Multivariable Calculus

Ben Woodruff[1]

Typeset on December 7, 2015

[1]Mathematics Faculty at Brigham Young University–Idaho, woodruffb@byui.edu

Introduction

This course may be like no other course in mathematics you have ever taken. We'll discuss in class some of the key differences, and eventually this section will contain a complete description of how this course works. For now, it's just a skeleton.

I received the following email about 6 months after a student took the course:

> Hey Brother Woodruff,
>
> I was reading *Knowledge of Spiritual Things* by Elder Scott. I thought the following quote would be awesome to share with your students, especially those in Math 215 :)
>
>> Profound [spiritual] truth cannot simply be poured from one mind and heart to another. It takes faith and diligent effort. Precious truth comes a small piece at a time through faith, with great exertion, and at times wrenching struggles.

Elder Scott's words perfectly describe how we acquire mathematical truth, as well as spiritual truth.

Contents

Chapter 1

Review

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Give a summary of the ideas you learned in 112, including graphing, derivatives (product, quotient, power, chain, trig, exponential, and logarithm rules), and integration (u-sub and integration by parts).

2. Compute the differential dy of a function and use it to approximate the change in a function.

3. Explain how to perform matrix multiplication and compute determinants of square matrices.

4. Illustrate how to solve systems of linear equations, including how to express a solution parametrically (in terms of t) when there are infinitely solutions.

5. Extend the idea of differentials to approximate functions using parabolas, cubics, and polynomials of any degree.

You'll have a chance to teach your examples to your peers prior to the exam.

1.1 Review of First Semester Calculus

1.1.1 Graphing

We'll need to know how to graph by hand some basic functions. If you have not spent much time graphing functions by hand before this class, then you should spend some time graphing the following functions by hand. When we start drawing functions in 3D, we'll have to piece together infinitely many 2D graphs. Knowing the basic shape of graphs will help us do this.

Problem 1.1 Provide a rough sketch of the following functions, showing their basic shapes:

$$x^2, x^3, x^4, \frac{1}{x}, \sin x, \cos x, \tan x, \sec x, \arctan x, e^x, \ln x.$$

Then use a computer algebra system, such as Wolfram Alpha, to verify your work. I suggest Wolfram Alpha, because it is now built into Mathematica 8.0. If you can learn to use Wolfram Alpha, you will be able to use Mathematica.

1.1.2 Derivatives

In first semester calculus, one of the things you focused on was learning to compute derivatives. You'll need to know the derivatives of basic functions (found on the end cover of almost every calculus textbook). Computing derivatives accurately and rapidly will make learning calculus in high dimensions easier. You'll want to be familiar with the power rule, sum rule, product rule, quotient rule, and chain rule, as well as implicit differentiation.

Problem 1.2 Compute the derivative of $e^{\sec x}\cos(\tan(x)+\ln(x^2+4))$. Show each step in your computation, making sure to show what rules you used.

See sections 3.2-3.6 for more practice with derivatives. The later problems in 3.6 review of most of the entire differentiation chapter.

Problem 1.3 If $y(p) = \dfrac{e^{p^3}\cot(4p+7)}{\tan^{-1}(p^4)}$ find dy/dp. Again, show each step in your computation, making sure to show what rules you used.

Problem 1.4 Given $c = \sqrt{a^2+b^2}$, Use implicit differentiation to compute $\frac{dc}{dt}$ in terms of a, b, $\frac{da}{dt}$, and $\frac{db}{dt}$.

The following problem will help you review some of your trigonometry, inverse functions, as well as implicit differentiation.

Problem 1.5 Use implicit differentiation to explain why the derivative of $y = \arcsin x$ is $y' = \dfrac{1}{\sqrt{1-x^2}}$. [Rewrite $y = \arcsin x$ as $x = \sin y$, differentiate both sides, solve for y', and then write the answer in terms of x].

See sections 3.7-3.9 for more examples involving inverse trig functions and implicit differentiation.

Problem 1.6 Compute $\frac{dy}{dx}$ if we know $5 = x^2 + 3xy - y^3$.

1.1.3 Integrals

Each derivative rule from the front cover of your calculus text is also an integration rule. In addition to these basic rules, we'll need to know three integration techniques. They are (1) u-substitution, (2) integration-by-parts, and (3) integration by using software. There are many other integration techniques, but we will not focus on them. If you are trying to compute an integral to get a number while on the job, then software will almost always be the tool you use. As we develop new ideas in this and future classes (in engineering, physics, statistics, math), you'll find that u-substitution and integrations-by-parts show up so frequently that knowing when and how to apply them becomes crucial.

Problem 1.7 Compute $\int x\sqrt{x^2+4}dx$.

For practice with u-substitution, see section 5.5 and 5.6.
For practice with integration by parts, see section 8.1.

Problem 1.8 Compute $\int x\sin 2x dx$.

Problem 1.9 Compute $\int \arctan x dx$.

Problem 1.10 Compute $\int x^2 e^{3x} dx$.

1.2 Differentials

The derivative of a function gives us the slope of a tangent line to that function. We can use this tangent line to estimate how much the output (y values) will change if we change the input (x-value). If we rewrite the notation $\frac{dy}{dx} = f'$ in the form $dy = f'dx$, then we can read this as "A small change in y (called dy) equals the derivative (f') times a small change in x (called dx)."

Definition 1.1. We call dx the differential of x. If f is a function of x, then the differential of f is $df = f'(x)dx$. Since we often write $y = f(x)$, we'll interchangeably use dy and df to represent the differential of f.

We will often refer to the differential notation $dy = f'dx$ as "a change in the output y equals the derivative times a change in the input x."

Problem 1.11 Let $f(x) = x^2 \ln(3x + 2)$ and $g(t) = e^{2t} \tan(t^2)$. Compute the derivatives $\frac{df}{dx}$ and $\frac{dg}{dt}$, and then state the differentials df and dg. If you skipped the defintion of a differential, you'll find it's directly above this problem.

See 3.10:19-38.

Most of higher dimensional calculus can quickly be developed from differential notation. Once we have the language of vectors and matrices at our command, we will develop calculus in higher dimensions by writing $d\vec{y} = Df(\vec{x})d\vec{x}$. Variables will become vectors, and the derivative will become a matrix.

This problem will help you see how the notion of differentials is used to develop equations of tangent lines. We'll use this same idea to develop tangent planes to surfaces in 3D and more.

Problem 1.12 Consider the function $y = f(x) = x^2$. This problem has multiple steps, but each is fairly short.

See 3.11:39-44. Also see problems 3.11:1-18. The linearization of a function is just an equation of the tangent line where you solve for y.

1. State the derivative of y with respect to x and the differential of y.
2. Give an equation of the tangent line to $f(x)$ at $x = 3$.
3. Draw a graph of $f(x)$ and the tangent line on the same axes. Place a dot at the point $(3, 9)$ and label it on your graph. Place another dot on the tangent line up and to the right of (3,9). Label the point (x, y), as it will represent any point on the tangent line.
4. From the point $(3, 9)$ to the point (x, y), the change in x, or run, is $dx = x - 3$. The change in y, or rise, is what? Use this to state the slope of the line connecting $(3, 9)$ and (x, y).
5. We already know the slope of the tangent line is the derivative $f'(3) = 6$. We also know the slope from the previous part. Set these two slope values equal, and verify that this gives an equation of the tangent line to $f(x)$ at $x = 3$.

Problem 1.13 The manufacturer of a spherical storage tank needs to create a tank with a radius of 5 m. Recall that the volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$. No manufacturing process is perfect, so the resulting sphere will have a radius of 5 m, plus or minus some small amount dr. The actual radius will be $5 + dr$. Find the differential dV. Then use differentials to estimate the change in the volume of the sphere if the actual radius is 5.02 m instead of the planned 5 m.

See 3.11:45-62.

Problem 1.14 A forest ranger needs to estimate the height of a tree. The ranger stands 50 feet from the base of tree and measures the angle of elevation to the top of the tree to be about $60°$.

1. If this angle of $60°$ is correct, then what is the height of the tree?
2. If the ranger's angle measurement could be off by as much as $5°$, then how much could his estimate of the height be off? Use differentials to give an answer.

If your answer here is quite large (much larger than the height of the tree), then look back at your work and see if using radians instead of degrees makes a difference. Why does it? Feel free to ask in class.

1.3 Matrices

We will soon discover that matrices represent derivatives in high dimensions. When you use matrices to represent derivatives, the chain rule is precisely matrix multiplication. For now, we just need to become comfortable with matrix multiplication.

We perform matrix multiplication "row by column". Wikipedia has an excellent visual illustration of how to do this. See Wikipedia for an explanation. See texample.net for a visualization of the idea.

The links will open your browser and take you to the web.

Problem 1.15 Compute the following matrix products.

- $\begin{bmatrix}3 & 2 & 1\end{bmatrix}\begin{bmatrix}-1\\2\\0\end{bmatrix}$
- $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5 & 0\\6 & 1\end{bmatrix}$

For extra practice, make up two small matrices and multiply them. Use Sage or Wolfram Alpha to see if you are correct (click the links to see how to do matrix multiplication in each system).

Problem 1.16 Compute the product $\begin{bmatrix}3 & 2 & 1\\0 & 1 & -4\end{bmatrix}\begin{bmatrix}-1 & 3 & 0\\2 & -1 & 0\\0 & 1 & 2\end{bmatrix}$.

1.3.1 Determinants

Determinants measure area, volume, length, and higher dimensional versions of these ideas. Determinants will appear as we study cross products and when we get to the high dimensional version of u-substitution.

Associated with every square matrix is a number, called the determinant, which is related to length, area, and volume, and we use the determinant to generalize volume to higher dimensions. Determinants are only defined for square matrices.

Definition 1.2. The determinant of a 2×2 matrix is the number

$$\det\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{vmatrix}a & b\\c & d\end{vmatrix} = ad - bc.$$

We use vertical bars next to a matrix to state we want the determinant, so $\det A = |A|$.

The determinant of a 3×3 matrix is the number

$$\begin{vmatrix}a & b & c\\d & e & f\\g & h & i\end{vmatrix} = a\det\begin{vmatrix}e & f\\h & i\end{vmatrix} - b\det\begin{vmatrix}d & f\\g & i\end{vmatrix} + c\det\begin{vmatrix}d & e\\g & h\end{vmatrix}$$
$$= a(ei - hf) - b(di - gf) + c(dh - ge).$$

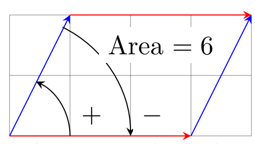
Notice the negative sign on the middle term of the 3×3 determinant. Also, notice that we had to compute three determinants of 2 by 2 matrices in order to find the determinant of a 3 by 3.

Problem 1.17 Compute $\begin{vmatrix}1 & 2\\3 & 4\end{vmatrix}$ and $\begin{vmatrix}1 & 2 & 0\\-1 & 3 & 4\\2 & -3 & 1\end{vmatrix}$.

For extra practice, create your own square matrix (2 by 2 or 3 by 3) and compute the determinant by hand. Then use Wolfram Alpha to check your work. Do this until you feel comfortable taking determinants.

What good is the determinant? The determinant was discovered as a result of trying to find the area of a parallelogram and the volume of the three dimensional version of a parallelogram (called a parallelepiped) in space. If we had a full semester to spend on linear algebra, we could eventually prove the following facts that I will just present here with a few examples.

Consider the 2 by 2 matrix $\begin{bmatrix}3 & 1\\0 & 2\end{bmatrix}$ whose determinant is $3\cdot 2-0\cdot 1=6$. Draw the column vectors $\begin{bmatrix}3\\0\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$ with their base at the origin (see figure 1.1). These two vectors give the edges of a parallelogram whose area is the determinant 6. If I swap the order of the two vectors in the matrix, then the determinant of $\begin{bmatrix}1 & 3\\2 & 0\end{bmatrix}$ is -6. The reason for the difference is that the determinant not only keeps track of area, but also order. Starting at the first vector, if you can turn counterclockwise through an angle smaller than 180° to obtain the second vector, then the determinant is positive. If you have to turn clockwise instead, then the determinant is negative. This is often termed "the right-hand rule," as rotating the fingers of your right hand from the first vector to the second vector will cause your thumb to point up precisely when the determinant is positive.



$$\begin{vmatrix}3 & 1\\0 & 2\end{vmatrix}=6 \text{ and } \begin{vmatrix}1 & 3\\2 & 0\end{vmatrix}=-6$$

Figure 1.1: The determinant gives both area and direction. A counter clockwise rotation from column 1 to column 2 gives a positive determinant.

