you can eliminate it from the problem.]

Problem 7.23 Repeat problem 7.17, but this time use Lagrange multipliers. Find the dimensions of the rectangular box of maximum volume that fits underneath the surface $z = f(x, y) = 9 - x^2 - y^2$ for $z \geq 0$.

[Hint: Let $f(x, y, z) = (2x)(2y)(z)$ and $g(x, y, z) = z + x^2 + y^2 = 9$. You'll get a system of 4 equations with 4 unknowns (x, y, z, λ). Try solving each equation for lambda. You know x, y, z can't be zero or negative, so ignore those possible cases.]

7.4 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 8

Line Integrals

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Describe how to integrate a function along a curve. Use line integrals to find the area of a sheet of metal with height $z = f(x, y)$ above a curve $\vec{r}(t) = (x, y)$ and the average value of a function along a curve.
2. Find the following geometric properties of a curve: centroid, mass, center of mass.
3. Compute the work (flow, circulation) and flux of a vector field along and across piecewise smooth curves.
4. Determine if a field is a gradient field (hence conservative), and use the fundamental theorem of line integrals to simplify work calculations.

You'll have a chance to teach your examples to your peers prior to the exam. Table 8.1 contains a summary of the key ideas for this chapter.

Surface Area	$\sigma = \int_C d\sigma = \int_C f ds = \int_a^b f \left\lvert \frac{d\vec{r}}{dt} \right\rvert dt$
Average Value	$\bar{f} = \frac{\int f ds}{\int ds}$
Work, Flow, Circulation	$W = \int_C d\text{Work} = \int_C (\vec{F} \cdot \vec{T}) ds = \int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy$
Flux	$\text{Flux} = \int_C d\text{Flux} = \int_C \vec{F} \cdot \vec{n} ds = \oint_C M dy - N dx$
Mass	$m = \int_C dm = \int_C \delta ds$
Centroid	$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\int x ds}{\int_C ds}, \frac{\int y ds}{\int_C ds}, \frac{\int z ds}{\int_C ds} \right)$
Center of Mass	$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\int x dm}{\int_C dm}, \frac{\int y dm}{\int_C dm}, \frac{\int z dm}{\int_C dm} \right)$
Fund. Thm of Line Int.	$f(B) - f(A) = \int_C \vec{\nabla} f \cdot d\vec{r}$

Table 8.1: A summary of the ideas in this unit.

I have created a YouTube playlist to go along with this chapter. Each video is about 4-6 minutes long.

- YouTube playlist for 08 - Line Integrals.
- A PDF copy of the finished product (so you can follow along on paper).

You'll also find the following links to Sage can help you speed up your time spent on homework. Thanks to Dr. Jason Grout at Drake university for contributing many of these (as well as being a constant help with editing, rewriting, and giving me great feedback). Thanks Jason.

- Sage Links
- Mathematica Notebook (If you have installed Mathematica)

If you would like homework problems from the text that line up with the ideas we are studying, please use the following table.

Topic (12th ed.)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Line integrals	16.1	1-8, 9-26, 33-42	27-32		43-46
Work, Flow, Circulation, Flux	16.2	7-12, 19-24, 31-36, 47-50	13-18, 25-30, 37-38,	51-54	55-60
Gradient Fields	16.2	1-6			
Gradient Fields	14.5	1-10			
Potentials	16.3	1-12,13-24	25-33	34-38	

8.1 Work, Flow, Circulation, and Flux

Now that we can describe motion, let's turn our attention to the work done by a vector field as we move through the field. Recall that work is a transfer of energy. Consider the following examples:

- A tornado picks up a couch and applies forces to the couch as it swirls around the center. Work is the transfer of the energy from the tornado to the couch, giving the couch its kinetic energy.
- When an object falls, gravity does work on the object. The work done by gravity converts potential energy to kinetic energy.
- If we consider the flow of water down a river, it is gravity that gives the water its kinetic energy. We can place a hydroelectric dam next to a river to capture a lot of this kinetic energy. Work transfers the kinetic energy of the river to rotational energy of the turbine, which eventually ends up as electrical energy available in our homes.

When we study work, we are really studying how energy is transferred. This is one of the key components of modern life.

Let's start with a review. Recall that the work done by a vector field $\vec{F}$ through a displacement $\vec{d}$ is the dot product $\vec{F}\cdot\vec{d}$.

Review An object moves from $A=(6,0)$ to $B=(0,3)$. Along the way, it encounters the constant force $\vec{F}=(2,5)$. How much work is done by $\vec{F}$ as the object moves from A to B? See [1].

Problem 8.1 An object moves from $A=(6,0)$ to $B=(0,3)$. A parametrization of the object's path is $\vec{r}(t)=(-6,3)t+(6,0)$ for $0\le t\le 1$.

[1] The displacement is $B-A=(-6,3)$. The work is $\vec{F}\cdot\vec{d}=(2,5)\cdot(-6,3)=-12+15=3$.

1. For $0 \leq t \leq .5$, the force encountered is $\vec{F} = (2, 5)$. For $.5 \leq t \leq 1$, the force encountered is $(2, 6)$. How much work is done in the first half second? How much work is done in the last half second? How much total work is done?

2. If we encounter a constant force $\vec{F}$ over a small displacement $d\vec{r}$, explain why the work done is $dW = \vec{F} \cdot d\vec{r} = F \cdot \frac{d\vec{r}}{dt}dt$.

3. Suppose that the force constantly changes as we move along the curve. At t, we'll assume we encountered the force $F(t) = (2, 5 + 2t)$, which we could think of as the wind blowing stronger and stronger to the north. Explain why the total work done by this force along the path is

$$W = \int \vec{F} \cdot d\vec{r} = \int_0^1 (2, 5 + 2t) \cdot (-6, 3)dt.$$

   Then compute this integral. It should be slightly larger than the first part.

4. (Optional) If you are familiar with the units of energy, complete the following. What are the units of $\vec{F}$, $d\vec{r}$, and dW.

You can visualize what's happening in this problem as follows. Attach a clothesline between the points (maybe representing two trees in your backyard). Put a cub scout space derby ship on the clothesline. Then the wind starts to blow. As the ship moves along the clothesline, the wind changes direction.

If a force F acts through a displacement d, then the most basic definition of work is $W = Fd$, the product of the force and the displacement. This basic definition has a few assumptions.

- The force F must act in the same direction as the displacement.
- The force F must be constant throughout the displacement.
- The displacement must be in a straight line.

We used the dot product to remove the first assumption when we showed that the work is simply the dot product

$$W = \vec{F} \cdot \vec{r},$$

where $\vec{F}$ is a force acting through a displacement $\vec{r}$. The previous problem showed that we can remove the assumption that $\vec{F}$ is constant to obtain

$$W = \int \vec{F} \cdot d\vec{r} = \int_a^b F \cdot \frac{d\vec{r}}{dt}dt,$$

provided we have a parametrization of $\vec{r}$ with $a \leq t \leq b$. The next problem gets rid of the assumption that $\vec{r}$ is a straight line.

Problem 8.2 Suppose that we move along the circle C parametrized by $\vec{r}(t) = (3\cos t, 3\sin t)$. As we move along C, we encounter a rotational force $\vec{F}(x, y) = (-2y, 2x)$.

Watch a YouTube video about work.

1. Draw C. Then at several points on the curve, draw the vector field $\vec{F}(x, y)$. For example, at the point $(3, 0)$ you should have the vector $\vec{F}(3, 0) = (-2(0), 2(3)) = (0, 6)$, a vector sticking straight up 6 units. Are we moving with the vector field, or against the vector field?

2. Explain why we can state that a little bit of work done over a small displacement is $dW = \vec{F} \cdot d\vec{r}$. Why does it not matter that $\vec{r}$ moves in a straight line?

3. Since a little work done by $\vec{F}$ along a small bit of C is $dW = \vec{F}\cdot d\vec{r}$, we know that the total work done is $\int dW = \int \vec{F}\cdot d\vec{r}$. This gives us

$$W = \int_C (-2y, 2x)\cdot d\vec{r} = \int_0^{2\pi} (-2(3\sin t), 2(3\cos t))\cdot(-3\sin t, 3\cos t)dt.$$

Complete the integral, showing that the work done by $\vec{F}$ along C is 36π.

We put the C under the integral $\int_C$ to remind us that we are integrating along the curve C. This means we need to get a parametrization of the curve C, and give bounds before we can integrate with respect to t.

Definition 8.1. The work done by a vector field $\vec{F}$, along a curve C with parametrization $\vec{r}(t)$ for $a \le t \le b$, is

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \frac{d\vec{r}}{dt}dt.$$

If we let $\vec{F} = (M, N)$ and we let $\vec{r}(t) = (x, y)$, so that $d\vec{r} = (dx, dy)$, then we can write work in the differential form

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C (M, N)\cdot(dx, dy) = \int_C Mdx + Ndy.$$

Review Consider the curve $y = 3x^2 - 5x$ for $-2 \le x \le 1$. Give a parametrization of this curve. See [2].