For a 3 by 3 matrix, the columns give the edges of a three dimensional parallelepiped and the determinant produces the volume of this object. The sign of the determinant is related to orientation. If you can use your right hand and place your index finger on the first vector, middle finger on the second vector, and thumb on the third vector, then the determinant is positive. For example, consider the matrix $A=\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$. Starting from the origin, each column represents an edge of the rectangular box $0\le x\le 1$, $0\le y\le 2$, $0\le z\le 3$ with volume (and determinant) $V=lwh=(1)(2)(3)=6$. The sign of the determinant is positive because if you place your index finger pointing in the direction (1,0,0) and your middle finger in the direction (0,2,0), then your thumb points upwards in the direction (0,0,3). If you interchange two of the columns, for example $B=\begin{bmatrix}0 & 1 & 0\\2 & 0 & 0\\0 & 0 & 3\end{bmatrix}$, then the volume doesn't change since the shape is still the same. However, the sign of the determinant is negative because if you point your index finger in the direction (0,2,0) and your middle finger in the direction (1,0,0), then your thumb points down in the direction (0,0,-3). If you repeat this with your left hand instead of right hand, then your thumb points up.

Problem 1.18 Compute the determinant of the matrix $\begin{bmatrix} -2 & 3 \\ 5 & 4 \end{bmatrix}$. Use your answer to find the area of the trianlge with vertices $(0,0)$, $(-2,5)$, and $(3,4)$.

Problem 1.19 Find the area of a triangle with vertices $(-3,1)$, $(-2,5)$, and $(3,4)$, using the determinant of an appropriate matrix.

1.4 Solving Systems of equations

Problem 1.20 Solve the following linear systems of equations.

For additional practice, make up your own systems of equations. Use Wolfram Alpha to check your work.

- $\begin{cases} x+y &= 3 \\ 2x-y &= 4 \end{cases}$
- $\begin{cases} -x+4y &= 8 \\ 3x-12y &= 2 \end{cases}$

Problem 1.21 Find all solutions to the linear system $\begin{cases} x+y+z &= 3 \\ 2x-y &= 4 \end{cases}$. Since there are more variables than equations, this suggests there is probably not just one solution, but perhaps infinitely many. One common way to deal with solving such a system is to let one variable equal t, and then solve for the other variables in terms of t. Do this three different ways.

This link will show you how to specify which variable is t when using Wolfram Alpha.

- If you let $x = t$, what are y and z. Write your solution in the form (x, y, z) where you replace x, y, and z with what they are in terms of t.
- If you let $y = t$, what are x and z (in terms of t).
- If you let $z = t$, what are x and y.

1.5 Higher Order Approximations

When you ask a calculator to tell you what $e^{.1}$ means, your calculator uses an extension of differentials to give you an approximation. The calculator only uses polynomials (multiplication and addition) to give you an answer. This same process is used to evaluate any function that is not a polynomial (so trig functions, square roots, inverse trig functions, logarithms, etc.) The key idea needed to approximate functions is illustrated by the next problem.

Problem 1.22 Let $f(x) = e^x$. You should find that your work on each step can be reused to do the next step.

- Find a first degree polynomial $P_1(x) = a + bx$ so that $P_1(0) = f(0)$ and $P_1'(0) = f'(0)$. In other words, give me a line that passes through the same point and has the same slope as $f(x) = e^x$ does at $x = 0$. Set up a system of equations and then find the unknowns a and b. The next two are very similar.

- Find a second degree polynomial $P_2(x) = a+bx+cx^2$ so that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$. In other words, give me a parabola that passes through the same point, has the same slope, and has the same concavity as $f(x) = e^x$ does at $x = 0$.
- Find a third degree polynomial $P_3(x) = a + bx + cx^2 + dx^3$ so that $P_3(0) = f(0)$, $P_3'(0) = f'(0)$, $P_3''(0) = f''(0)$, and $P_3'''(0) = f'''(0)$. In other words, give me a cubic that passes through the same point, has the same slope, the same concavity, and the same third derivative as $f(x) = e^x$ does at $x = 0$.
- Now compute $e^{.1}$ with a calculator. Then compute $P_1(.1)$, $P_2(.1)$, and $P_3(.1)$. How accurate are the line, parabola, and cubic in approximating $e^{.1}$?

Problem 1.23 Now let $f(x) = \sin x$. Find a 7th degree polynomial so that the function and the polynomial have the same value and same first seven derivatives when evaluated at $x = 0$. Evaluate the polynomial at $x = 0.3$. How close is this value to your calculator's estimate of $\sin(0.3)$? You may find it valuable to use the notation

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_7x^7.$$

The polynomial you are creating is often called a Taylor polynomial. (I'm giving you the name so that you can search online for more information if you are interested.)

The previous two problems involved finding polynomial approximations to the function at $x = 0$. The next problem shows how to move this to any other point, such as $x = 1$.

Problem 1.24 Let $f(x) = e^x$.

- Find a second degree polynomial
$$T(x) = a + bx + cx^2$$
so that $T(1) = f(1)$, $T'(1) = f'(1)$, and $T''(1) = f''(1)$. In other words, give me a parabola that passes through the same point, has the same slope, and the same concavity as $f(x) = e^x$ does at $x = 1$.
- Find a second degree polynomial written in the form
$$S(x) = a + b(x-1) + c(x-1)^2$$
so that $S(1) = f(1)$, $S'(1) = f'(1)$, and $S''(1) = f''(1)$. In other words, find a quadratic that passes through the same point, has the same slope, and the same concavity as $f(x) = e^x$ does at $x = 1$.

Notice that we just replaced x with $x - 1$. This centers, or shifts, the approximation to be at $x = 1$. The first part will be much simpler now when you let $x = 1$.

- Find a third degree polynomial written in the form
$$P(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3$$
so that $P(1) = f(1)$, $P'(1) = f'(1)$, $P''(1) = f''(1)$, and $P'''(1) = f'''(1)$. In other words, give me a cubic that passes through the same point, has the same slope, the same concavity, and the same third derivative as $f(x) = e^x$ does at $x = 1$.

1.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 2

Vectors

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Define, draw, and explain what a vector is in 2 and 3 dimensions.
2. Add, subtract, and multiply (scalar, dot product, cross product) vectors. Be able to illustrate each operation geometrically, where possible.
3. Use vector products to find angles, length, area, projections, and work.
4. Use vectors to give equations of lines and planes, and be able to draw lines and planes in 3D.

You'll have a chance to teach your examples to your peers prior to the exam.

2.1 Vectors and Lines

Learning to work with vectors will be a key tool we need for our work in high dimensions. Let's start with some problems related to finding distance in 3D, drawing in 3D, and then we'll be ready to work with vectors.

To find the distance between two points (x_1, y_1) and (x_2, y_2) in the plane, we create a triangle connecting the two points. The base of the triangle has length $\Delta x = (x_2 - x_1)$ and the vertical side has length $\Delta y = (y_2 - y_1)$. The Pythagorean theorem gives us the distance between the two points as $\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Problem 2.1 The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in 3-dimensions is $\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. Construct an appropriate picture and show how to use the Pythagorean theorem repeatedly to prove this fact about distance in 3D.

Problem 2.2 Find the distance between the two points $P = (2, 3, -4)$ and $Q = (0, -1, 1)$. Then give an equation of the sphere passing though point Q whose center is at P. See 12.1:41-58.

Problem 2.3 For each of the following, construct a rough sketch of the set of points in space (3D) satisfying: See 12.1:1-40.

1. $2 \leq z \leq 5$
2. $x = 2, y = 3$
3. $x^2 + y^2 + z^2 = 25$

Definition 2.1. A vector is a magnitude in a certain direction. If P and Q are points, then the vector $\vec{PQ}$ is the directed line segment from P to Q. This definition holds in 1D, 2D, 3D, and beyond. If $V = (v_1, v_2, v_3)$ is a point in space, then to talk about the vector $\vec{v}$ from the origin O to V we'll use any of the following notations:

Most textbooks use a bold font to write vectors. When writing vectors by hand, it's common to use an arrow above a letter to represent that it's a vector.

$$\vec{v} = \mathbf{v} = \vec{OV} = \langle v_1, v_2, v_3 \rangle = (v_1, v_2, v_3) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$
$$= \underbrace{v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}}_{\text{common in engineering}} = \underbrace{v_1\hat{\mathbf{x}} + v_2\hat{\mathbf{y}} + v_3\hat{\mathbf{z}}}_{\text{common in physics}}.$$

We call v_1, v_2, and v_3 the x, y, and z components of the vector, respectively.

Note that (v_1, v_2, v_3) could refer to either the point V or the vector $\vec{v}$. The context of the problem we are working on will help us know if we are dealing with a point or a vector.

Definition 2.2. Let $\mathbb{R}$ represent the set real numbers. Real numbers are actually 1D vectors. Let $\mathbb{R}^2$ represent the set of vectors (x_1, x_2) in the plane. Let $\mathbb{R}^3$ represent the set of vectors (x_1, x_2, x_3) in space. There's no reason to stop at 3, so let $\mathbb{R}^n$ represent the set of vectors $(x_1, x_2, \ldots, x_n)$ in n dimensions.

In first semester calculus and before, most of our work dealt with problem in $\mathbb{R}$ and $\mathbb{R}^2$. Most of our work now will involve problems in $\mathbb{R}^2$ and $\mathbb{R}^3$. We've got to learn to visualize in $\mathbb{R}^3$.

Definition 2.3. The **magnitude**, or **length**, or **norm** of a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. It is just the distance from the point (v_1, v_2, v_3) to the origin.

A **unit vector** is a vector whose length is one unit. We commonly place a hat above unit vectors, as in $\hat{v}$ or $\hat{\mathbf{v}}$, The standard unit vectors are vectors of lenght one that point in the positive x, y, and z directions, namely

$$\mathbf{i} = \langle 1, 0, 0 \rangle = \hat{\mathbf{x}}, \quad \mathbf{j} = \langle 0, 1, 0 \rangle = \hat{\mathbf{y}}, \quad \mathbf{k} = \langle 0, 0, 1 \rangle = \hat{\mathbf{z}}.$$

Note that in 1D, the length of the vector $\langle -2 \rangle$ is simply $|-2| = \sqrt{(-2)^2} = 2$, the distance to 0. Our use of the absolute value symbols is appropriate, as it generalizes the concept of absolute value (distance to zero) to all dimensions.

Definition 2.4. Suppose $\vec{x} = \langle x_1, x_2, x_3 \rangle$ and $\vec{y} = \langle y_1, y_2, y_3 \rangle$ are two vectors in 3D, and c is a real number. We define vector addition and scalar multiplication as follows:

- Vector addition: $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ (add component-wise).
- Scalar multiplication: $c\vec{x} = (cx_1, cx_2, cx_3)$.

Problem 2.4 Consider the vectors $\vec{u} = (1, 2)$ and $\vec{v} = \langle 3, 1 \rangle$. Draw $\vec{u}$, $\vec{v}$, $\vec{u} + \vec{v}$, and $\vec{u} - \vec{v}$ with their tail placed at the origin. Then draw $\vec{v}$ with its tail at the head of $\vec{u}$.

See 12.2:23-24.

Problem 2.5 Consider the vector $\vec{v} = (3, -1)$. Draw $\vec{v}$, $-\vec{v}$, and $3\vec{v}$. Suppose a donkey travels along the path given by $(x, y) = \vec{v}t = (3t, -t)$, where t represents time. Draw the path followed by the donkey. Where is the donkey at time $t = 0, 1, 2$? Put markers on your graph to show the donkey's location. Then determine how fast the donkey is traveling.

See 11.1: 3,4.

In the previous problem you encountered $(x, y) = (3t, -t)$. This is an example of a function where the input is t and the output is a vector (x, y). For each input t, you get a single vector output (x, y). Such a function we call a **parametrization** of the donkey's path. Because the output is a vector, we call the function a **vector-valued function**. Often, we'll use the variable $\vec{r}$ to represent the radial vector (x, y), or (x, y, z) in 3D which points from the origin outwards. So we could rewrite the position of the donkey as $\vec{r}(t) = (3, -1)t$. We use $\vec{r}$ instead of r to remind us that the output is a vector.

Problem 2.6 Suppose a horse races down a path given by the vector-valued function $\vec{r}(t) = (1, 2)t + (3, 4)$. (Remember this is the same as writing $(x, y) = (1, 2)t + (3, 4)$ or similarly $(x, y) = (1t + 3, 2t + 4)$.) Where is the horse at time $t = 0, 1, 2$? Put markers on your graph to show the horse's location. Draw the path followed by the horse. Give a unit vector that tells the horse's direction. Then determine how fast the horse is traveling.

See 12.2: 1.

Problem 2.7 Consider the two points $P = (1, 2, 3)$ and $Q = (2, -1, 0)$. Write the vector $\vec{PQ}$ in component form (a, b, c). Find the length of vector $\vec{PQ}$. Then find a unit vector in the same direction as $\vec{PQ}$. Finally, find a vector of length 7 units that points in the same direction as $\vec{PQ}$.

See 12.2: 9,17,25,33 and surrounding.

Problem 2.8 A raccoon is sitting at point $P = (0, 2, 3)$. It starts to climb in the direction $\vec{v} = \langle 1, -1, 2 \rangle$. Write a vector equation $(x, y, z) = (?, ?, ?)$ for the line that passes through the point P and is parallel to $\vec{v}$. [Hint, study problem 2.6, and base your work off of what you saw there. It's almost identical.]