Problem 8.3 Consider the parabolic curve $y = 4 - x^2$ for $-1 \le x \le 2$, and the vector field $\vec{F}(x, y) = (2x + y, -x)$.

Please use this Sage link to check your work.

1. Give a parametrization $\vec{r}(t)$ of the parabolic curve that starts at $(-1, 3)$ and ends at $(2, 0)$. See the review problem above if you need a hint.
2. Compute $d\vec{r}$ and state dx and dy. What are M and N in terms of t?
3. Compute the work done by $\vec{F}$ on an object that moves along the parabola from $(-1, 3)$ to $(2, 0)$ (i.e. compute $\int_C Mdx + Ndy$).
4. How much work is done by $\vec{F}$ to move an object along the parabola from $(2, 0)$ to $(-1, 3)$. In other words, if you traverse along a path backwards, how much work is done?

Click the link to check your answer with Sage.

Problem 8.4 Again consider the vector field $\vec{F}(x, y) = (2x + y, -x)$. In the previous problem we considered how much work was done by $\vec{F}$ as an object moved along the the parabolic curve $y = 4 - x^2$ for $-1 \le x \le 2$. We now want to know how much work is done to move an object along a straight line from $(-1, 3)$ to $(2, 0)$.

1. Give a parametrization $\vec{r}(t)$ of the straight line curve that starts at $(-1, 3)$ and ends at $(2, 0)$. Make sure you give bounds for t.
2. Compute $d\vec{r}$ and state dx and dy. What are M and N in terms of t?
3. Compute the work done by $\vec{F}$ to move an object along the straight line path from $(-1, 3)$ to $(2, 0)$. Check your answer with Sage.

When you enter your curve in Sage, remember to type the times symbol in "(3*t-1, ...)". Otherwise, you'll get an error.

[2] Whenever you have a function of the form $y = f(x)$, you can always use $x = t$ and $y = f(t)$ to parametrize the curve. So we can use $\vec{r}(t) = (t, 3t^2 - 5t)$ for $-2 \le t \le 1$ as a parametrization.

4. Optional (we'll discuss this in class if you don't have it). How much work does it take to go along the closed path that starts at $(2,0)$, follows the parabola $y = 4 - x^2$ to $(-1,3)$, and then returns to $(2,0)$ along a straight line. Show that this total work is $W = -9$.

The examples above showed us that we can compute work along any curve. All we have to do is parametrize the curve, take a derivative, and then compute $dW = \vec{F} \cdot d\vec{r}$. This gives us a little bit of work along a curve, and we sum up the little bits of work (integrate) to find the total work.

In the examples above, the vector fields represented forces. However, vector fields can represent much more than just forces. The vector field might represent the flow of water down a river, or the flow of air across an airplane wing. When we think of the vector field as a velocity field, we might ask how much of the fluid flows along our curve. Alternately, we could ask how much of the fluid flows across our curve. These two questions lead to flow along a curve, and flux across a curve. The flow along a curve is directly related to the lift of an airplane wing (which occurs when the flow along the top of the wing is different than the flow below the wing). The flux across a curve takes us to powering a wind mill as wind flows across the surface of a blade (once we hit 3D integrals).

Review Consider the unit vector $\vec{T} = \dfrac{(3,4)}{5}$. Give two unit vectors that are orthogonal to $\vec{T}$. See [3].

Problem 8.5: Intro to Flux Consider the curve $\vec{r}(t) = (5\cos t, 5\sin t)$ and the vector field $\vec{F}(x,y) = (3x, 3y)$. This is a radial field that pushes things straight outwards (away from the origin).

1. Compute the work $W = \int_C (M,N) \cdot (dx,dy)$ and show it equals zero. (Can you give a reason why it should be zero?)

   See Sage for the work calculation.

2. Compute the integral $\Phi = \int_C (M,N) \cdot (dy,-dx)$ and show it equals 150π.

   See Sage for this flux computation.

3. How does the vector $(dy,-dx)$ relate to the tangent vector (dx,dy)? Make a guess as to what $\Phi = \int_C (M,N) \cdot (dy,-dx)$ measures.

Definition 8.2: Flow, Circulation, and Flux. Suppose C is a smooth curve with parametrization $\vec{r}(t) = (x,y)$. Suppose that $\vec{F}(x,y)$ is a vector field that represents the velocity of some fluid (like water or air).

- Watch a YouTube video about flow and circulation.
- Watch a YouTube video about flux.

- We say that C is a closed curve if C begins and ends at the same point.
- We say that C is a simple curve if C does not cross itself.
- The flow of $\vec{F}$ along C is the integral

$$\text{Flow} = \int_C (M,N) \cdot (dx,dy) = \int_C M\,dx + N\,dy.$$

- If C is a simple closed curve parametrized counter clockwise, then the flow of $\vec{F}$ along C is called circulation, and we write

$$\text{Circulation} = \oint_C M dx + N dy.$$

Any time you see a circle around an integral, it means that you're integrating along a closed curve.

- The flux of $\vec{F}$ across C is the flow of the fluid across the curve (an area/second). If C is a simple closed curve parametrized counter clockwise, then the outward flux is the integral

$$\text{Flux} = \Phi = \oint_C (M, N) \cdot (dy, -dx) = \oint_C M dy - N dx.$$

Problem 8.6 Consider the vector field $\vec{F}(x, y) = (2x + y, -x + 2y)$. When you construct a plot of this vector field, you'll notice that it causes objects to spin outwards in the clockwise direction. Suppose an object moves counterclockwise around a circle C of radius 3 that is centered at the origin. (You'll need to parameterize the curve.)

If you haven't yet, please watch the YouTube videos for

- work,
- flow and circulation, and
- flux.

1. Should the circulation of $\vec{F}$ along C be positive or negative? Make a guess, and then compute the circulation $\oint_C M dx + N dy$. Whether your guess was right or wrong, explain why you made the guess.
2. Should the flux of $\vec{F}$ across C be positive or negative? Make a guess, and then actually compute the flux $\oint_C M dy - N dx$. Whether your guess was right or wrong, explain why you made the guess.
3. Please use this Sage link to check both computations.

We'll tackle more work, flow, circulation, and flux problems as we proceed through this chapter.

8.2 Area and Average Value

In first semester calculus, we learned that the area under a function $f(x)$ above the x-axis is given by $A = \int_a^b f(x)dx$. The quantity $dA = f(x)dx$ represents a small bit of area whose length is dx and whose height is $f(x)$. To get the total area, we just added up the little bits of area, which is why

$$A = \int dA = \int_a^b f(x)dx.$$

We found earlier this semester that we can obtain the surface area of a region that lies above a curve $\vec{r}(t)$ in the xy plane with height $f(x, y)$ by using the formula

$$\sigma = \int d\sigma = \int_C f(x, y)ds = \int_C f(x(t), y(t)) \left| \frac{d\vec{r}(t)}{dt} \right| dt.$$

The next exercise reminds us of this process.

Exercise Consider the surface in space that is below the function $f(x, y) = 9 - x^2 - y^2$ and above the curve C parametrized by $\vec{r}(t) = (2\cos t, 3\sin t)$ for $t \in [0, 2\pi]$. Think of this region as a metal sheet that has been stood up with its edge along C, where the height above each spot is given by $z = f(x, y)$. Follow

Watch a YouTube video.

this Sage link for a picture of the sheet. If we cut the curve up into lots of tiny bits, the length of each bit is approximately given by the arc length differential

See Problem 6.19

$$ds = \left|\frac{d\vec{r}}{dt}\right| dt = \sqrt{(-2\sin t)^2 + (3\cos t)^2}dt.$$

The height of the surface above each little arc is given by $f(x, y)$, and so the surface area of a little part of the surface is

$$d\sigma = f ds = (9 - 4\cos^2 t - 9\sin^2 t)\sqrt{(-2\sin t)^2 + (3\cos t)^2}dt.$$

This means the total area of the metal sheet that lies above C and under f is given by the integral

$$\sigma = \int_C f ds = \int_0^{2\pi} \underbrace{(9 - (2\cos t)^2 - (3\sin t)^2)}_{f(\vec{r}(t))} \underbrace{\sqrt{(-2\sin t)^2 + (3\cos t)^2}dt}_{ds}.$$

This Sage worksheet will compute the integral for you.

Our results from the exercise above suggest the following definition.

Definition 8.3: Line Integral. Let f be a function and let C be a piecewise smooth curve whose parametrization is $\vec{r}(t)$ for $t \in [a, b]$. We'll require that the composition $f(\vec{r}(t))$ be continuous for all $t \in [a, b]$. Then we define the line integral of f over C to be the integral

The line integral is also called the path integral, contour integral, or curve integral.

$$\int_C f ds = \int_a^b f(\vec{r}(t))\frac{ds}{dt}dt = \int_a^b f(\vec{r}(t))\left|\frac{d\vec{r}}{dt}\right| dt.$$

Notice that this definition suggests the following four steps. These four steps are the key to computing any line integral.