Then generalize your work to give an equation of the line that passes through the point $P = (x_1, y_1, z_1)$ and is parallel to the vector $\vec{v} = (v_1, v_2, v_3)$.

See 12.5: 1-12.

Make sure you ask me in class to show you how to connect the equation developed above to what you have been doing since middle school. If you can remember $y = mx + b$, then you can quickly remember the equation of a line. If I don't show you in class, make sure you ask me (or feel free to come by early and ask before class).

Problem 2.9 Let $P = (3, 1)$ and $Q = (-1, 4)$.

See 12.5: 13-20.

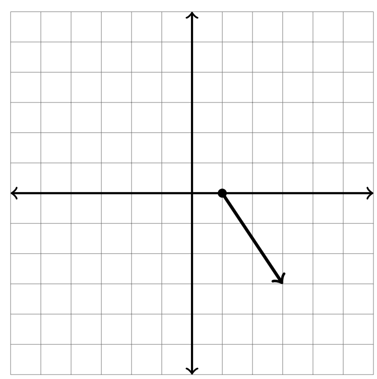
- Write a vector equation $\vec{r}(t) = (?, ?)$ for (i.e, give a parametrization of) the line that passes through P and Q, with $\vec{r}(0) = P$ and $\vec{r}(1) = Q$.
- Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is twice the speed of the first line.
- Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is one unit per second.

If you want to analyze how a river is flowing, one way to do so would be to construct a plot of the river and at each point in the river draw a vector to represent the velocity at that point. This create a collection of many vectors drawn all at once, where the base of each velocity vector is placed at the point where the velocity occurs. The next problem has you construct your first vector field. We'll come back to vector fields more as the semester progresses. Eventually vector fields will be one of the most important ideas in this course. I want you to see one now.

Problem 2.10: Vector Fields Consider the function $\vec{F}(x, y) = \langle 2x + y, -3x + 2y\rangle$. This is a function where the input is a point (x, y) in the plane, and the output is the vector $\langle 2x + y, -3x + 2y\rangle$. For example, if we input the point $(1, 0)$, then the output is $\langle 2(1) + 0, -3(1) + 0\rangle$. To construct a vector field, you draw the output with its base located at the input. In the picture below, based at (1,0) we draw a vector that points right 2 and down 3.

1. Complete the table below and add the other 7 vectors to the graph.

(x, y)	$\langle 2x + y, -3x + 2y\rangle$
$(1, 0)$	$\langle 2, -3\rangle$
$(1, 1)$	
$(1, -1)$	
$(0, 1)$	
$(0, -1)$	
$(-1, 0)$	
$(-1, 1)$	
$(-1, -1)$	



2. Repeat the above for the vector field $\vec{F}(x, y) = (-2y, 3x)$, constructing a vector field plot consisting of 8 vectors.

2.2 The Dot Product

Now that we've learned how to add and subtract vectors, stretch them by scalars, and use them to find lines, it's time to introduce a way of multiplying vectors called the dot product. The dot product arises naturally when we try to find the angle between two vectors. We'll need to recall the law of cosines, stated below.

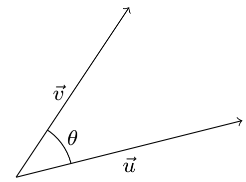
Theorem (The Law of Cosines)**.** *Consider a triangle with side lengths a, b, and c. Let θ be the angle between the sides of length a and b. Then the law of cosines states that*

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

If $\theta = 90°$, then $\cos\theta = 0$ and this reduces to the Pythagorean theorem.

Problem 2.11 Sketch in $\mathbb{R}^2$ the vectors $\langle -1, 2\rangle$ and $\langle 3, 5\rangle$. Then use the law of cosines to find the angle between the vectors. See 12.3: 9-12.

Problem 2.12 Consider the two vectors $\vec{u}$ and $\vec{v}$ in the plane (so $\vec{u}, \vec{v} \in \mathbb{R}^2$) shown in margin to the right.



1. Add the vector $\vec{u} - \vec{v}$ to the picture to the right.
2. Use the law of cosines to explain why $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}| \cos\theta$.

Notice that in your work on the previous problem, the fact that $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}| \cos\theta$ did not require ever referring to the fact that the vectors were in $\mathbb{R}^2$. This fact is true for vectors in general.

Problem 2.13 Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$ (which we write as $\vec{u}, \vec{v} \in \mathbb{R}^3$).

See page 693 if you are struggling.

1. First use the result of the previous problem to explain why
$$|\vec{u}||\vec{v}| \cos\theta = \frac{|\vec{u}|^2 + |\vec{v}|^2 - |\vec{u} - \vec{v}|^2}{2}.$$
2. Now use the coordinates (u_1, u_2, u_3) and (v_1, v_2, v_3) to simplify the right hand side of the equation above. For example, you'll replace $|\vec{u}|^2$ with $\left(\sqrt{u_1^2 + u_2^2 + u_3^2}\right)^2 = u_1^2 + u_2^2 + u_3^2$. For the difference $|\vec{u} - \vec{v}|$, you'll need to subtract coordinates and then compute the magnitude, which gives something like $|\vec{u} - \vec{v}| = \sqrt{(u_1 - v_1)^2 + \cdots}$. When you are done simplifying you should end up with something quite simple.

Definition 2.5: The Dot Product. Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$. We define the dot product of these two vectors to be
$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$
A similar definition holds for vectors in $\mathbb{R}^n$, where $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$. You just multiply corresponding components together and then add. It is the same process that we use in matrix multiplication.

With the definition of the dot product, we can rewrite the law of cosines as
$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta.$$

Problem 2.14 Use our new rule $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$ to find the angle between each pair of vectors below. If the angle is messy, first write the answer in terms of arccos and then use a calculator to approximate the angle.

See 12.3: 9-12.

1. $1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $-2\mathbf{i} + 1\mathbf{j} + 4\mathbf{k}$
2. $(1, 2, 3)$ and $(-2, 1, 0)$

In the previous problem, you should have found that one of the pairs of vectors had a dot product that was zero.

Definition 2.6. We say two vectors $\vec{u}$ and $\vec{v}$ are orthogonal when $\vec{u} \cdot \vec{v} = 0$.

Problem 2.15 Find two vectors orthogonal to $(1, 2)$. Then find 4 vectors orthogonal to $(3, 2, 1)$.

The dot product provides a really easy way to determine when two vectors meet at a right angle. The dot product is precisely zero when this happens. The next problem has you justify this fact.

Problem 2.16 Show that if two nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal, then the angle between them is $90°$. Then show that if the angle between them is $90°$, then the vectors are orthogonal.

See page 694.

Note: There are two things to show above. First, assume that the vectors are orthogonal (so their dot product is zero) and use this to compute the angle. Then second, assume that the angle between them is $90°$ and use this to compute the dot product.

Let's end this section by looking at some properties of the dot product.

Problem 2.17 Mark each statement true or false. Then make up an example to illustrate why you gave your answer. I have done the first as an example. You can assume that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ and that $c \in \mathbb{R}$.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

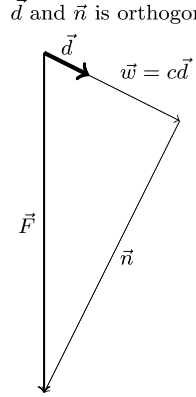
   Solution: This is true. If $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$, then we know $\vec{u} \cdot \vec{v} = (a, b) \cdot (c, d) = ac + bd$ and $\vec{v} \cdot \vec{u} = (a, b) \cdot (c, d) = ca + db$. Since $ab = ba$ and $cd = dc$, these two are clearly true.
2. $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$.
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$.
4. $\vec{u} + (\vec{v} \cdot \vec{w}) = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{w})$.
5. $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$.
6. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

The last property above is extremely important, namley it connects the length of a vector to the dot product. We have now seen that we can compute both lengths and angles from the dot product. Any time you are working with either lengths or angles, there is a dot product hiding in the background. On a side note, in dimension 4 and higher, we define lenths and angles directly from the dot product.

2.2.1 Projections and Work

Suppose a heavy box needs to be lowered down a ramp. The box exerts a downward force of say 200 Newtons, which we could write in vector notation as $\vec{F} = \langle 0, -200 \rangle$. If the ramp was placed so that the box needed to be moved right 6 m, and down 3 m, then we'd need to get from the origin $(0, 0)$ to the point $(6, -3)$. This displacement can be written as $\vec{d} = \langle 6, -3 \rangle$. The force $\vec{F}$ acts straight down, rather than with the displacement. Our goal in this section is to find out how much of the force $\vec{F}$ acts in the direction of the displacement. This will tell us precisely the force needed to prevent the box from sliding down the ramp (neglecting friction). We are going to break the force $\vec{F}$ into two components, one component in the direction of $\vec{d}$, and another component orthogonal to $\vec{d}$.

In the diagram below, we have $\vec{F} = \vec{w} + \vec{n}$ where $\vec{w}$ is parallel to $\vec{d}$ and $\vec{n}$ is orthogonal to $\vec{d}$.



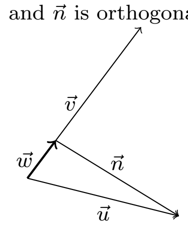
Problem 2.18 Read the preceding paragraph. Rather than working with the specific numbers given in that paragraph, please use $\vec{F}$ and $\vec{d}$ to represent any vector, so that when we are done with this problem we'll have a symbolic solution.

We want to write $\vec{F}$ as the sum of two vectors $\vec{F} = \vec{w} + \vec{n}$, where $\vec{w}$ is parallel to $\vec{d}$ and $\vec{n}$ is orthogonal to $\vec{d}$. Since $\vec{w}$ is parallel to $\vec{d}$, we can write $\vec{w} = c\vec{d}$ for some unknown scalar c. This means that $\vec{F} = c\vec{d} + \vec{n}$. Use the fact that $\vec{n}$ is orthogonal to $\vec{d}$ to show that $c = \dfrac{\vec{F}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}$.

[Hint: Dot each side of $\vec{F} = c\vec{d} + \vec{n}$ with $\vec{d}$ and distribute. You'll need to use the fact that $\vec{n}$ and $\vec{d}$ are orthogonal to remove $\vec{n}\cdot\vec{d}$ from the problem. This should turn the vectors into numbers, so you can use division and solve for c directly. Don't spent more than 10 minutes on this problem.]

Problem 2.19 Consider the vectors $\vec{u}$ and $\vec{v}$ in the diagram to the right. We can write $\vec{u}$ as the sum of a vector that is parallel to $\vec{v}$ (called $\vec{w}$ below) and a vector that is orthogonal to $\vec{v}$ (called $\vec{n}$ below). This gives us $\vec{u} = \vec{w} + \vec{n}$.

In the diagram below, we have $\vec{u} = \vec{w} + \vec{n}$ where $\vec{w}$ is parallel to $\vec{v}$ and $\vec{n}$ is orthogonal to $\vec{v}$.



Notice the right angle where vectors $\vec{n}$ and $\vec{w}$ meet.

1. Let θ be the angle between $\vec{u}$ and $\vec{v}$. Use right triangle trigonometry to explain why the length of $\vec{w}$ is given by $|\vec{w}| = |\vec{u}|\cos\theta$.
2. Now that we know the length of $\vec{w}$, explain why $\vec{w} = (|\vec{u}|\cos\theta)\dfrac{\vec{v}}{|\vec{v}|}$. See problem 2.7 if you need help.
3. We have a formula that connects the dot product to the cosine of the angle between two vectors. Show the steps that transform the equation above into the equation

$$\vec{w} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|}.$$

Can you explain why this also means

$$\vec{w} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}?$$

The previous two problems give us the definition of a projection.

Definition 2.7. The projection of $\vec{F}$ onto $\vec{d}$, written $\text{proj}_{\vec{d}}\vec{F}$, is defined as

$$\text{proj}_{\vec{d}}\vec{F} = \underbrace{\left(\frac{\vec{F}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\right)\vec{d}}_{\text{quick computation method}} = \underbrace{\left(\frac{\vec{F}\cdot\vec{d}}{|\vec{d}|}\right)\frac{\vec{d}}{|\vec{d}|}}_{\substack{\text{geometric method}\\\text{magnitude times direction}}}.$$

When we wish to write $\vec{F}$ as the sum of a vector parallel to $\vec{d}$ plus a vector orthogonal to $\vec{d}$, the projection of $\vec{F}$ onto $\vec{d}$ is precisely the portion of $\vec{F}$ that is parallel to $\vec{d}$.

Problem 2.20 Let $\vec{u} = (-1, 2)$ and $\vec{v} = (3, 4)$. Draw $\vec{u}$, $\vec{v}$, and $\text{proj}_{\vec{v}}\vec{u}$. Then draw a line segment from the head of $\vec{u}$ to the head of the projection.

See 12.3:1-8 (part d).