When we ask you to set up a line integral, it means that you should do steps 1–3, so that you get an integral with a single variable and with bounds that you could plug into a computer or complete by hand.

1. Start by getting a parametrization $\vec{r}(t)$ for $a \le t \le b$ of the curve C.
2. Find the speed by computing the velocity $\frac{d\vec{r}}{dt}$ and then the speed $\left|\frac{d\vec{r}}{dt}\right|$.
3. Multiply f by the speed, and replace each x, y, and/or z with what it equals in terms of t.
4. Integrate the product from the previous step. Practice doing this by hand on every problem, unless it specifically says to use technology. Some of the integrals are impossible to do by hand.

You should use the Sage line integral calculator to check all your answers.

Problem 8.7 Let $f(x, y, z) = x^2 + y^2 - 2z$ and let C be two coils of the helix $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$, starting at $t = 0$. Remember that the parameterization means $x = 3\cos t$, $y = 3\sin t$, and $z = 4t$. Compute $\int_C f ds$. [You will have to find the end bound yourself. How much time passes to go around two coils?]

See 16.1: 9-32. Some problems give you a parametrization, some expect you to come up with one on your own.

Check your answer with Sage.

Problem 8.8 Consider the function $f(x, y) = 3xy + 2$. Let C be a circle of radius 4 centered at the origin. Compute $\int_C f ds$. [You'll have to come up with your own parameterization.]

To practice matching parameterizations to curves, try 16.1:1-8.

Check your answer with Sage.

Problem 8.9 Let $f(x, y, z) = x^2 + 3yz$. Let C be the straight line segment from $(1, 0, 0)$ to $(0, 4, 5)$. Compute $\int_C f ds$.

If you've forgotten how to parametrize line segments, see 2.9.

Check your answer with Sage.

Problem 8.10 Let $f(x, y) = x^2 + y^2 - 25$. Let C be the portion of the parabola $y^2 = x$ between $(1, -1)$ and $(4, 2)$. We want to compute $\int_C f ds$.

See 5.20 if you forgot how to parametrize plane curves.
Check your answer with Sage.

1. Draw the curve C and the function $f(x, y)$ on the same 3D xyz axes.
2. Without computing the line integral $\int_C f ds$, determine if the integral should be positive or negative. Explain why this is so by looking at the values of $f(x, y)$ at points along the curve C. Is $f(x, y)$ positive, negative, or zero, at points along C?
3. Parametrize the curve and set up the line integral $\int_C f ds$. [Hint: if you let $y = t$, then $x =$? What bounds do you put on t?]
4. Use technology to compute $\int_C f ds$ to get a numeric answer. Was your answer the sign that you determined above?

8.3 Average Value

The concept of averaging values together has many applications. In first-semester calculus, we saw how to generalize the concept of averaging numbers together to get an average value of a function. We'll review both of these concepts. Later, we'll generalize average value to calculate centroids and center of mass.

Problem 8.11 Suppose a class takes a test and there are three scores of 70, five scores of 85, one score of 90, and two scores of 95. We will calculate the average class score, $\bar{s}$, four different ways to emphasize four ways of thinking about the averages. We are emphasizing the pattern of the calculations in this problem, rather than the final answer, so it is important to write out each calculation completely in the form $\bar{s} =$ ______ before calculating the number $\bar{s}$.

1. Compute the average by adding 11 numbers together and dividing by the number of scores. Write down the whole computation before doing any arithmetic.

   $\bar{s} = \frac{\sum \text{values}}{\text{number of values}}$

2. Compute the numerator of the fraction in the previous part by multiplying each score by how many times it occurs, rather than adding it in the sum that many times. Again, write down the calculation for $\bar{s}$ before doing any arithmetic.

   $\bar{s} = \frac{\sum(\text{value}\cdot\text{weight})}{\sum \text{weight}}$

3. Compute $\bar{s}$ by splitting up the fraction in the previous part into the sum of four numbers. This is called a "weighted average" because we are multiplying each score value by a weight.

   $\bar{s} = \sum(\text{value} \cdot (\% \text{ of stuff}))$

4. Another way of thinking about the average $\bar{s}$ is that $\bar{s}$ is the number so that if all 11 scores were the same value $\bar{s}$, you'd have the same sum of scores. Write this way of thinking about these computations by taking the formulas for $\bar{s}$ in the first two parts and multiplying both sides by the denominator.

   $(\text{number of values})\bar{s} = \sum \text{values}$
   $(\sum \text{weight})\bar{s} = \sum(\text{value} \cdot \text{weight})$

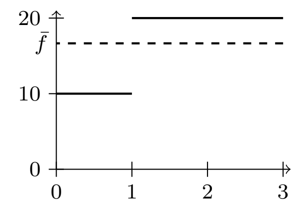
[3] We just reverse the order and change a sign to get $\vec{N}_1 = \frac{(-4, 3)}{5}$ and $\vec{N}_1 = \frac{(4, -3)}{5}$ as orthogonal unit vectors.

In the next problem, we generalize the above ways of thinking about averages from a discrete situation to a continuous situation. You did this in first-semester calculus when you did average value using integrals.

Problem 8.12 Suppose the price of a stock is \$10 for one day. Then the price of the stock jumps to \$20 for two days. Our goal is to determine the average price of the stock over the three days.

1. Why is the average stock price not \$15? Use any of the methods from the previous problem to show that the average price is $\bar{f} = 50/3$.

2. We can use the function $f(t) = \begin{cases} 10 & 0 \leq t < 1 \\ 20 & 1 \leq t \leq 3 \end{cases}$ to model the price of the stock for the three-day period. The graph to the right shows both f and $\bar{f}$ over the three day period. Compute the area under both f and $\bar{f}$ for $0 \leq t \leq 3$, and show that the two areas are equal.



The solid line shows the graph of f while the dashed line shows the average price $\bar{f}$.

3. The average value of a function over an interval $[a, b]$ is a constant value $\bar{f}$ so that the areas under both f and $\bar{f}$ are equal, which means

$$\int_a^b \bar{f} dx = \int_a^b f dx.$$

Solve for $\bar{f}$ symbolically (without doing any of the integrals). This quantity we call the average value of f over $[a, b]$.

Ask me in class about the "ant farm" approach to average value.

Problem 8.13 Consider the elliptical curve C given by the parametrization $\vec{r}(t) = (2\cos t, 3\sin t)$. Let f be the function $f(x, y) = 9 - x^2 - y^2$.

Watch a YouTube video.

1. Draw the surface f in 3D. Add to your drawing the curve C in the xy plane. Then draw the sheet whose area is given by the integral $\int_C f ds$. Please head to http://bmw.byuimath.com/dokuwiki/doku.php?id=surface_area_example to check your work.

2. What's the maximum height and minimum height of the sheet?

See problem 6.19.

3. We'd like to find a constant height $\bar{f}$ so that the area under f, above C, is the same as the area under $\bar{f}$, above C. This height $\bar{f}$ we call the average value of f along C. Explain why the average value of f along C is

Please read Isaiah 40:4 and Luke 3:5. These scriptures should help you remember how to find average value.

$$\bar{f} = \frac{\int_C f ds}{\int_C ds}.$$

[Hint: The area under $\bar{f}$ above C is $\int_C \bar{f} ds$. The area under f above C is $\int_C f ds$. Set them equal and solve for $\bar{f}$.]

4. Use a computer to evaluate the integrals $\int_C f ds$ and $\int_C ds$, and then give an approximation to the average value of f along C. How does your average value compare to the maximum and minimum of f along C?

Problem 8.14 The temperature $T(x, y, z)$ at points on a wire helix C given by $\vec{r}(t) = (\sin t, 2t, \cos t)$ is known to be $T(x, y, z) = x^2 + y + z^2$. What are the temperatures at $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$ and $t = 2\pi$? You should notice the temperature is constantly changing. Make a guess as to what the average temperature is (share with the class why you made the guess you made—it's OK if you're wrong). Then compute the average temperature of the wire using the integral formula from the previous problem. Do the computations by hand.

8.4 Physical Properties

A number of physical properties of real-world objects can be calculated using the concepts of averages and line integrals. We explore some of these in this section. Additionally, many of these concepts and calculations are used in statistics.

8.4.1 Centroids

Definition 8.4: Centroid. Let C be a curve. If we look at all of the x-coordinates of the points on C, the "center" x-coordinate, $\bar{x}$, is the average of all these x-coordinates. Likewise, we can talk about the averages of all of the y coordinates or z coordinates of points on the function ($\bar{y}$ or $\bar{z}$, respectively). The *centroid* of an object is the geometric center $(\bar{x}, \bar{y}, \bar{z})$, the point with coordinates that are the average x, y, and z coordinates. The average value formula gives the coordinates of the centroid as

Watch a YouTube video.

These are the formulas for the centroid.

$$\bar{x} = \frac{\int_C x ds}{\int_C ds}, \quad \bar{y} = \frac{\int_C y ds}{\int_C ds}, \quad \text{and} \quad \bar{z} = \frac{\int_C z ds}{\int_C ds}.$$

Notice that the denominator in each case is just the arc length $s = \int_C ds$.