Now let $\vec{u} = (-2, 0)$ and keep $\vec{v} = (3, 4)$. Draw $\vec{u}$, $\vec{v}$, and $\text{proj}_{\vec{v}}\vec{u}$. Then draw a line segment from the head of $\vec{u}$ to the head of the projection.

One final application of projections pertains to the concept of work. Work is the transfer of energy. If a force F acts through a displacement d, then the most basic definition of work is $W = Fd$, the product of the force and the displacement. This basic definition has a few assumptions.

- The force F must act in the same direction as the displacement.
- The force F must be constant throughout the entire displacement.
- The displacement must be in a straight line.

Before the semester ends, we will be able to remove all 3 of these assumptions. The next problem will show you how dot products help us remove the first assumption.

Recall the set up to problem 2.18. We want to lower a box down a ramp (which we will assume is frictionless). Gravity exerts a force of $\vec{F} = \langle 0, -200\rangle$ N. If we apply no other forces to this system, then gravity will do work on the box through a displacement of $\langle 6, -3\rangle$ m. The work done by gravity will transfer the potential energy of the box into kinetic energy (remember that work is a transfer of energy). How much energy is transferred?

Problem 2.21 Find the amount of work done by the force $\vec{F} = \langle 0, -200\rangle$ through the displacement $\vec{d} = \langle 6, -3\rangle$. Find this by doing the following:

See 12.3: 24, 41-44.

1. Find the projection of $\vec{F}$ onto $\vec{d}$. This tells you how much force acts in the direction of the displacement. Find the magnitude of this projection.
2. Since work equals $W = Fd$, multiply your answer above by $|\vec{d}|$.
3. Now compute $\vec{F} \cdot \vec{d}$. You have just shown that $W = \vec{F} \cdot \vec{d}$ when $\vec{F}$ and $\vec{d}$ are not in the same direction.

Th dot product gives us the work done by $\vec{F}$ through a displacement $\vec{d}$ when $\vec{F}$ and $\vec{d}$ are not in the same direction. Remember that the dot product is a number, which means it may be hard to visual. Connecting the dot product to work done by one vector in the direction of another can often lead to a good geometric description of the dot product.

Problem Answer each of the following, assuming that none of the vectors are the zero vector.

1. Suppose $\vec{u} \cdot \vec{v} = 0$. What do you know about the two vectors?
2. Suppose $\vec{u} \cdot \vec{v} > 0$. What do you know about the two vectors?
3. Suppose $\vec{u} \cdot \vec{v} < 0$. What do you know about the two vectors?

See [1] for a solution.

---

[1] When the dot product is zero, we know that the two vectors meet at a 90° angle. Thinking about this in terms of work, this means that the force has no portion in the direction of the displacement, hence there is no work done. If the dot product is positive, then the force has a portion acting in the direction of the displacement. This means that the angle between the two vectors is acute. Similarly if the dot product is negative then the angle must be obtuse (greater than 90°.)

2.3 The Cross Product and Planes

The dot product gave us a way of multiplying two vectors together, but the result was a number, not a vector. We now define the cross product, which will allow us to multiply two vectors together to give us another vector. We were able to define the dot product in all dimensions. The cross product is only defined in $\mathbb{R}^3$.

Definition 2.8: The Cross Product. The cross product of two vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ is a new vector $\vec{u}\times\vec{v}$. This new vector is (1) orthogonal to both $\vec{u}$ and $\vec{v}$, (2) has a length equal to the area of the parallelogram whose sides are these two vectors, and (3) points in the direction your thumb points as you curl the base of your right hand from $\vec{u}$ to $\vec{v}$.

A formula for the cross product is

$$\vec{u}\times\vec{v} = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\rangle = \det\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{bmatrix}.$$

The definition of a the cross product tells us what kind of vector we need (orthogonal to both, magnitude equal to an area, and direction following the right hand rule), but doesn't give us a formula for computing it. The formula given here is nontrivial to develop from the definition. Wikipedia (see this link) gives a decent explanation, but does skip one difficult step in their computation. We will use the formula given here without proof.

See 12.4: 1-8.

Problem 2.22 Let $\vec{u} = (1, -2, 3)$ and $\vec{v} = (2, 0, -1)$.

- Compute $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$. How are they related?
- Compute $\vec{u}\cdot(\vec{u}\times\vec{v})$ and $\vec{v}\cdot(\vec{u}\times\vec{v})$. Why did you get the answer you got?
- Compute $\vec{u}\times(2\vec{u})$. Why did you get the answer you got?

Problem 2.23 Let $P = (2, 0, 0)$, $Q = (0, 3, 0)$, and $R = (0, 0, 4)$. Find a vector that orthogonal to both $\vec{PQ}$ and $\vec{PR}$. Then find the area of the triangle PQR. Construct a 3D graph of this triangle.

See 12.4: 15-18. Remember, the magnitude of the cross product gives the area of the parallelogram formed using the two vectors as the edges.

Problem 2.24 Consider $\vec{\imath} = (1, 0, 0)$, $2\vec{\jmath} = (0, 2, 0)$, and $3\vec{k} = (0, 0, 3)$.

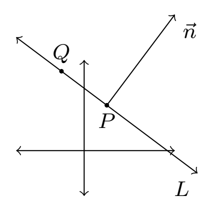
See 12.3: 9-14.

- Compute $\vec{\imath}\times 2\vec{\jmath}$ and $2\vec{\jmath}\times\vec{\imath}$. Try to do this without using the determinant formula, but instead using the definition.
- Compute $\vec{\imath}\times 3\vec{k}$ and $3\vec{k}\times\vec{\imath}$.
- Compute $2\vec{\jmath}\times 3\vec{k}$ and $3\vec{k}\times 2\vec{\jmath}$.

Give a geometric reason as to why some vectors above have a plus sign, and some have a minus sign.

We will now combine the dot product with the cross product to develop an equation of a plane in 3D. Before doing so, let's look at what information we need to obtain a line in 2D, and a plane in 3D. To obtain a line in 2D, one way is to have 2 points. The next problem introduces the new idea by showing you how to find an equation of a line in 2D.

Problem 2.25 Suppose the point $P = (1, 2)$ lies on line L. Suppose that the angle between the line and the vector $\vec{n} = \langle 3, 4\rangle$ is 90° (whenever this happens we say the vector $\vec{n}$ is normal to the line). Let $Q = (x, y)$ be another point on the line L. Use the fact that $\vec{n}$ is orthogonal to $\vec{PQ}$, together with the dot product, to obtain an equation of the line L.



Problem 2.26 Let $P = (a, b, c)$ be a point on a plane in 3D. Let $\vec{n} = (A, B, C)$ be a normal vector to the plane (so the angle between the plane and and $\vec{n}$ is $90°$). Let $Q = (x, y, z)$ be another point on the plane. Show that an equation of the plane through point P with normal vector $\vec{n}$ is See page 709.

$$A(x-a) + B(y-b) + C(z-c) = 0.$$

Problem 2.27 Consider the three points $P = (1, 0, 0), Q = (2, 0, -1), R = (0, 1, 3)$. Find an equation of the plane which passes through these three points. [Hint: first find a normal vector to the plane.] See 12.5: 21-28.

Problem 2.28 Consider the two planes $x + 2y + 3z = 4$ and $2x - y + z = 0$. These planes meet in a line. Find a vector that is parallel to this line. Then find a vector equation of the line. See 12.5: 57-60.

Problem 2.29 Find an equation of the plane containing the lines $\vec{r}_1(t) = (1, 3, 0)t + (1, 0, 2)$ and $\vec{r}_2(t) = (2, 0, -1)t + (2, 3, 2)$.

Problem 2.30 Consider the points $P = (2, -1, 0)$, $Q = (0, 2, 3)$, and $R = (-1, 2, -4)$.

1. Give an equation $(x, y, z) = (?, ?, ?)$ of the line through P and Q.
2. Give an equation of the line through P and R.
3. Give an equation of the plane through P, Q, and R.

Problem 2.31 Consider the points $P = (2, 4, 5)$, $Q = (1, 5, 7)$, and $R = (-1, 6, 8)$.

1. What is the area of the triangle PQR.
2. Give a normal vector to the plane through these three points.
3. What is the distance from the point $A = (1, 2, 3)$ to the plane PQR. [Hint: What does the projection of $\vec{PA}$ onto $\vec{n}$ have to do with this problem?]

2.4 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 3

Curves

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Be able to graph and equations of conic sections (parabolas, ellipses, hyperbolas). Find the focus of parabolas.

2. Use a change-of-coordinates involving translation and stretching to give an equation of and graph a curve.

3. Model motion in the plane using parametric equations. In particular, describe conic sections using parametric equations.

4. Find derivatives and tangent lines for parametric equations. Explain how to find velocity, speed, and acceleration from parametric equations.

5. Use integrals to find the length of a parametric curve and related quantities.

You'll have a chance to teach your examples to your peers prior to the exam.

3.1 Creating Good Graphs in the Plane

Before we jump fully into $\mathbb{R}^3$, we need some good examples of planar curves (curves in $\mathbb{R}^2$) that we'll extend to objects in 3D. For now, we'll focus on parabolas, circles, ellipses, and hyperbolas. We need to become comfortable drawing these graphs, as well as translating, streching (rescaling), and reflecting them about lines.

Given a graph of a function $y = f(x)$, how do we modify the equation $y = f(x)$ to obtain a new function that has been shifted? You might recall several rules that allow you to translate functions left and right, up and down, or even rescale (stretch) the functions vertically and horizontally. For example, if we start with the parabola $y = x^2$, then the equation $y = (x-2)^2 + 3$, or equivalently $y - 3 = (x-2)^2$, is the same parabola except we have shifted it right 2 and up 3.

In this section, we'll revisit the concepts of translating and stretching functions. All of these ideas are part of a bigger picture which we'll refer to as changing coordinates. In the example above we had two curves, namely $y = x^2$ and the translated $y - 3 = (x-2)^2$. To simplify our work, let's use the variables u and v for the starting equation and x and y for the translated equation. Notice then that we have $v = u^2$ and $y - 3 = (x-2)^2$. If we just let $v = y - 3$ and $u = x - 2$, or equivalently $x = u + 2$ and $y = v + 3$, then we have equations

In practice, we generally don't use new variables but might instead write the change-of-coordinates as $x_n = x_o + 2$ and $y_n = y_o + 3$ where n stands for "new" and o stands for "old". After making the change, we just drop subscripts.

that allow us to change between uv and xy coordinatse. We call each pair of equations a change-of-coordinates. We'll often write our changes of coordinates by solving for x and y, as the equations $x = u + 2$ and $y = v + 3$ clearly show us that the x-values should be the old u-values shifted 2 units right and the y-values should be the old v-values shifted 3 units up.

Problem 3.1 Consider the circle $u^2 + v^2 = 1$ and the change-of-coordinates given by $x = 2u + 1$ and $y = 3v + 4$.

1. Draw the curve $u^2 + v^2 = 1$ in the uv plane.
2. Solve the change-of-coordinate equations for u and v and then give an equation of the curve using x and y coordinates.
3. Construct a graph of the curve in the xy plane. You should have an ellipse.
4. Use the same change-of-coordinates with the curve $v = u^2$ to state the equation using x and y coordinates, and then draw the parabola in both the uv plane and the xy plane.
5. How would you describe the connection between the graphs you made in the uv plane and their corresponding graph in the xy plane?

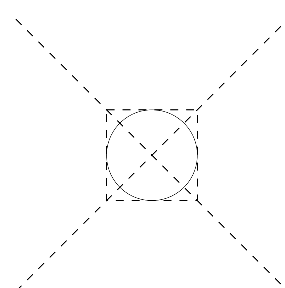
In the previous problem you were given a curve using uv coordaintes, and then asked to use a change-of-coordinates to construct a graph in the xy plane. The next problem has you do this in reverse, namely gives you curve in the xy plane and asks you to state the change of coordinates that would reduce the curve to a simple object in the uv plane.

Problem 3.2 Start by graphing the parabola $y = 3(x-1)^2 + 2$.

1. Give a change-of-coordinates of the form $x = ?u+?$, $y = ?v+?$ that will transform the curve $v = u^2$ in the uv plane to the parabola $y = 3(x-1)^2+2$.
2. Which of $y = 3(x-1)^2 + 2$ or $\dfrac{y-2}{3} = (x-1)^2$ makes it easier to see the change of coordinates?
3. Construct a graph of the parabola $\dfrac{y+1}{2} = \left(\dfrac{x-3}{4}\right)^2$. Optionally, state the change-of-coordinates you used.

Problem 3.3 Consider the curve $x^2 - y^2 = 1$, which we call a hyperbola.

1. Show that $y = \pm x\sqrt{1 - \frac{1}{x^2}}$, and then use this fact to explain why y approaches the lines $y = \pm x$ as x gets large. We call these two lines the asymptotes of the hyperbola, and any good graph of a hyperbola should include them.
2. We'll now construct a graph of the hyperbola. One simple way to draw the asymptotes is to start by constructing a rectangular box with corners at $(1, \pm 1)$ and $(-1, \pm 1)$. Connecting opposing corners of this box gives the asymptotes $y = \pm x$. The circle $x^2 + y^2 = 1$ should fit nicely inside your box (see the picture on the right). Now use software to view a graph of the hyperbola $x^2 - y^2 = 1$ and add it to your picture, making sure the hyperbola follows the asymptotes as $|x|$ gets large. When you construct your graph on your paper, make sure your sketch includes the box, lines, and circle, as well as the hyperbola.



3. Now construct a graph of $\frac{(x-1)^2}{4}-\frac{(y-4)^2}{9}=1$, including an appropriate box and asymptotes. If you want to find the box easily, start by drawing the ellipse $\frac{(x-1)^2}{4}+\frac{(y-4)^2}{9}=1$, and then add the box, the asymptotes, and finally the hyperbola.

Problem 3.4 The equation $4x^2+4y^2+6x-8y-1=0$ represents a circle (though initially it does not look like it). Use the method of completing the square to rewrite the equation in the form $(x-a)^2+(y-b)^2=r^2$ (hence telling you the center and radius). Then generalize your work to find the center and radius of any circle written in the form $x^2+y^2+Dx+Ey+F=0$.

Problem 3.5 Consider the parabola $v=u^2$ and the hyperbola $u^2-v^2=1$.

1. Using the change of coordinates $x=v$, $y=u$, draw the corresponding parabola and hyperbola in the xy-plane.
2. Using the change of coordinates $x=2v+1$, $y=3u+4$, draw the corresponding parabola in the xy-plane.
3. Draw the hyperbola $\frac{(y-4)^2}{9}-\frac{(x-1)^2}{4}=1$ in the xy-plane.

Problem 3.6 Consider the change of coordiantes $x=au+h$, $y=bv+k$.

1. Use this change of coordinates to rewrite the parabola $v=u^2$, the ellipse $u^2+v^2=1$, and the hyperbola $u^2-v^2=1$ using xy coordinates.
2. In your own words, how do each of the values of a, b, h, and k, change the graph of the curve in the uv plane when you draw the graph in the xy plane. Include pictures to accompany your words.

When you hear the word parabola, what comes to mind? For me, the answer is $y=x^2$ and objects whose graphs are similar. This is not a definition. Before we can formally define a parabola, we need to define the distance between a point and a line.

Definition 3.1. Let P be a point and L be a line. Define the distance between P and L to be the length of the shortest line segment that has one end on L and the other end at P. Note: This segment will always be perpendicular to L.

Definition 3.2. Given a point P (called the focus) and a line L (called the directrix) which does not pass through P, we define a parabola as the set of all points Q in the plane so that the distance from P to Q equals the distance from Q to L. The vertex is the point on the parabola that is closest to the directrix.

The definition above is the formal definition of a parabola. There are similar definitions for an ellipse and a hyperbola, but you can view those out of class. The basic ideas we need for this semester come by starting with the basic equations $u^2+v^2=1$ and $u^2-v^2=1$ and then using a change-of-coordinates. This next problem has you use the definition above to give an equation of a parabola if you know the location of the focus and directrix.

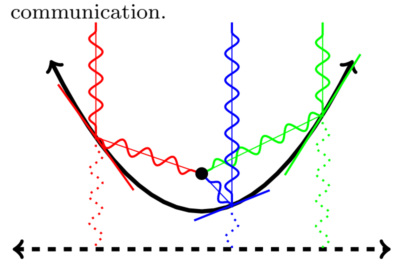
Problem 3.7 Consider the line $L : y = -p$, the point $P = (0, p)$, and another point $Q = (x, y)$. Use the distance formula to show that an equation of a parabola with directrix L and focus P is $x^2 = 4py$. Then use your work to explain why an equation of a parabola with directrix $x = -p$ and focus $(p, 0)$ is $y^2 = 4px$.

See page 658.

Please ask me about the reflective properties of parabolas, ellipses, and hyperbolas in class, if I have not told you already. We use the objects in satellite dishes, long range telescopes, solar ovens, and more. The following optional problem provides the basis to these reflective properties. If you wish to present it, let me know.

Problem: Optional Consider the parabola $x^2 = 4py$ with directrix $y = -p$ and focus $(0, p)$. Let $Q = (a, b)$ be some point on the parabola. Let T be the tangent line to L at point Q. Show that the angle between PQ and T is the same as the angle between the line $x = a$ and T. This shows that a vertical ray coming down towards the parabola will reflect off the wall of a parabola and head straight towards the vertex, as shown to the right.

A ray from space travling towards a satelite dish will bounce off the dish's wall and head towards the focus. The satelite dish focuses many rays at a single point, which allows for long range communication.



The next two problems will help you use the basic equations of a parabola, together with shifting and reflecting, to study all parabolas whose axis of symmetry is parallel to either the x or y axis.

Problem 3.8 Once the directrix and focus are known, we can give an equation of a parabola. For each of the following, give an equation of the parabola with the stated directrix and focus. Provide a sketch of each parabola.

See 11.6: 9-14

1. The focus is $(0, 3)$ and the directrix is $y = -3$.
2. The focus is $(0, 3)$ and the directrix is $y = 1$.
3. The focus is $(2, -5)$ and the directrix is $y = 3$.
4. The focus is $(1, 2)$ and the directrix is $x = 3$.

Problem 3.9 Each equation below represents a parabola. Find the focus, directrix, and vertex of each parabola, and then provide a rough sketch.

See 11.6: 9-14

1. $y = x^2$
2. $(y - 2)^2 = 4(x - 1)$
3. $y = -8x^2 + 3$
4. $y = x^2 - 4x + 5$ (You'll need to complete the square.)

Problem 3.10 For each ellipse below, graph the ellipse.

See 11.6: 17-24. Feel free to use technology to help you.

1. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$
2. $16x^2 + 25y^2 = 400$ [Hint: divide by 400.]
3. $\dfrac{(x-1)^2}{5} + \dfrac{(y-2)^2}{9} = 1$

4. $x^2 + 2x + 2y^2 - 8y = 9$ (You'll need to complete the square.)

Problem 3.11 For each hyperbola below, graph the hyperbola (include the box and asymptotes).

See 11.6: 27-34. Feel free to use technology to help you.

1. $\frac{x^2}{25} - \frac{y^2}{9} = 1$ and $\frac{y^2}{9} - \frac{x^2}{25} = 1$
2. $25y^2 - 16x^2 = 400$ [Hint: divide by 400.]
3. $\frac{(x-1)^2}{5} - \frac{(y-2)^2}{9} = 1$
4. $x^2 + 2x - 2y^2 + 8y = 9$ (You'll need to complete the square.)

3.2 Parametric Equations

In middle school, you learned to write an equation of a line as $y = mx + b$. In the vector unit, we learned to write this in vector form as $(x, y) = (1, m)t + (0, b)$. The equation to the left is called a vector equation. It is equivalent to writing the two equations

$$x = 1t + 0, y = mt + b,$$

which we will call parametric equations of the line. We were able to quickly develop equations of lines in space, by just adding a third equation for z.

Parametric equations provide us with a way of specifying the location (x, y, z) of an object by giving an equation for each coordinate. We will use these equations to model motion in the plane and in space. In this section we'll focus mostly on planar curves.

Definition 3.3. If each of f and g are continuous functions, then the curve in the plane defined by $x = f(t), y = g(t)$ is called a parametric curve, and the equations $x = f(t), y = g(t)$ are called parametric equations for the curve. You can generalize this definition to 3D and beyond by just adding more variables.

Problem 3.12 By plotting points, construct graphs of the three parametric curves given below (just make a t, x, y table, and then plot the (x, y) coordinates). Place an arrow on your graph to show the direction of motion.

See 11.1: 1-18. This is the same for all the problems below.

1. $x = \cos t, y = \sin t$, for $0 \leq t \leq 2\pi$.
2. $x = \sin t, y = \cos t$, for $0 \leq t \leq 2\pi$.
3. $x = \cos t, y = \sin t, z = t$, for $0 \leq t \leq 4\pi$.

Problem 3.13 Plot the path traced out by the parametric curve $x = 1 + 2\cos t, y = 3 + 5\sin t$. Then use the trig identity $\cos^2 t + \sin^2 t = 1$ to give a Cartesian equation of the curve (an equation that only involves x and y).

Problem 3.14 Find parametric equations for a line that passes through the points $(0, 1, 2)$ and $(3, -2, 4)$.

What we did in the previous chapter should help here.

Problem 3.15 Plot the path traced out by the parametric curve $\vec{r}(t) = (t^2+1, 2t-3)$. Give a Cartesian equation of the curve (eliminate the parameter t), and then find the focus of the resulting curve.

Problem 3.16 Consider the parametric curve given by $x = \tan t, y = \sec t$. Plot the curve for $-\pi/2 < t < \pi/2$. Give a Cartesian equation of the curve. (A trig identity will help - what identity involves both tangent and secant?) [Hint: this problem will probably be easier to draw if you first find the Cartesian equation, and then plot the curve.]

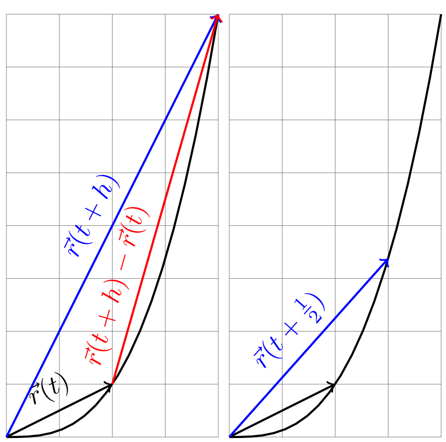
3.2.1 Derivatives and Tangent lines

We're now ready to discuss calculus on parametric curves. The derivative of a vector valued function is defined using the same definition as first semester calculus.

Definition 3.4. If $\vec{r}(t)$ is a vector equation of a curve (or in parametric form just $x = f(t), y = g(t)$), then we define the derivative to be

$$\frac{d\vec{r}}{dt} = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

Problem 3.17 Consider the curve $\vec{r}(t) = (2t, t^3)$. We'll analyze this curve at $t = 1$, where $\vec{r}(1) = (2,1)$. When $h = 1$, we have $\vec{r}(t+h) = \vec{r}(2) = (2,8)$ and the difference quotient $\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$ equals the difference $\vec{r}(2) - \vec{r}(1)$ and simply connects the heads of these two vectors, as shown below on the left.



1. The picture above on the right shows $\vec{r}(t)$ and $\vec{r}(t+h)$ when $t = 1$ and $h = 1/2$. Add to this picture the difference $\vec{r}(t+h) - \vec{r}(t)$ and the difference quotient $\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$.
2. Leaving $t = 1$ but changing h to $h = 1/4$ and then $h = 1/8$, construct a third and fourth picture that shows $\vec{r}(t), \vec{r}(t+h)$, the difference, and the difference quotient.
3. Letting $t = 1$, as $h \to 0$ what happens to $\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$? Draw this vector.

The previous problem gave a geometric intuition of the derivative, and emphasizes why the derivative is tangent to a curve. The following problem will provide a simple way to compute derivatives.

Problem 3.18 Let $\vec{r}(t) = (f(t), g(t))$. Show that $\frac{d\vec{r}}{dt} = (f'(t), g'(t))$. See page 728.

[The definition of the derivative is $\frac{d\vec{r}}{dt} = \lim_{h\to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$. We were told $\vec{r}(t) = (f(t), g(t))$, so use this in the derivative definition. Perform the vector arithmetic componentwise, and you should obtain $\frac{d\vec{r}}{dt} = (f'(t), g'(t))$.]

The previous problem shows you can take the derivative of a vector valued function by just differentiating each component separately. The next problem shows you that velocity and acceleration are still connected to the first and second derivatives.

Problem 3.19 Consider the parametric curve given by $\vec{r}(t) = (3\cos t, 3\sin t)$. See 13.1:5-8 and 13.1:19-20

1. Graph the curve $\vec{r}$, and compute $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$.
2. On your graph, draw the vectors $\frac{d\vec{r}}{dt}\left(\frac{\pi}{4}\right)$ and $\frac{d^2\vec{r}}{dt^2}\left(\frac{\pi}{4}\right)$ with their tail placed on the curve at $\vec{r}\left(\frac{\pi}{4}\right)$. These vectors represent the velocity and acceleration vectors.
3. Give a vector equation of the tangent line to this curve at $t = \frac{\pi}{4}$. (You know a point and a direction vector.)

Definition 3.5. If an object moves along a path $\vec{r}(t)$, we can find the velocity and acceleration by just computing the first and second derivatives. The velocity is $\frac{d\vec{r}}{dt}$, and the acceleration is $\frac{d^2\vec{r}}{dt^2}$. Speed is a scalar, not a vector. The speed of an object is just the length of the velocity vector.

Problem 3.20 Consider the curve $\vec{r}(t) = (2t+3, 4(2t-1)^2)$.