Problem 8.15 Let C be the semicircular arc $\vec{r}(t) = (a \cos t, a \sin t)$ for $t \in [0, \pi]$. Without doing any computations, make an educated guess for the centroid $(\bar{x}, \bar{y})$ of this arc. Then compute the centroid using the integral formulas above. Share with the class your guess, even if it was incorrect.

8.4.2 Mass and Center of Mass

Density is generally a mass per unit volume. However, when talking about a curve or wire, as in this chapter, it's simpler to let density be the mass per unit length. Sometimes an object is made out of a composite material, and the density of the object is different at different places in the object. For example, we might have a straight wire where one end is aluminum and the other end is copper. In the middle, the wire slowly transitions from being all aluminum to all copper. The centroid is the midpoint of the wire. However, since copper has a higher density than aluminum, the balance point (the center of mass) would not be at the midpoint of the wire, but would be closer to the denser and heavier copper end. In this section, we'll develop formulas for the mass and center of mass of such a wire. Such composite materials are engineered all the time (though probably not our example wire). In future mechanical engineering courses, you would learn how to determine the density δ (mass per unit length) at each point on such a composite wire.

Problem 8.16: Mass Suppose a wire C has the parameterization $\vec{r}(t)$ for $t \in [a, b]$. Suppose the wire's density (mass per unit length) at a point (x, y, z) on the wire is given by the function $\delta(x, y, z)$. Since density is a mass per length, multiplying density by a small length ds gives us the mass of a small portion of the curve. We represent this symbolically using $dm = \delta(\vec{r}(t_0))ds$.

Watch a YouTube video.

1. Explain why the mass m of the wire is given by the formulas below (explain why each equal sign is true):

$$m = \int_C dm = \int_C \delta ds = \int_a^b \delta(\vec{r}(t)) \left|\frac{d\vec{r}}{dt}\right| dt.$$

2. Now suppose a wire lies along the straight segment from $(0, 2, 0)$ to $(1, 1, 3)$. A parametrization of this line is $\vec{r}(t) = (t, -t+2, 3t)$ for $t \in [0, 1]$. The wire's density (mass per unit length) at a point (x, y, z) is $\delta(x, y, z) = x + y + z$.

   (a) Is the wire heavier at $(0, 2, 0)$ or at $(1, 1, 3)$?

   (b) What is the total mass of the wire?

The center of mass of an object is the point where the object balances. In order to calculate the x-coordinate of the center of mass, we average the x-coordinates, but we weight each x-coordinate with its mass. Similarly, we can calculate the y and z coordinates of the center of mass.

Wikipedia has some interesting applications of center of mass.

The next problem helps us reason about the center of mass of a collection of objects. Calculating the center of mass of a collection of objects is important, for example, in astronomy when you want to calculate how two bodies orbit each other.

Problem 8.17 Suppose two objects are positioned at the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. Our goal in this problem is to understand the difference between the centroid and the center of mass.

1. Find the centroid of two objects.

2. Suppose both objects have the same mass of 2 kg. Find the center of mass.

3. Suppose we now have 7 objects, all with the same mass of 2 kg. Two of the objects are at P_1, and 5 of the objects are at P_2. Find the center of mass of the 7 objects.

4. Suppose there are two objects, but the mass of the object at point P_1 is 2 kg, and the mass of the object at point P_2 is 5 kg. Will the center of mass be closer to P_1 or P_2? Give a physical reason for your answer before doing any computations. Then find the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the two points. [Hint: You should get $\bar{x} = \frac{2x_1+5x_2}{2+5}$.]

Problem 8.18: Center of mass In problem 8.17, we focused on a system with two points (x_1, y_1) and (x_2, y_2) with masses m_1 and m_2. The center of mass in the x direction is given by

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$$\bar{x} = \frac{x_1m_1 + x_2m_2}{m_1 + m_2} = \frac{\sum_{i=1}^{2} x_i m_i}{\sum_{i=1}^{2} m_i}$$

1. If we consider a system with 3 points, what formula gives the center of mass in the x direction? What if there are 4 points, 5 points, or n points?

2. Suppose now that we have a wire located along a curve C. The density of the wire is known to be $\delta(x, y, z)$ (which could be different at different points on the curve). Imagine cutting the wire into a thousand or more tiny chunks. Each chunk would be centered at some point (x_i, y_i, z_i) and have length ds_i. The mass of each little chunk would be approximately $dm_i \approx \delta ds_i$. Give a formula for the center of mass in the y direction of these thousands of points (x_i, y_i, z_i), each with mass dm_i. [This should almost be an exact copy of the first part.]

3. Explain why
$$\bar{y} = \frac{\int_C y dm}{\int_C dm} = \frac{\int_C y\delta ds}{\int_C \delta ds}.$$

Ask me in class to show you another way to obtain the formula for center of mass. It involves looking at masses weighted by their distance (called a moment of mass). Many of you will have already seen an idea similar to this in statics, but in that class you are talking about moments of force, not moments of mass.

For quick reference, the formulas for the centroid of a wire along C are

$$\bar{x} = \frac{\int_C x ds}{\int_C ds}, \quad \bar{y} = \frac{\int_C y ds}{\int_C ds}, \quad \text{and} \quad \bar{z} = \frac{\int_C z ds}{\int_C ds}. \quad \text{(Centroid)}$$

If the wire has density δ, then the formulas for the center of mass are

$$\bar{x} = \frac{\int_C x dm}{\int_C dm}, \quad \bar{y} = \frac{\int_C y dm}{\int_C dm}, \quad \text{and} \quad \bar{z} = \frac{\int_C z dm}{\int_C dm}, \quad \text{(Center of mass)}$$

where $dm = \delta ds$. Notice that the denominator in each case is just the mass $m = \int_C dm$.

We'll often use the notation $(\bar{x}, \bar{y}, \bar{z})$ to talk about both the centroid and the center of mass. If no density is given in a problem, then $(\bar{x}, \bar{y}, \bar{z})$ is the centroid. If a density is provided, then $(\bar{x}, \bar{y}, \bar{z})$ refers to the center of mass. If the density is constant, it doesn't matter (the centroid and center of mass are the same).

The quantity $\int_C x dm$ is sometimes called the first moment of mass about the yz-plane (so $x = 0$). Notationally, some people write $M_{yz} = \int_C x ds$. Similarly, we could write $M_{xz} = \int_C y dm$ and $M_{xy} = \int_C z dm$. With this notation, we could write the center of mass formulas as
$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right).$$

Problem 8.19 Suppose a wire with density $\delta(x, y) = x^2 + y$ lies along the curve C which is the upper half of a circle around the origin with radius 7.

1. Parametrize C (find $\vec{r}(t)$ and the domain for t).

2. Where is the wire heavier, at $(7, 0)$ or $(0, 7)$? [Compute δ at both.]

3. In problem 8.15, we showed that the centroid of the wire is $(\bar{x}, \bar{y}) = \left(0, \frac{2(7)}{\pi})\right)$. We now seek the center of mass. Before computing, will $\bar{x}$ change? Will $\bar{y}$ change? How will each change? Explain.

4. Set up the integrals needed to find the center of mass. Then use technology to compute the integrals. Give an exact answer (involving fractions), rather than a numerical approximation.

8.5 The Fundamental Theorem of Line Integrals

In this final section we'll return to the concept of work. Many vector fields are actually the derivative of a function. When this occurs, computing work along a curve is extremely easy. All you have to know is the endpoints of the curve, and the function f whose derivative gives you the vector field. This function is called a potential for a vector field. Once we are comfortable finding potentials, we'll show that the work done by such a vector field is the difference in the potential at the end points. This makes finding work extremely fast.

Definition 8.5: Gradients and Potentials. Let $\vec{F}$ be a vector field.

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- A potential for the vector field is a function f whose derivative equals $\vec{F}$. So if $Df = \vec{F}$, then we say that f is a potential for $\vec{F}$.
- When we want to emphasize that the derivative of f is a vector field, we call Df the gradient of f and write $Df = \vec{\nabla} f$.
- If $\vec{F}$ has a potential, then we say that $\vec{F}$ is a gradient field.

The symbol $\vec{\nabla} f$ is read "the gradient of f" or "del f."

We'll quickly see that if a vector field has a potential, then the work done by the vector field is the difference in the potential. If you've ever dealt with kinetic and potential energy, then you hopefully recall that the change in kinetic energy is precisely the difference in potential energy. This is the reason we use the word "potential."

Problem 8.20 Let's practice finding gradients and potentials.

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1. Let $f(x, y) = x^2 + 3xy + 2y^2$. Find the gradient of f, i.e. find $Df(x, y)$. Then compute $D^2 f(x, y)$ (you should get a square matrix). What are f_{xy} and f_{yx}?
2. Consider the vector field $\vec{F}(x, y) = (2x + y, x + 4y)$. Find the derivative of $\vec{F}(x, y)$ (it should be a square matrix). Then find a function $f(x, y)$ whose gradient is $\vec{F}$ (i.e. $Df = \vec{F}$). What are f_{xy} and f_{yx}?
3. Consider the vector field $\vec{F}(x, y) = (2x + y, 3x + 4y)$. Find the derivative of $\vec{F}$. Why is there no function $f(x, y)$ so that $Df(x, y) = \vec{F}(x, y)$? [Hint: look at f_{xy} and f_{yx}.]