1. Construct a graph of $\vec{r}$ for $0 \leq t \leq 2$.
2. If this curve represented the path of a horse running through a pasture, find the velocity of the horse at any time t, and then specifically at $t = 1$. What is the horse's speed at $t = 1$?
3. Give a vector equation of the tangent line to $\vec{r}$ at $t = 1$. Include this on your graph.
4. Explain how to obtain the slope of the tangent line, and then write an equation of the tangent line using point-slope form. [Hint: How can you turn the direction vector, which involves (dx/dt) and (dy/dt), into the number given by the slope (dy/dx)?]

Problem 3.21 Suppose an object travels along the path given by $\vec{r}(t) = (3t, -2t^2)$. The velocity is $\vec{v}(t) = (3, -4t)$ and the acceleration is $\vec{a}(t) = (0, -4)$. At time $t = 1$, these vectors are $\vec{v}(1) = (3, -4)$ and $\vec{a}(1) = (0, -4)$.

1. Why do we know that the acceleration and velocity vectors are not in the same direction?
2. What is the vector component of the acceleration vector that points in the same direction as the velocity vector? In other words, what is $\text{proj}_{\vec{v}}\vec{a}$. We'll call this vector $\vec{a}_{\parallel\vec{v}}$.
3. What is the vector component of the acceleration vector that is orthogonal to the velocity vector? We'll call this vector $\vec{a}_{\perp\vec{v}}$.
4. Draw a picture that shows the relationship among $\vec{v}$, $\vec{a}$, $\vec{a}_{\parallel\vec{v}}$, and $\vec{a}_{\perp\vec{v}}$.

3.2.2 Integration, Arc Length, and More

In this section, we will develop ways to integrate along paths. Everything in this section is a generalization of integration from first semester calculus. Try the following exercise whose solution is provided in the footnotes.

Exercise Consider a function $y = f(x)$ for $a \leq x \leq b$ and assume that $f(x) \geq 0$. Imagine cutting the x-axis up into many little bits, where we use dx to represent the length of each little bit. See [1] for a solution.

1. If we pick one of the tiny bits of length dx whose left endpoint is located at x, what does the quantity $dA = f(x)dx$ give us? Construct a picture to illustrate this.
2. Why is the total area given by $A = \int_a^b f(x)dx$.

If an object moves at a constant speed, then the distance traveled is

$$\text{distance} = \text{speed} \times \text{time}.$$

This requires that the speed be constant. What if the speed is not constant? Over a really small time interval dt, the speed is almost constant, so we can still use the idea above. The following problem will help you develop the key formula for arc length.

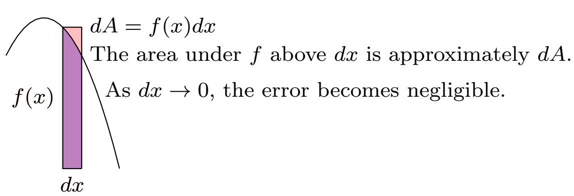
Problem 3.22: Derivation of the arc length formula Suppose an object moves along the path given by $\vec{r}(t) = (x(t), y(t))$ for $a \leq t \leq b$. We know that the velocity is $\frac{d\vec{r}}{dt}$, and so the speed is just the magnitude of this vector.

1. Show that we can write the object's speed at any time t as $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.
2. If you move at constant speed $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ for a time length dt, what's the distance ds you have traveled.
3. Explain why the length of the path given by $\vec{r}(t)$ for $a \leq t \leq b$ is

$$s = \int ds = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

This is the arc length formula. Ask me in class for an alternate way to derive this formula.

[1] The quantity $dA = f(x)dx$ is the area of a rectangle whose base is dx wide and whose height is $f(x)$. If dx is really small, then the function f is almost constant, so $f(x)$ and $f(x + dx)$ are really close. The little bit of area dA is extremely close the actual area under f that lies above the x axis between x and $x + dx$, off by the small amount of the rectangle that lies above the curve as shown below. This extra area becomes negligible as $dx \to 0$.



To find the total area under the curve, all we have to do is add up the little bits of area. In terms of Riemann sums, we would write $\sum dA$. The integral symbol just means that we're letting $dx \to 0$, and so the total area is found using $A = \int dA$. To obtain the total area, we just add up the little bits of area. When we replace dA with $f(x)dx$, we put the bounds $x = a$ to $x = b$ on the integral to obtain $A = \int dA = \int_a^b f(x)dx$.

Problem: Alternate derivation of arc length formula Suppose an object moves along the path given by $\vec{r}(t) = (x(t), y(t))$ for $a \leq t \leq b$. Imagine slicing the path up into hundreds of tiny slices. Let ds represent the length of each tiny slice.

1. Draw an appropriate diagram showing an arbitrary curve, a tiny chunk of the curve of length ds, and a triangle so that the Pythagorean theorem gives the approximation $ds = \sqrt{(dx)^2 + (dy)^2}$.
2. Use algebra to show that $\sqrt{(dx)^2 + (dy)^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$.
3. Explain why the length of the path given by $\vec{r}(t)$ for $a \leq t \leq b$ is

$$s = \int ds = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Now that we have a formula for computing arc length, let's practice using it with a few problems. The next two problems just have you use the formula above. The first asks you to actually perform an integral, the second asks you to set up several integrals. You'll find that arc length problems can become quite messy and sometimes impossible to compute exactly because of the square root term in the integrand.

Problem 3.23 Find the length of the curve $\vec{r}(t) = \left(t^3, \dfrac{3t^2}{2}\right)$ for $t \in [1, 3]$. The notation $t \in [1, 3]$ means $1 \leq t \leq 3$. Be prepared to show us your integration steps in class (you'll need a substitution).

See 11.2: 25-30

Problem 3.24 For each curve below, set up an integral formula which would give the length, and sketch the curve. Do not worry about integrating them.

1. The parabola $\vec{r}(t) = (t, t^2)$ for $t \in [0, 3]$.
2. The ellipse $\vec{r}(t) = (4\cos t, 5\sin t)$ for $t \in [0, 2\pi]$.
3. The hyperbola $\vec{r}(t) = (\tan t, \sec t)$ for $t \in [-\pi/4, \pi/4]$.

The reason I don't want you to actually compute the integrals is that they will get ugly really fast. Try doing one in Wolfram Alpha and see what the computer gives.

To actually compute the integrals above and find the lengths, we would use a numerical technique to approximate the integral (something akin to adding up the areas of lots and lots of rectangles as you did in first semester calculus).

Let's finish this chapter with some examples that illustrate how the arc length formula gives us much more than just length. This first problem comes from physics and asks you to find the total charge on a rod if you know the charge per length. The same type of problem shows up in engineering as finding the total mass of wire whose density (mass per length) is known. Since we know that density is mass per length, then all we have to do is times density by length to obtain the mass.

Problem 3.25 A wire lies along the curve $\vec{r}(t) = (7\cos t, 7\sin t)$ for $0 \leq t \leq \pi$. The wire contains charged particles where the charge per unit length at location (x, y) is given by $q(x, y) = y$. In this problem we'll compute the total charge on the wire.

If the wire were a conductor, then the charged particles (electrons) would not stay put, but rather flow freely along the wire until the repulsive forces are minimized. This wire is an insulator.

1. Why is the charge over a small distance ds approximately given by $dQ = q(x, y)ds$?

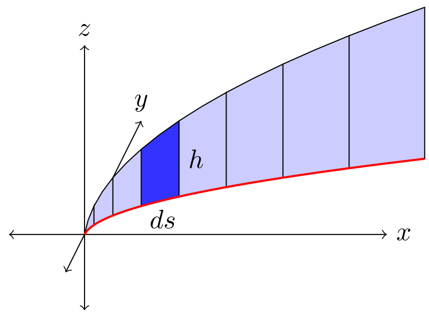
2. The total charge is the sum of the charges over all the little pieces on the rod. This gives us the total charge as

$$Q_{\text{total}} = \int_C dQ = \int_C q(x,y)ds = \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

   Replace x and y with what they are in terms of t and then finish by computing the integral above.

We can also use the arc length formula to find the surface area of some types of surfaces. Expanding this idea, we could use the formula developed in the next problem to compute the charge on a surface, the mass of a surface, and much more. For now, let's just compute the surface area.

Problem 3.26 A metal sheet lies above the parabola $\vec{r}(t) = (t^2, t)$ for $0 \le t \le 2$. Above the point (x, y), the height of the metal sheet just $h(x, y) = y$, the y-value. The picture below shows the sheet, sliced into 8 bits.



1. If we slice the surface into many tiny vertical strips with base length ds, explain why the surface area of each vertical strip is approximately $d\sigma = h(x, y)ds$.

2. The total surface area is the sum of the surface areas over all the vertical strips. This gives us the total surface area as

$$\sigma = \int_C d\sigma = \int_C h(x,y)ds = \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

   Replace x and y with what they are in terms of t and then finish by computing the integral above.

3.3 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 4

New Coordinates

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Use a change-of-coordinates to convert between rectangular and another coordinate system. In particular, be able to convert points and equations between rectangular and polar coordinates.

2. Graph polar functions $r = f(\theta)$ in the xy plane, and set up the arc length formula to find their length.

3. Given a change-of-coordinates, find the differentials dx and dy and write them in both vector and matrix form. Use these to compute tangent vectors, slope $\frac{dy}{dx}$, and equations of tangent lines.

4. Compute double integrals to find the area of regions in the xy plane, and use the determinant to explain how area between different coordinate systems is related.

5. Shade regions in the plane bounded by $\alpha \leq \theta \leq \beta$ and $r_1(\theta) \leq r \leq r_2(\theta)$, and use double integrals to compute their area.

You'll have a chance to teach your examples to your peers prior to the exam.

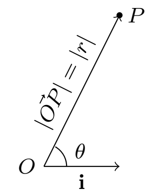
4.1 Polar Coordinates

Up to now, we most often give the location of a point (or coordinates of a vector) by stating the (x, y) coordinates. These are called the Cartesian (or rectangular) coordinates. Some problems are much easier to work with if we know how far a point is from the origin, together with the angle between the x-axis and a ray from the origin to the point.

Problem 4.1 There are two parts to this problem. See 11.3:5-10.

1. Consider the point P with Cartesian (rectangular) coordinates $(2, 1)$. Find the distance r from P to the origin. Consider the ray $\vec{OP}$ from the origin through P. Find an angle between $\vec{OP}$ and the x-axis.

2. Given a generic point $P = (x, y)$ in the plane, write a formula to find the distance r from P to the origin (in terms of x and y) as well as a formula to find the angle θ between the vector $(1, 0)$ (the positive x-axis) and the vector from the origin to P. [Hint: A picture of a triangle will help here.]



Definition 4.1. Let P be be a point in the plane with Cartesian coordinates (x, y). Let $O = (0, 0)$ be the origin. We say that (r, θ) is a polar coordinates of P if (1) we have $|\vec{OP}| = |r|$, and (2) the angle between $\mathbf{i} = (1, 0)$ and $\vec{OP}$ is θ, or coterminal with θ.

Problem 4.2 The following points are given using polar coordinates. Plot the points in the Cartesian plane, and give the Cartesian (rectangular) coordinates of each point. The points are

See 11.3:5-10.

$$(r, \theta) = (1, \pi), \left(6, \frac{\pi}{4}\right), \left(-3, \frac{\pi}{4}\right), \left(3, \frac{5\pi}{4}\right), \text{ and } \left(-2, -\frac{\pi}{6}\right).$$

Finish by explaining why a general formula for x and y if we know a point has polar coordinates (r, θ) is $x = r\cos\theta$ and $y = r\sin\theta$.

See page 647.

The equations above, namely

$$x = r\cos\theta, \quad y = r\sin\theta$$

are a typical example of what we call a change-of-coordinates. We've seen that these equations allow us transfer points back and forth between Cartesian coordinates and polar coordinates. We can also use this change-of-coordinates to transfer equations back and forth between coordinate system. The next two problems have you do this.

Problem 4.3 Each of the following equations is written in the Cartesian (rectangular) coordinate system. Convert each to an equation in polar coordinates, and then solve for r so that the equation is in the form $r = f(\theta)$. You'll want to use the change-of-coordinates to replace any x and y you see so that it is in terms of r and θ.

See 11.3: 53-66.

1. $x^2 + y^2 = 7$
2. $2x + 3y = 5$
3. $x^2 = y$

Problem 4.4 Each of the following equations is written using polar coordinates. Convert each to an equation in using Cartesian coordinates (sometimes called rectangular coordinates). You'll want to use the change-of-coordinates to replace any r and θ you see so that it is in terms of x and y.

See 11.3: 27-52. I strongly suggest that you do many of these as practice.

1. $r = 9\cos\theta$
2. $r = \dfrac{4}{2\cos\theta + 3\sin\theta}$
3. $\theta = 3\pi/4$

We've been writing the change-of-coordinates by listing the two equations $x = r\cos\theta$, $y = r\sin\theta$. We can also write this in vector notation as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}.$$

This is a vector equation in which you input polar coordinates (r, θ) and get out Cartesian coordinates (x, y). So you input one thing to get out one thing, which means that we have a function. We could also write $\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta)$, where we've used the letter T as the name for the function because the function is a $t$ransformation between coordinate systems.

To emphasize that the domain and range are both two dimensional systems, we could also write $\vec{T} : \mathbb{R}^2 \to \mathbb{R}^2$. In the next chapter, we'll spend more time with this notation. The following problem will show you one way to graph a change-of-coordinates, or coordinate transformation. When you're done, you should essentially have polar graph paper.

Problem 4.5 Consider the polar coordinate transformation

$$\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta).$$

For this problem, you are just drawing many parametric curves. This is what we did in the previous chapter.

1. Let $r = 3$ and then graph $\vec{T}(3,\theta) = (3\cos\theta, 3\sin\theta)$ for $\theta \in [0, 2\pi]$ in the xy plane. Remember, the notation $\theta \in [0, 2\pi]$ just means $0 \leq \theta \leq 2\pi$. If you get a circle, you're doing this right.
2. Let $\theta = \frac{\pi}{4}$ and then, on the same axes as above, add the graph of $\vec{T}\left(r, \frac{\pi}{4}\right) = \left(r\frac{\sqrt{2}}{2}, r\frac{\sqrt{2}}{2}\right)$ for $r \in [0, 5]$.
3. To the same axes as above, add the graphs of $\vec{T}(1,\theta), \vec{T}(2,\theta), \vec{T}(4,\theta)$ for $\theta \in [0, 2\pi]$ and $\vec{T}(r, 0), \vec{T}(r, \pi/2), \vec{T}(r, 3\pi/4), \vec{T}(r, \pi)$ for $r \in [0, 5]$.

If you ended up circles and rays, then you're doing this correctly. Congrats, you just drew a four dimensional graph (we'll talk more about this in class).

Make sure you ask me in class to show you the corresponding graph in the $r\theta$ plane, or come to class with it drawn and ready to share.

Problem 4.6 We have two equations $x = r\cos\theta$ and $y = r\sin\theta$. Suppose that a point is moving through space and x, y, r, θ all depend on time t.

1. Explain why $\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}$. Obtain a similar equation for $\frac{dy}{dt}$.

   Hint: Use implicit differentiation.
2. We can obtain the differential dx and dy in terms of r, θ, dr, and $d\theta$ if we multiply through by dt. This gives $dx = \cos\theta dr - r\sin\theta d\theta$ and $dy = ?$. Write you answer as the vector equation
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta \\ ? \end{pmatrix} dr + \begin{pmatrix} -r\sin\theta \\ ? \end{pmatrix} d\theta.$$
3. Find a 2 by 2 matrix so that we can write the above vector equation as the matrix equation
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}.$$

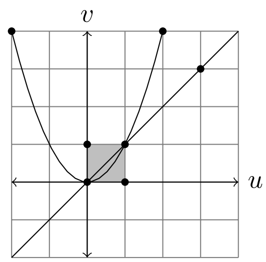
The vector equation above is the sum of vectors times scalars. Matrix multiplication was invented to abbreviate this type of sums. The vector are placed in the columns of the matrix, and the scalars are placed in a column vector to the right of the matrix.

Let's try the last two problems with a different change-of-coordinates, of the form $x = au + bv, y = cu + dv$. Any change of coordinates of this form we call a linear change-of-coordinates. You should see that lines map to lines in your work below.

Problem 4.7 Consider the change-of-coordinates $x = u-v$, $y = u+v$, which we could also write as the coordinate transformation $\vec{T}(u,v) = (u-v, u+v)$.

1. In the table below, you're given several (u,v) points. Find the corresponding (x,y) pair.

(u,v)	(x,y)
$(0,0)$	$(0,0)$
$(1,0)$	$(1-0,1+0)=(1,1)$
$(0,1)$	$(0-1,0+1)=(-1,1)$
$(1,1)$	
$(3,3)$	
$(2,4)$	
$(-2,4)$	



2. In the graph above is a plot of the points from the table, graphed in the uv plane. In addition, we see the parabola $v = u^2$, the line $v = u$, and the shaded box whose corners are the first few points. Constructed a plot (please make a grid) in the xy planes that contains the points from above. Connect the points in your xy plot to show how the parabola, line, and shaded box transform because of this change-of-coordinates.

Problem 4.8 Consider the change-of-coordinates from the problem above, namely $x = u - v$, $y = u + v$, or equivalently $\vec{T}(u,v) = (u-v, u+v)$.

1. If we assume x, y, u, v are all functions of t, we can compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Do so and then multiply your equations on both sides by dt to obtain the differentials dx and dy. Write your answer as the vector equation
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} du + \begin{pmatrix} ? \\ ? \end{pmatrix} dv.$$

2. Find a 2 by 2 matrix so that we can write the above vector equation as the matrix equation
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

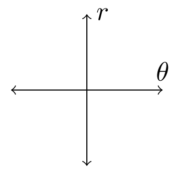
3. If we use the change-of-coordinates $x = 2u + 3v$, $y = 4u + 5v$, then find the differential dx and dy and write your answer as both
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} du + \begin{pmatrix} ? \\ ? \end{pmatrix} dv \quad \text{and} \quad \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

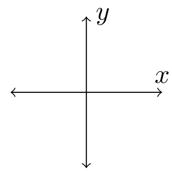
4.2 Graphing Transformed Equations

You've spent a lot of time in your past graphing equations of the form $y = f(x)$. Let's now graph equations of the form $r = f(\theta)$ in the xy plane.

Problem 4.9 In the θr plane, graph the curve $r = \sin\theta$ for $\theta \in [0, 2\pi]$ (make a table where you pick several values for θ and then compute r). Then graph the curve $r = \sin\theta$ for $\theta \in [0, 2\pi]$ in the xy plane (add to your table the corresponding x and y values). The graphs should look very different. If one looks like a sine wave, and the other looks like a circle, you're on the right track. Here's the start of a table to help you, as well as the axes you'll need to put your graphs on.

θ	r	$x = r\cos\theta$	$y = r\sin\theta$
0	$\sin(0) = 0$	0	0
$\frac{\pi}{6}$	$\sin\frac{\pi}{6} = \frac{1}{2}$	$\frac{1}{2}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{4}$	$\frac{1}{2}\sin\frac{\pi}{6} = \frac{1}{4}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}\cos\frac{\pi}{4} = \frac{1}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$			
$\frac{\pi}{2}$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$





In general, to construct a graph of a polar curve in the xy plane, we create an r, θ table. We choose values for θ that will make it easy to compute any trig functions involved. If you need to, add x and y to your table before plotting the location of the polar point in the xy plane. Then connect the points in a smooth manner, making sure that your radius grows or shrinks appropriately as your angle increases. Ask me in class to show you some animations of this, or you can see these animations before class if you open up the Mathematica Technology Introduction.

Problem 4.10 Graph the polar curve $r = 2 + 2\cos\theta$ in the xy plane.

See 11.4: 1-20.

Problem 4.11 Graph the polar curve $r = 2\sin 3\theta$ in the xy plane. [Hint: You'll want to chose values for theta so that 3θ hits all multiples of ninety degrees, the places where r attains it maximums and minimums.]

Problem 4.12 Graph the polar curve $r = 3\cos 2\theta$ in the xy plane.

4.3 Calculus with Change-of-Coordinates

Problem 4.13 We saw in some previous problems that we can express the differential dx and dy as the matrix product

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}.$$

1. Use the matrix equation above to compute $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of r and $\frac{dr}{d\theta}$, if we assume that r is a function of θ.

Hint: Just multiply everything out and divide by $d\theta$.

2. Explain why the slope of a tangent line in the xy plane to the curve $r = f(\theta)$ is
$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

For parametric curves $\vec{r}(t) = (x(t), y(t))$, to find the slope of the curve we just compute

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

A polar curve of the form $r = f(\theta)$ is just the parametric curve $(x, y) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. The previous problem showed us that we can find the slope by computing

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Problem 4.14 Consider the polar curve $r = 1 + 2\cos\theta$, graphed in the xy plane. (It wouldn't hurt to provide a quick sketch of the curve.) See 11.2: 1-14.

1. Compute both $dx/d\theta$ and $dy/d\theta$.
2. Find the slope dy/dx of the curve at $\theta = \pi/2$.
3. Give both a vector equation of the tangent line, and a Cartesian equation of the tangent line at $\theta = \pi/2$.

Problem 4.15 Consider the parabola $v = u^2$ and the change-of-coordinates $x = 2u + v$, $y = u - 2v$.

1. Construct a graph of the parabola in the xy plane.
2. Compute both dx/du and dy/du. Then find the slope dy/dx of the parabola at $u = 1$.
3. Give both a vector equation of the tangent line, and a Cartesian equation of the tangent line at $u = 1$.

We showed in the curves section that you can find the arc length for a parametric curve by using the formula

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

If we replace t with θ, this becomes a formula for the arc length of a curve given in polar coordinates. The next problem has you show that this formula can be greatly simplified.

Problem 4.16 Recall that $x = r\cos\theta$ and $y = r\sin\theta$. Suppose that r is a continuously differentiable function of θ. Let C be the path in the xy plane traces out for $\alpha \leq \theta \leq \beta$. We know that the length of C is See 11.5: 29.

$$s = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

Simplify the formula above to show that we can compute the arc length using

$$s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

[Hint: The product rule and several applications of the Pythagorean identity will help. Just plug everything in, expand (it gets ugly), and then simplify.]

Problem 4.17 Set up (do not evaluate) an integral formula to compute each of the following (draw the curve to be sure your bounds are correct - getting the right bounds is perhaps the toughest part of this problem.):

See 11.5: 21-28.

1. The length of one petal of the rose $r = 3\cos 2\theta$.
2. The length of the entire rose $r = 2\sin 3\theta$.

We've now seen one example of how we can use a change-of-coordinates to compute an integral, namely to find arc length. You've actually been using a change-of-coordinates since first semester calculus, every time you performed a substitution to complete an integral. The next problem has you revisit this, and notice something crucial about differentials.

Problem 4.18 Consider the integral $\displaystyle\int_{-1}^{2} e^{-3x}\,dx$.

1. To complete this integral we use the substitution $u = -3x$. Solve for x and compute the differential dx.
2. Now perform the substitution, filling in the missing parts of
$$\int_{x=-1}^{x=2} e^{-3x}\,dx = \int_{u=?}^{u=?} e^{u}?du.$$
To find the u bounds, just ask, "If $x = -1$, then $u = ?$" Don't spend any time completing the integral, rather just focus on completing the substitution above.

Note: When a definite integral ends with du, the bounds should be in terms of u. Many of you have always ignored this step, and instead would first compute $\int e^u?du$ without bounds, replace u with $-3x$, and then finish. We need the approach on the left in high dimensions.

3. The x values range from -1 to 2. This is an interval whose width is 3 units along the x-axis. Our substitution $u = -3x$ gives us an interval along the u-axis. How long is this interval, and what does your differential equation $dx = -\frac{1}{3}du$ have to do with this?
4. The substitution $u = -3x$ is a one-dimensional change-of-coordinates. We can write the differential $dx = -\frac{1}{3}du$ in the matrix form
$$\begin{pmatrix} dx \end{pmatrix} = \begin{bmatrix} -\frac{1}{3} \end{bmatrix} \begin{pmatrix} du \end{pmatrix}.$$
We have not defined the determinant of a 1 by 1 matrix. What would you define the determinant of a 1 by 1 matrix to be, and why?

We've now seen that the differential equation $dx = \frac{dx}{du}du$ tells us how to relate lengths along the u-axis to lengths along the x-axis. The next two problems have you focus on how a two dimensional change-of-coordinates helps us connect areas in the uv plane to areas in the xy plane.

Problem 4.19 Consider the change-of-coordinates $x = 2u$, $y = 3v$.

1. The box in the uv plane with $0 \le u \le 1$ and $1 \le v \le 2$ should correspond to a box in the xy plane. Draw and shade this box in the xy plane and find its area.

2. Compute the differentials dx and dy. State these differentials using both the vector and matrix forms

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} du + \begin{pmatrix} ? \\ ? \end{pmatrix} dv \quad \text{and} \quad \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

3. What's the determinant of the matrix above, and what does the determinant have to do with your picture?
4. Consider the box given by $-1 \leq u \leq 1$ and $-1 \leq v \leq 1$. State the area of this box in both the uv plane and the xy plane.
5. Consider the circle $u^2 + v^2 = 1$. The area inside this circle in the uv plane is $A = \pi$. Guess the area inside the corresponding ellipse in the xy plane.

Problem 4.20 Consider the change-of-coordinates $x = 2u + v$, $y = u - 2v$.