See problem 6.13.

Based on your observations in the previous problem, we have the following key theorem.

Theorem 8.6. *Let $\vec{F}$ be a vector field that is everywhere continuously differentiable. Then $\vec{F}$ has a potential if and only if the derivative $D\vec{F}$ is a symmetric matrix. We say that a matrix is symmetric if interchanging the rows and columns results in the same matrix (so if you replace row 1 with column 1, and row 2 with column 2, etc., then you obtain the same matrix).*

Problem 8.21 For each of the following vector fields, start by computing the derivative. Then find a potential, or explain why none exists.

If you haven't yet, please watch this YouTube video.

1. $\vec{F}(x, y) = (2x - y, 3x + 2y)$
2. $\vec{F}(x, y) = (2x + 4y, 4x + 3y)$
3. $\vec{F}(x, y) = (2x + 4xy, 2x^2 + y)$

4. $\vec{F}(x,y,z) = (x+2y+3z, 2x+3y+4z, 2x+3y+4z)$
5. $\vec{F}(x,y,z) = (x+2y+3z, 2x+3y+4z, 3x+4y+5z)$
6. $\vec{F}(x,y,z) = (x+yz, xz+z, xy+y)$
7. $\vec{F}(x,y) = \left(\dfrac{x}{1+x^2} + \arctan(y), \dfrac{x}{1+y^2}\right)$

If a vector field has a potential, then there is an extremely simple way to compute work. To see this, we must first review the fundamental theorem of calculus. The second half of the fundamental theorem of calculus states,

> If f is continuous on $[a,b]$ and F is an anti-derivative of f, then $F(b) - F(a) = \int_a^b f(x)dx$.

If we replace f with f', then an anti-derivative of f' is f, and we can write,

> If f is continuously differentiable on $[a,b]$, then
> $$f(b) - f(a) = \int_a^b f'(x)dx.$$

This last version is the version we now generalize.

Theorem 8.7 (The Fundamental Theorem of Line Integrals)**.** *Suppose f is a continuously differentiable function, defined along some open region containing the smooth curve C. Let $\vec{r}(t)$ be a parametrization of the curve C for $t \in [a,b]$. Then we have*
$$f(\vec{r}(b)) - f(\vec{r}(a)) = \int_a^b Df(\vec{r}(t))D\vec{r}(t)\ dt.$$

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Notice that if $\vec{F}$ is a vector field, and has a potential f, which means $\vec{F} = Df$, then we could rephrase this theorem as follows.

> Suppose $\vec{F}$ is a a vector field that is continuous along some open region containing the curve C. Suppose $\vec{F}$ has a potential f. Let A and B be the start and end points of the smooth curve C. Then the work done by $\vec{F}$ along C depends only on the start and end points, and is precisely
> $$f(B) - f(A) = \int_C \vec{F} \cdot d\vec{r} = \int_C Mdx + Ndy.$$
> The work done by $\vec{F}$ is the difference in a potential.

If you are familiar with kinetic energy, then you should notice a key idea here. Work is a transfer of energy. As an object falls, energy is transferred from potential energy to kinetic energy. The total kinetic energy at the end of a fall is precisely equal to the difference between the potential energy at the top of the fall and the potential energy at the bottom of the fall (neglecting air resistance). So work (the transfer of energy) is exactly the difference in potential energy.

Problem 8.22: Proof of Fundamental Theorem Suppose $f(x,y)$ is continuously differentiable, and suppose that $\vec{r}(t)$ for $t \in [a,b]$ is a parametrization of a smooth curve C. Prove that $f(\vec{r}(b)) - f(\vec{r}(a)) = \int_a^b Df(\vec{r}(t))D\vec{r}(t)\ dt$. [Let $g(t) = f(\vec{r}(t))$. Why does $g(b) - g(a) = \int_a^b g'(t)dt$? Use the chain rule (matrix form) to compute $g'(t)$. Then just substitute things back in.]

The proof of the fundamental theorem of line integrals is quite short. All you need is the fundamental theorem of calculus, together with the chain rule (6.7).

Problem 8.23 For each vector field and curve below, find the work done by $\vec{F}$ along C. In other words, compute the integral $\int_C Mdx + Ndy$ or $\int_C Mdx + Ndy + Pdz$.

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1. Let $\vec{F}(x, y) = (2x + y, x + 4y)$ and C be the parabolic path $y = 9 - x^2$ for x from -3 to 2.

   See Sage for a picture.

2. Let $\vec{F}(x, y, z) = (2x + yz, 2z + xz, 2y + xy)$ and C be the straight segment from $(2, -5, 0)$ to $(1, 2, 3)$.

   See Sage for a picture.

[Hint: If you parametrize the curve, then you've done the problem the HARD way. You don't need any parameterizations at all. Did you find a potential, and then plug in the end points?]

Problem 8.24 Let $\vec{F} = (x, z, y)$. Let C_1 be the curve which starts at $(1, 0, 0)$ and follows a helical path $(\cos t, \sin t, t)$ to $(1, 0, 2\pi)$. Let C_2 be the curve which starts at $(1, 0, 2\pi)$ and follows a straight line path to $(2, 4, 3)$. Let C_3 be any smooth curve that starts at $(2, 4, 3)$ and ends at $(0, 1, 2)$.

See Sage—C_1 and C_2 are in blue, and several possible C_3 are shown in red.

- Find the work done by $\vec{F}$ along each path C_1, C_2, C_3.
- Find the work done by $\vec{F}$ along the path C which follows C_1, then C_2, then C_3.
- If C is any path that can be broken up into finitely many smooth sub-paths, and C starts at $(1, 0, 0)$ and ends at $(0, 1, 2)$, what is the work done by $\vec{F}$ along C?

If you are parameterizing the curves, you're doing this the really hard way. Are you using the potential of the vector field?

In the problem above, the path we took to get from one point to another did not matter. The vector field had a potential, which meant that the work done did not depend on the path traveled.

Definition 8.8: Conservative Vector Field. We say that a vector field is conservative if the integral $\int_C \vec{F} \cdot d\vec{r}$ does not depend on the path C. We say that a curve C is piecewise smooth if it can be broken up into finitely many smooth curves.

Review Compute $\displaystyle\int \frac{x}{\sqrt{x^2+4}}dx$. See [4].

Problem 8.25 The gravitational vector field is directly related to the radial field $\vec{F} = \dfrac{(-x, -y, -z)}{(x^2 + y^2 + z^2)^{3/2}}$. Show that this vector field is conservative, by finding a potential for $\vec{F}$. Then compute the work done by an object that moves from $(1, 2, -2)$ to $(0, -3, 4)$ along ANY path that avoids the origin.

[See the review problem just before this if you're struggling with the integral.]

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[4] Let $u = x^2 + 4$, which means $du = 2xdx$ or $dx = \frac{du}{2x}$. This means

$$\int \frac{x}{\sqrt{x^2+4}}dx = \int \frac{x}{\sqrt{u}}\frac{du}{2x} = \frac{1}{2}\int u^{-1/2}du = \frac{1}{2}\frac{u^{1/2}}{1/2} = \sqrt{u} = \sqrt{x^2+4}.$$

Problem 8.26 Suppose $\vec{F}$ is a gradient field. Let C be a piecewise smooth closed curve. Compute $\int_C \vec{F} \cdot d\vec{r}$ (you should get a number). Explain how you know your answer is correct.

8.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 9

Integration

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Set up and compute iterated double, triple, and surface integrals to compute area, volume, and surface area.

2. Explain how to interchange the bounds of integration, and recognize when swapping the bounds is needed.

3. Set up and compute integrals in different coordinate systems using a Jacobian. In particular, show how to change between any two of rectangular, polar, cylindrical, and spherical coordinates.

4. Find mass, centroids, centers of mass.

You'll have a chance to teach your examples to your peers prior to the exam.

9.1 Double and Triple Integrals

It's time to revisit a topic we began earlier in the semester, namely double integrals. Let's start by describing regions in the plane. In first semester calculus, we often use the inequalities $a \leq x \leq b$ and $f(x) \leq y \leq g(x)$ to describe the region above f below g for x between a and b. We trapped x between two constants, and y between two functions. Sometimes we wrote $c \leq y \leq d$ where $f(y) \leq x \leq g(y)$ to describe the region to the right of f and left of g for y between c and d. We need to practice writing inequalities in both forms, as these inequalities provide us the bounds of integration for double integrals.

Problem 9.1 Consider the region R in the xy-plane that is below the line $y = x + 2$, above the line $y = 2$, and left of the line $x = 5$. We can describe this region by saying for each x with $0 \leq x \leq 5$, we want y to satisfy $2 \leq y \leq x + 2$. In set builder notation, we write

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 5, 2 \leq y \leq x + 2\}.$$

We use the symbols $\{$ and $\}$ to enclose sets and the symbol $\mid$ for "such that". We read the above line as "R equals the set of (x, y) in the plane such that zero is less than x which is less than 5, and 2 is less than y which is less than $x + 2$." We can use the iterated double integral $\int_0^5 \int_2^{x+2} dydx$ to compute the area of this region.

1. Draw this region.
2. Describe the region R by saying for each y with $c \leq y \leq d$, we want x to satisfy $a(y) \leq x \leq b(y)$. In other words, find constants c and d, and functions $a(y)$ and $b(y)$, so that for each y between c and d, the x values must be between the functions $a(y)$ and $b(y)$. Write your answer using the set builder notation
$$R = \{(x, y) \mid c \leq y \leq d, a(y) \leq x \leq b(y)\}.$$
3. Finish setting up the iterated double integral $\int_?^? \int_?^? dxdy$.

[Hint: If you're struggling, then draw the 4 curves given by $0 = x$, $x = 5$, $2 = y$ and $y = x + 2$. Then shade either above, below, left, or right of the line (as appropriate).]

Definition 9.1: Double and Iterated Integrals. Given a region R, we write $\iint_R f(x, y)dA$ for the **double integral** of f over R. We just have to state what the region R is to talk about a double integral. The formal definition of a double integrals involves slicing the region R up into tiny rectangles of area $dxdy$, multiplying each rectangle by a function f, and then summing over all rectangles. This process is repeated as the length and width of the rectangles shrinks to zero at similar rates, with the double integral being the limit of this process.

An **iterated integral** is an integral where we have actually specified the order of integration and given bounds for each integral. For double integral there are two options, namely

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y)dydx \quad \text{and} \quad \int_c^d \int_{a(y)}^{b(y)} f(x, y)dxdxy.$$

We'll focus mostly on setting up iterated integrals in this course.

Just as with line integrals, we can compute the mass of a region by adding up the little bits of mass given by $dm = \delta(x, y)dA$ to obtain the mass as

$$m = \iint_R \delta(x, y)dA = \iint_R \delta(x, y)dxdy = \iint_R \delta(x, y)dxdy.$$

Note that if $\delta(x, y) = 1$, then this reduces to the formula for the area of R.

Problem 9.2 For each region R below, draw the region. Then use the given density to set up an iterated double integral which would give the mass. You do not need to fully compute each integral, rather just set it up.

1. The region R is above the line $x + y = 1$ and inside the circle $x^2 + y^2 = 1$. The density is $\delta(x, y) = x$.
2. The region R is below the line $y = 8$, above the curve $y = x^2$, and to the right of the y-axis. The density is $\delta(x, y) = xy^2$.
3. The region R is bounded by $2x + y = 3$, $y = x$, and $x = 0$. The density is $\delta(x, y) = 7$.

Problem 9.3 Consider the iterated integral $\int_0^3 \int_x^3 e^{y^2} dydx$.

1. Write the bounds as two inequalities ($0 \leq x \leq 3$ and $? \leq y \leq ?$). Then draw and shade the region R described by these two inequalities.

2. Swap the order of integration from $dydx$ to $dxdy$. This forces you to describe the region using two inequalities of the form $c \leq y \leq d$ and $a(y) \leq x \leq b(y)$.

3. Use your new bounds to compute the integral by hand.

4. Why is the original integral $\int_0^3 \int_x^3 e^{y^2} dydx$ impossible to compute without first swapping the order of integration? [Hint: Try computing the inner integral $\int_x^3 e^{y^2} dy$ – why can't you?]

Problem 9.4 Compute the iterated integral $\int_0^{2\sqrt{\pi}} \int_{y/2}^{\sqrt{\pi}} \sin(x^2) dxdy$.

Problem 9.5 Consider the region in the plane that is bounded by the curves $x = y^2 - 3$ and $x = y - 1$. A metal plate occupies this region in space, and its temperature function on the plate is given by the function $T(x, y) = 2x + y$. Find the average temperature of the metal plate. Compute any integrals.

Just as we've used double integral to compute the area and mass for regions in the plane, we can use triple integrals to compute volume and mass for solids in space. A triple integral is an integral of the form $\iiint_D dV$, where dV represents a small portion of volume of the solid region D. However, now there are six different possible orders of integration when we want to create iterated integrals. For example if we pick the order $dzdydx$, then to set up the integral we'll need $a \leq x \leq b$, $c(x) \leq y \leq d(x)$, and $e(x, y) \leq z \leq f(x, y)$. Note that the outermost bounds must be always be constant, whereas the innermost bounds can depend on all of the other variables.

Problem 9.6 Do not evaluate the integrals below. Our focus is setting up the bounds for triple integrals.

1. The iterated triple integral $\int_{-1}^1 \int_0^4 \int_0^{y^2} dzdxdy$ gives the volume of the solid D that lies under the surface $z = y^2$, above the xy-plane, and bounded by the planes $y = -1$, $y = 1$, $x = 0$, and $x = 4$. Sketch this region.

2. Set up an iterated triple integral that gives the volume of the solid in the first octant that is bounded by the coordinate planes ($x = 0$, $y = 0$, $z = 0$), the plane $y + z = 2$, and the surface $x = 4 - y^2$, using the order of integration $dxdzdy$. Make sure you sketch the region.

3. Set up an integral to give the volume of the pyramid in the first octant that is below the planes $\frac{x}{3} + \frac{z}{2} = 1$ and $\frac{y}{5} + \frac{z}{2} = 1$. [Hint, don't let z be the inside bound. Try an order such as $dydxdz$.]

Just as we computed centroids in the line integral chapter, we can compute centroids for planar regions and solids in space. Recall that to find $\bar{x}$, we solved for $\bar{x}$ in the equation $\int_C \bar{x}dm = \int_C xdm$ to obtain $\bar{x} = \frac{\int_C xdm}{\int_C dm}$. As this process has nothing to do with the curve or little bit of mass, for solids D in space we obtain $\bar{x} = \frac{\iiint_D xdm}{\iiint_D dm}$. Since mass if found by taking a density (mass per unit volume) and multiplying it by a volume, we know $dm = \delta dV = \delta dxdydz$. For solid regions in space this gives the formulas

$$\bar{x} = \frac{\iiint_D x\delta dV}{\iiint_D \delta dV}, \quad \bar{y} = \frac{\iiint_D y\delta dV}{\iiint_D \delta dV}, \quad \text{and} \quad \bar{z} = \frac{\iiint_D z\delta dV}{\iiint_D \delta dV}.$$

Similar formulas hold for planar regions, using double integrals and dA instead.

Problem 9.7 Consider the triangular wedge D that is in the first octant, bounded by the planes $\frac{y}{7} + \frac{z}{5} = 1$ and $x = 12$. In the yz plane, the wedge forms a triangle that passes through the points $(0, 0, 0)$, $(0, 7, 0)$, and $(0, 0, 5)$. Draw the solid and then set up integral formulas that give the centroid $(\bar{x}, \bar{y}, \bar{z})$ of D. Actually compute the integrals for $\bar{y}$. Then state $\bar{x}$ and $\bar{z}$ from symmetry.

Problem 9.8 Let R be the region in the plane with $a \leq x \leq b$ and $g(x) \leq y \leq f(x)$. Let A be the area of R.

1. Set up an iterated integral to compute the area of R. Then compute the inside integral. You should obtain a familiar formula from first-semester calculus.

2. Set up an iterated integral formula to compute $\bar{x}$ for the centroid. By computing the inside integral, show that $\bar{x} = \frac{1}{A}\int_a^b x(f - g)dx$.

3. If the density depends only on x, so $\delta = \delta(x)$, set up an iterated integral formula to compute $\bar{y}$ for the center of mass. Compute the inside integral and show that

$$\bar{y} = \frac{1}{\text{mass}}\int_a^b \frac{1}{2}(f^2 - g^2)\delta(x)dx = \frac{1}{\text{mass}}\int_a^b \frac{1}{2}(f + g)(f - g)\delta(x)dx.$$

When we use double integrals to find centroids, the formulas for the centroid are similar for both $\bar{x}$ and $\bar{y}$. In other courses, you may see the formulas on the left, because the ideas are presented without requiring knowledge of double integrals. Integrating the inside integral from the double integral formula gives the single variable formulas.

Problem 9.9 Consider the iterated integral

$$\int_{-3}^{3}\int_0^{\sqrt{9-y^2}}\int_0^{9-x^2-y^2} dzdxdy.$$

There are 5 other iterated integrals that are equal to this integral, by switching the order of the bounds. One of the integrals is $\int_0^9\int_0^{\sqrt{9-z}}\int_{-\sqrt{9-x^2-z}}^{\sqrt{9-x^2-z}} dydxdz$. Set up the equivalent integrals using the bound $dydzdx$ and $dxdzdy$. [Optional: Give the remaining two.]

Problem 9.10 Consider the iterated integral

$$\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{y} dzdydx.$$

The bounds for this integral describe a region in space which satisfies the 3 inequalities $-1 \leq x \leq 1$, $0 \leq y \leq 1-x^2$, and $0 \leq z \leq y$.

1. Draw the solid domain D in space described by the bounds of the iterated integral.
2. There are 5 other iterated integrals equivalent to this one. Set up the two integrals that use the order $dydxdz$ and $dxdzdy$.
3. Optional: Give the remaining three.

9.2 Changing Coordinates - Substitution

Earlier in the semester we explored what happens if we change from Cartesian coordinates to another coordinate system. We found that if we want to change from rectangular to polar coordinates, then we used

$$\iint_R f(x,y)dxdy = \iint_R rdrd\theta,$$

provided the values for r were never negative. The next problem has you review why.

Problem 9.11 Consider the polar change of coordinates $x = r\cos\theta$ and $y = r\sin\theta$, which we write in vector form as

$$\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta).$$

1. Compute the derivative $D\vec{T}(r,\theta)$. You should have a 2 by 2 matrix.
2. We need a single number from this matrix that tells us something about area. Determinants are connected to area. Compute the determinant of $D\vec{T}(r,\theta)$ and simplify.
3. Let $\vec{T}_2(\theta,r) = (r\cos\theta, r\sin\theta)$, so the same change of coordinates except now θ is the first variable. Compute $D\vec{T}_2(\theta,r)$ and then the determinant of this 2 by 2 matrix. Verify that you obtain $-r$.

The problem above showed that the order in which we list variables can alter the sign of the determinant of the derivative. Taking an absolute value gets rid of any such possible sign changes. The Jacobian of a transformation, in general, is the absolute value of the determinant of the derivative of the transformation. Any time we want to change coordinates from rectangular, we'll need to insert the Jacobian into our computations. Let's summarize these results in a theorem.

Theorem 9.2. *Consider the polar coordinate transformation*

$$\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta).$$

Ask me in class to give you an informal picture approach that explains why $dA = rdrd\theta$.

The Jacobian of x and y with respect to r and θ is the absolute value of the determinant of the derivative of the transformation, which we write as

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \left|\det(D\vec{T}(r,\theta))\right| = \left|\det(D(x,y)(r,\theta))\right|.$$

If we require all bounds for r to be nonnegative, we can ignore the absolute value. If R_{xy} is a region in the xy plane that corresponds to the region $R_{r\theta}$ in the $r\theta$ plane (where $r \geq 0$), then we have

$$\iint_{R_{xy}} f(x,y)dxdy = \iint_{R_{r\theta}} f(r\cos\theta, r\sin\theta)r\ drd\theta.$$

Problem 9.12 The double integral $\displaystyle\int_0^1 \int_{\sqrt{3}y}^{\sqrt{4-y^2}} (x^2+y^2)dxdy$ computes the mass of a region in the plane with density $\delta = x^2+y^2$ that is bounded by the curves $y=0$, $y=1$, $x=\sqrt{3}y$, and $x=\sqrt{4-y^2}$. Start by drawing the region. Then convert this integral to an integral in polar coordinates (don't forget the Jacobian), and finish by actually computing the integral to get $2\pi/3$.

Problem 9.13 For each region R below, draw the region in the xy-plane. Then use the given density to set up an iterated double integral in polar coordinates which would give the mass. You do not need to fully compute each integral, rather just set it up. For example, if the region is the inside of the circle $x^2+y^2=9$, and the density is $\delta(x,y)=y$, then the mass is

$$m = \iint_R \delta dA = \int_0^{2\pi}\int_0^3 \underbrace{(r\sin\theta)}_{\delta=y}\underbrace{rdrd\theta}_{dA}.$$

1. The region R is the quarter circle in the first quadrant inside the circle $x^2+y^2=25$. The density is $\delta(x,y)=x$.
2. The region R is below $y=\sqrt{9-x^2}$, above $y=x$, and to the right of $x=0$. The density is $\delta(x,y)=xy^2$.
3. The region R is the triangular region below $y=\sqrt{3}x$, above the x-axis, and to the left of $x=1$. The density is $\delta(x,y)=7$.

Problem 9.14 Compute the integral $\displaystyle\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}dydx$. [Hint: Try switching coordinate systems to polar coordinates. This will require you to first draw the region of integration, and then then obtain bounds for the region in polar coordinates.]

Problem 9.15 Show that the centroid of a semicircular disc of radius a (with $y \geq 0$) is $(\bar{x},\bar{y}) = (0, \frac{4a}{3\pi})$. Set up all the integral using polar coordinates, and then actually perform the computations.

Let's now look at some three dimensional coordinate transformations, and their corresponding Jacobians. Just as in two dimensions, the Jacobian of a transformation $(x,y,z)=\vec{T}(u,v,w)$ is the absolute value of the determinant of the derivative of the transformation, or simply

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\left|\det(D\vec{T}(u,v,w))\right|=\left|\det(D(x,y,z)(u,v,w)))\right|.$$

We've already seen both cylindrical and spherical coordinates. The next problem has you compute their respective Jacobians.

Problem 9.16 Recall that the cylindrical change of coordinates is

$$(x,y,z)=\vec{C}(r,\theta,z)=(r\cos\theta,r\sin\theta,z),$$

and the spherical change of coordinates is

$$(x,y,z)=\vec{S}(\rho,\phi,\theta)=(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi).$$

1. Compute the Jacobian of the cylindrical transformation by hand, and verify that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}=|r|$. If we want to drop the absolute values, what must we require?
2. Write down the 3 by 3 matrix whose determinant you would need to compute to obtain the Jacobian of the spherical coordinate transformation. Use software to compute and simply the Jacobian to verify that $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}=|\rho^2\sin\phi|$. If we want to drop the absolute values, what must we require?

The previous problem shows us that we can write

$$\iiint dV=\iiint dxdydz=\iiint rdrd\theta dz=\iiint \rho^2\sin\phi d\rho d\phi d\theta,$$

provided we require $r\geq 0$ and $0\leq\phi\leq\pi$. Cylindrical coordinates are useful for problems which involve cylinders, paraboloids, and cones. Spherical coordinates are often useful for problems which involve cones and spheres.

Problem 9.17 We can find the volume of the solid D under $f(x,y)=9-x^2-y^2$ for $x\geq 0$ and $z\geq 0$ by computing the triple integral See Sage.

$$V=\iiint_D dV=\int_{y=-3}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_0^{9-x^2-y^2}dzdxdy.$$

1. Draw the solid and give bounds using cylindrical coordinates.
2. Rewrite the integral using cylindrical coordinates. Remember the Jacobian.
3. Compute the integral in the previous part by hand. [Suggestion: Simplify $9-x^2-y^2=9-(x^2+y^2)=9-r^2$ before integrating.]

Problem 9.18 Set up an integral in spherical coordinates that gives the volume of the solid ball that is inside $x^2+y^2+z^2=a^2$, so inside a sphere of radius a. Then actually compute the integral and simplify your result to obtain the common formula $V=\frac{4}{3}\pi a^3$ for the volume of a ball of radius a.

As a final part to this problem consider just the top half of the ball. Set up a formula that would give $\bar{z}$, the height of the centroid of top half of the ball. Then use software to check that you are correct, as the answer is $\bar{z}=\frac{3}{8}a$.

Problem 9.19 Consider the solid domain D in space which is above the cone $z=\sqrt{x^2+y^2}$ and below the paraboloid $z=6-x^2-y^2$. Sketch the region by hand, and then use cylindrical coordinates to set up an iterated triple integral that would give the volume of the region. You'll need to find where the surfaces intersect, as their intersection will help you determine the appropriate bounds.

See Sage for a picture of the region.

Problem 9.20 Consider the solid D in space that is both inside the sphere $x^2+y^2+z^2=9$ and yet outside the cylinder $x^2+y^2=4$. Start by drawing the region.

1. Set up an iterated integral in cylindrical coordinates that would give the volume of D, using the order $dzdrd\theta$.
2. Repeat the first part, but use the order $d\theta drdz$.
3. Set up an iterated integral in spherical coordinates that would give the volume of D.

Problem 9.21 The integral $\int_0^{\pi}\int_0^1\int_{\sqrt{3}r}^{\sqrt{4-r^2}} rdzdrd\theta$ represents the volume of solid domain D in space. Set up integrals in both rectangular coordinates and spherical coordinates that would give the volume of the exact same region.

9.3 Surface Integrals

In first-semester calculus, we learned how to compute integrals $\int_a^b fdx$ along straight (flat) segments $[a,b]$. This semester, in the line integral unit, we learned how to change the segment to a curve, which allowed us to compute integrals $\int_C fds$ along any curve C, instead of just along curves (segments) on the x-axis. The integral $\int_a^b dx=b-a$ gives the length of the segment $[a,b]$. The integral $\int_C ds$ gives the length s of the curve C.

This semester we've learned how to compute double integrals $\iint_R fdA$ along flat regions R in the plane. We'll now learn how to change the flat region R into a curved surface S, and then compute integrals of the form $\iint_S fd\sigma$ along curved surfaces. The differential $d\sigma$ stands for a little bit of surface area. We already know that $\iint_R dA$ gives the area of R. We'll define $\iint_S d\sigma$ so that it gives the surface area of S.

Problem 9.22 Consider the surface S given by $z=9-x^2-y^2$, an upside down parabola that intersect the xy plane in a circle of radius 3. A parametrization of the portion of this surface that lies above the xy-plane is

$$\vec{r}(x,y)=(x,y,9-x^2-y^2)\quad\text{for}\quad -3\leq x\leq 3, -\sqrt{9-x^2}\leq y\leq\sqrt{9-x^2}.$$

1. Draw the surface S. Add to your surface plot the parabolas given by $\vec{r}(x,0)$, $\vec{r}(x,1)$, and $\vec{r}(x,2)$, as well as the parabolas given by $\vec{r}(0,y)$, $\vec{r}(1,y)$, and $\vec{r}(2,y)$. You should have an upside down paraboloid, with at least 6 different parabolas drawn on the surface. These parabolas should divide the surface up into a bunch of different patches. Our goal is to find the area of each patch, where each patch is almost like a parallelogram.

See Sage for a solution.

2. Find both $\dfrac{\partial\vec{r}}{\partial x}$ and $\dfrac{\partial\vec{r}}{\partial y}$. Then at the point $(2,1)$, draw both of these partial derivatives with their bases at $(2,1)$. These vectors form the edges of a parallelogram. Add that parallelogram to your picture.

3. Show that the area of a parallelogram whose edges are the vectors $\dfrac{\partial\vec{r}}{\partial x}$ and $\dfrac{\partial\vec{r}}{\partial y}$ is $\sqrt{1+4x^2+4y^2}$. [Hint: think about the cross product.]

4. Notice in your work above that we drew parabolas by changing both x and y by 1 unit. If instead we had drawn parabolas at increments of .5 instead of 1, then we'd need to multiply our partial derivatives by .5 before finding the area of the parallelogram. If we use increments of dx and dy, then the edges of our parallelogram are the vectors $\vec{r}_x dx$ and $\vec{r}_y dy$. Find the area of this parallelogram.

In the previous problem, you showed that the area of the parallelogram with edges given by $\frac{\partial\vec{r}}{\partial x}dx$ and $\frac{\partial\vec{r}}{\partial y}dy$ is

$$d\sigma = \left|\frac{\partial\vec{r}}{\partial x}\times\frac{\partial\vec{r}}{\partial y}\right| dxdy = |\vec{r}_x\times\vec{r}_y|\, dxdy.$$

This little bit of area approximates the surface area of a tiny patch on the surface. When we add all these areas up, we obtain the surface area.

Definition 9.3. Let S be a surface. Let $\vec{r}(u,v)=(x,y,z)$ be a parametrization of the surface, where the bounds on u and v form a region R in the uv plane. Then the surface area element (representing a little bit of surface) is

$$d\sigma = \left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right| dudv = |\vec{r}_u\times\vec{r}_v|\, dudv.$$

The surface integral of a continuous function $f(x,y,z)$ along the surface S is

$$\iint_S f(x,y,z)d\sigma = \iint_R f(\vec{r}(u,v))\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right| dudv.$$

If we let $f=1$, then the surface area of S is simply

$$\sigma = \iint_S d\sigma = \iint_R \left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right| dudv.$$

This definition tells us how to compute any surface integral. The steps are almost identical to the line integral steps.

1. Start by getting a parametrization $\vec{r}$ of the surface S where the bounds form a region R.

2. Find a little bit of surface area by computing $d\sigma = \left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right| dudv$.

3. Multiply f by $d\sigma$, and replace each x, y, z with what they equals from the parametrization.

4. Integrate the previous function along R, your parameterization's bounds.

Example 9.4. Consider again the surface S given by $z = 9 - x^2 - y^2$, for $z \geq 0$. We used the parametrization

$$\vec{r}(x,y) = (x, y, 9 - x^2 - y^2) \quad \text{for} \quad -3 \leq x \leq 3, -\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2}.$$

to obtain $d\sigma = |\vec{r}_x \times \vec{r}_y|\, dxdy = \sqrt{4x^2 + 4y^2 + 1} dxdy$. This means that the surface area is

$$\sigma = \iint_S d\sigma = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{4x^2 + 4y^2 + 1} dydx.$$

At this point we now have an iterated double integral. As the region described by the integral is a circle, we can swap to polar coordinates to simplify the computations. The bounds are $0 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$, which means

$$\sigma = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{4(x^2 + y^2) + 1}\; dydx = \int_{0}^{2\pi} \int_{0}^{3} \sqrt{4r^2 + 1}\; r dr d\theta.$$

Problem 9.23 Consider again the surface S from the example above. A different parametrization of this surface is

$$\vec{r}(r,\theta) = (r\cos\theta, r\sin\theta, 9 - r^2), \quad \text{where } 9 - r^2 \geq 0.$$

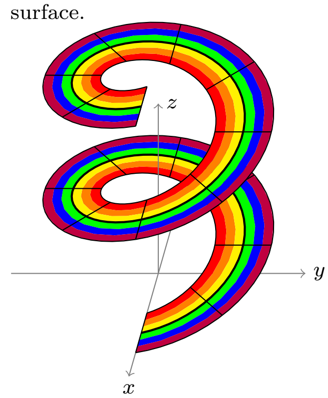
1. Give a set of inequalities for r and θ that describe the region $R_{r\theta}$ over which we need to integrate.
2. Find the surface area element $d\sigma = |\vec{r}_r \times \vec{r}_\theta|\, dr d\theta$. Simplify your work to show that $d\sigma = r\sqrt{4r^2 + 1}\; dr d\theta$.
3. Set up the surface integral $\iint_S d\sigma$ as an iterated double integral over $R_{r\theta}$, and then actually compute the integral by hand.

Problem 9.24 Consider the parametric surface

$$\vec{r}(a,t) = (a\cos t, a\sin t, t) \quad \text{for} \quad 2 \leq a \leq 4 \text{ and } 0 \leq t \leq 4\pi.$$

Find $\vec{r}_a$ and $\vec{r}_t$. Then compute the surface area element $d\sigma = |\vec{r}_a \times \vec{r}_t| dadt$. Set up an iterated integral for the surface area. Don't compute the integral.

Here's a rough sketch of the surface.



Problem 9.25 If a surface S is parametrized by $\vec{r}(x,y) = (x, y, f(x,y))$, show that $d\sigma = \sqrt{1 + f_x^2 + f_y^2}\; dxdy$ (compute a cross product). If $\vec{r}(x,z) = (x, f(x,z), z)$, what does $d\sigma$ equal (compute a cross product - you should see a pattern)? Use the pattern you've discovered to quickly compute $d\sigma$ for the surface $x = 4 - y^2 - z^2$, and then set up an iterated double integral that would give the surface area of S for $x \geq 0$.

Problem 9.26 Consider the sphere $x^2 + y^2 + z^2 = a^2$. We'll find $d\sigma$ using two different parameterizations.

1. Consider the rectangular parametrization $\vec{r}(x,y) = (x, y, \sqrt{a^2 - x^2 - y^2})$. Compute $d\sigma$? [Hint, use the previous problem.] Why can this parametrization only be use if the surface has positive z-values?

2. Consider the spherical parametrization

$$\vec{r}(\phi,\theta) = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi).$$

Show that

$$d\sigma = (a^2|\sin\phi|)d\phi d\theta = (a^2\sin\phi)d\phi d\theta,$$

where we can ignore the absolute values if we require $0 \leq \phi \leq \pi$. Along the way, you'll show that

$$\vec{r}_\phi \times \vec{r}_\theta = a^2\sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

We can compute mass, average value, centroids, and center of mass for surfaces. We just replace dA with $d\sigma$, and all the formulas are the same.

Problem 9.27 Consider the hemisphere $x^2 + y^2 + z^2 = a^2$ for $z \geq 0$.

1. Set up a formula that would give $\bar{z}$ for the centroid of the hemisphere. I suggest you use a spherical parametrization, as then the bounds are fairly simple, and we know $d\sigma = (a^2\sin\phi)d\phi d\theta$ from the previous problem.

2. Compute both the integrals in your formula. The combine your work to show that $\bar{z} = \frac{a}{2}$.

3. One of the integrals you computed gave the surface area of a hemisphere of radius a. Which is it? Use that result to give the surface area of a sphere of radius a.

Problem 9.28 Consider the surface S that is the portion of the cone $x^2 = y^2 + z^2$ with $1 \leq x \leq 4$.

1. Give a parametrization of the cone, including bounds.

2. Use your parametrization to compute the surface area element $d\sigma$.

3. Compute the surface area of S. Yes, actually compute the integral.

4. Setup a formula that would give the center of mass $\bar{x}$ of the cone if the density is $\delta(x,y,z) = x$. Don't spend any time computing the integrals.