1. The lines $u = 0, u = 1, u = 2$ and $v = 0, v = 1, v = 2$ correspond to lines in the xy plane. Draw these lines in the xy-plane. [Hint: One option is to find the xy coordinates of the (u, v) points $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, etc., and then just connect the dots to make a grid.]
2. The box in the uv plane with $0 \leq u \leq 1$ and $1 \leq v \leq 2$ should correspond to a parallelogram in the xy plane. Shade this parallelogram in your picture above and find the area of the parallelogram.
3. Compute the differentials dx and dy. State these differentials using the vector form

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} du + \begin{pmatrix} ? \\ ? \end{pmatrix} dv$$

What do the two vectors above have to do with your picture?

4. Write the differentials above in the matrix form

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

What's the determinant of the matrix above, and what does the determinant have to do with your picture?

The next three problems have you analyze the integrals $\int_C dx$ and $\int_C dy$, and from them develop a way to compute area using double integrals.

Problem 4.21 Consider the polar curve C given by the equation $r = 3 + 2\sin\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$. Start by drawing the curve in the xy plane.

1. The integral $\int_C dx$ adds up little changes in x. Adding up lots of little changes in x should give us a total change in x. Verify this is true by computing

$$\begin{aligned} \int_C dx &= \int_0^{\frac{\pi}{2}} \frac{d}{d\theta}(r\cos\theta)\, d\theta = \int_0^{\frac{\pi}{2}} \frac{d}{d\theta}((3 + 2\sin\theta)\cos\theta)\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \left((2\cos\theta)\cos\theta - (3 + 2\sin\theta)\sin\theta\right)\, d\theta \end{aligned}$$

and compare your answer to the change in x. Feel free to use technology to compute the integral.

2. Explain what the integral $\int_C dy$ gives, and then verify your answer by computing the integral. You are welcome to use software.

Problem 4.22 Consider the region R between the functions $y = x^2$ and $y = -x$ for $0 \leq x \leq 2$. Draw both functions and shade the region R.

1. The integral $\displaystyle\int_0^2 dx$ adds up little changes in x. Adding up lots of little changes in x should give us a total change in x. Verify this is true by computing $\displaystyle\int_0^2 dx$ and compare your answer to the change in x.

2. For our given bounds, we know the curve $y = x^2$ starts at $(0, 0)$ and moves to $(2, 4)$. Explain why, without computation, we know the integral $\displaystyle\int_{x=0}^{x=2} dy$ equals 4 when we assume $y = x^2$. Then actually compute the integral to obtain this value.

   Since the bounds are in terms of x, you'll need to replace dy with what it equals in terms of x.

3. If we fix x to be a specific number, the integral $\displaystyle\int_{y=-x}^{y=x^2} dy$ adds up little changes in y for that specific x value. Compute this integral. What does this integral have to do with the region R?

   The bounds are now in terms of y, so don't replace dy to be in terms of x.

4. Explain why $\displaystyle\int_{x=0}^{x=2}\left(\int_{y=-x}^{y=x^2} dy\right) dx$ gives the area of the region R. Finish by computing this double integral.

Problem 4.23 Consider the double integral

$$\int_{y=-1}^{y=2}\left(\int_{x=y^2}^{x=y+2} dx\right) dy.$$

1. The bounds in the integral above describe a region in xy plane where $-1 \leq y \leq 2$ and $y^2 \leq x \leq y + 2$. Sketch this region.

2. Consider the inner integral $\int_{x=y^2}^{x=y+2} dx$. For a specific y value, what physical quantity does this integral compute. Hint: if you add up little changes in x (so dx's), what do you get?

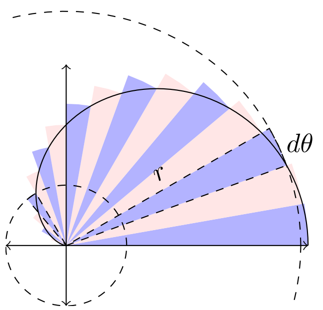
3. When we multiply $\int_{x=y^2}^{x=y+2} dx$ by a small height dy, explain why this gives us a little bit of area. Draw a small rectangle whose area is given by $dA = \left(\int_{x=y^2}^{x=y+2} dx\right) dy$.

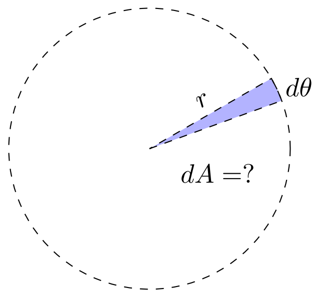
4. Adding up little bits of area gives total area, so the double integral at the start of this problem gives an area. Compute the integral.

Problem 4.24 The double integral $\displaystyle\int_{x=a}^{x=b}\left(\int_{y=g(x)}^{y=f(x)} dy\right) dx$ computes the area of a region in the xy plane that you should be quite familiar with. Compute the inner integral $\int_{y=g(x)}^{y=f(x)} dy$ to obtain the single variable formula you should be more familiar with. Provide a sketch of the region, using some specific functions to illustrate this abstract idea.

The previous problems showed us how to compute the area of a region using double integrals. The remaining problems in this chapter have you compute the area inside polar curves, which we will find first by using a single integral, and then by using a double integral.

Problem 4.25 In this problem, you will develop a formula for finding area inside a polar curve C. For illustration purposes, the curve C is drawn below in black for $0 \leq \theta \leq \pi$. We've sliced the total bounds for θ up into several tiny angles of measure $d\theta$. We can approximate the area inside the polar curve by instead finding the area inside several circular sectors, as illustrated below. See page 653.





Angle	Area enclosed
2π	πr^2
π	$\frac{1}{2}\pi r^2$
$\pi/2$	
$\pi/4$	
$\pi/10$	
$d\theta$	$dA =$

1. Consider the circular sector (the blue wedge on the right above) whose radius is r and angle is $d\theta$. Explain why the area of this circular sector is
$$dA = \frac{1}{2}r^2 d\theta.$$
[Hint: One way to do this is to compare the area enclosed based on the angle. For example, we know that if the angle is 2π, then the area enclosed is πr^2. You can fill out the table to the right and look for a pattern.]

2. Explain why the area bounded by the polar curve $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$ is
$$A = \int dA = \int_\alpha^\beta \frac{1}{2}r^2 d\theta.$$
What must be true about the curve $r = f(\theta)$ for this formula to be valid?

Problem 4.26 Use the integral formula above to compute the area inside of the polar curve $r = \sin\theta$. Show how you completed each step of the integral. [Hint: Construct a graph to determine the appropriate bounds for the integral. When you integrate, you'll need to use a half-angle identity - look it up on Wikipedia if you don't remember the identity.] See 11.5: 1-20.

For the rest of the semester, rather than use the half angle identity as you did above, you are welcome to just use the fact that
$$\int_0^{2\pi} \sin^2\theta d\theta = \int_0^{2\pi} \cos^2\theta d\theta = \pi.$$
Ask me in class for a nice visual way to remember this fact that has to do with average value.

Problem 4.27 Set up (do not evaluate) an integral to compute each of the following (make sure you draw the curve to be sure your bounds are correct):

1. The area inside the cardioid $r = 2 + 2\sin\theta$.
2. The area inside the circle $r = 4\cos\theta$.

Problem 4.28 Consider the change-of-coordinates $x = r\cos\theta$, $y = r\sin\theta$.

1. The lines $r = 1, r = 2, r = 3$ and $\theta = 0, \theta = \frac{\pi}{6}, \theta = \frac{\pi}{3}$ correspond to circles and lines in the xy plane. Draw these circles and lines in the xy-plane. The box in the $r\theta$ plane with $2 \leq r \leq 3$ and $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$ corresponds to a region in the xy plane. Shade this region in the xy plane.

2. Compute the differentials dx and dy. State these differentials using the vector form
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} dr + \begin{pmatrix} ? \\ ? \end{pmatrix} d\theta.$$
Let $r = 2$ and $\theta = \frac{\pi}{6}$ and then draw the two vectors above with their base at the xy coordinate whose polar point is $(2, \frac{\pi}{6})$.

3. Write the differentials above (at arbitrary r and θ) in the matrix form
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}.$$
Compute the determinant of the matrix above. Make a guess about what the determinant has to do with your picture.

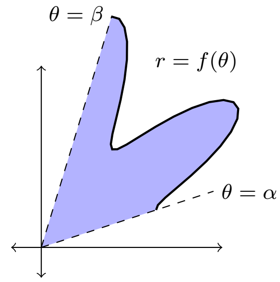
In this chapter we saw that for the change of coordinates $x = 2u + v$, $y = u - 2v$, we can write the differentials dx and dy in the matrix form
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$
We also saw that the determinant of the matrix above, namely 5, gives us a scale factor that connects areas in the xy plane to areas in the uv plane. If we find the area of a region in the uv plane, we can times that area by 5 to obtain the area in the xy plane. We can write this relationship between areas as $dxdy = 5dudv$.

We've also seen in this chapter that we can use double integrals to compute the area of regions in the xy plane. For example, if we want the area of a region R in the xy plane that is bounded by $a \leq x \leq b$ and $g(x) \leq y \leq f(x)$, then we just compute the double integral
$$A = \int\int_R dA = \int\int_R dydx = \int_a^b \int_{g(x)}^{f(x)} dydx.$$
Computing the inner integral gives us
$$A = \int_a^b \int_{g(x)}^{f(x)} dydx = \int_a^b \left(y\Big|_{g(x)}^{f(x)} \right) dx = \int_a^b (f(x) - g(x))\, dx,$$
a familiar formula we used in first semester calculus.

The next problem has you combine these two results, namely determinants of matrices and double integrals, to develop a formula for the area bounded by polar curves.

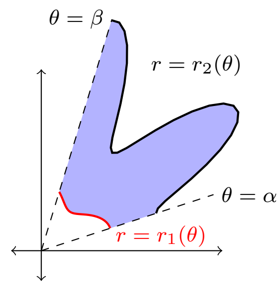
Problem 4.29 Consider the region R in the xy plane bounded by and $\alpha \le \theta \le \beta$ and $0 \le r \le f(\theta)$.

Here's a typical curve with $\alpha \le \theta \le \beta$ and $0 \le r \le f(\theta)$.



1. Explain why $dydx = rdrd\theta$. (Read the previous paragraphs and see the last part of the previous problem.)
2. We've seen that the area of a region in the xy plane can be found using the double integral $\int\int_R dydx$. If we swap to polar coordinates, then this gives us $\int\int_R rdrd\theta$. Using the bounds for the region, which are given above, we have
$$A = \int_\alpha^\beta \int_0^{f(\theta)} rdrd\theta.$$
Compute the inside integral to reduce this to a single integral.
3. Let's now find the area of the region R bounded by $\alpha \le \theta \le \beta$ and $r_1(\theta) \le r \le r_2(\theta)$. Set up a double integral that would give the area of this region R, and then compute the inner integral to obtain a single integral formula $A = \int_\alpha^\beta \frac{r_2^2}{2} - \frac{r_1^2}{2} d\theta$. Can you think of a way to obtain this without using double integrals?

Here's a typical curve with $\alpha \le \theta \le \beta$ and $r_1(\theta) \le r \le r_2(\theta)$.



In this chapter, we've introduced several different coordinate systems that people have used over the centuries. Sometimes a problem can't be solved until the correct coordinate system is chosen. Problem 4.5 showed you how to graph the coordinate transformation given by polar coordinates. The following problem has you graph a different coordinate system.

Problem 4.30 Consider the coordinate transformation
$$T(a, \omega) = (a \cos \omega, a^2 \sin \omega)$$
which is the same as writing $x = a \cos \omega$, $y = a^2 \sin \omega$.

1. Let $a = 3$ and graph the curve $\vec{T}(3, \omega) = (3 \cos \omega, 9 \sin \omega)$ for $\omega \in [0, 2\pi]$ in the xy plane. Another way to say this is to just graph the curve $a = 3$ (which is a line in the aw coordinate system) in the xy plane.

   See Sage. Click on the link to see how to check your answer in Sage.
2. Let $\theta = \frac{\pi}{4}$ and then, on the same axes as above, add the graph of $\vec{T}\left(a, \frac{\pi}{4}\right) = \left(a\frac{\sqrt{2}}{2}, a^2\frac{\sqrt{2}}{2}\right)$ for $a \in [0, 4]$.

   See Sage. Notice that you can add the two plots together to superimpose them on each other.
3. To the same axes as above, add the graphs of $\vec{T}(1, \omega), \vec{T}(2, \omega), \vec{T}(4, \omega)$ for $\omega \in [0, 2\pi]$ and $\vec{T}(a, 0), \vec{T}(a, \pi/2), \vec{T}(a, -\pi/6)$ for $a \in [0, 4]$.

   Use Sage to check your answer.

[Hint: when you're done, you should have a bunch of parabolas and ellipses.]

Problem 4.31 Use the change-of-coordinates from the previous problem, namely $x = a \cos \omega$, $y = a^2 \sin \omega$.

1. Compute the differentials dx and dy and write them in the matrix form
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} da \\ d\omega \end{pmatrix}.$$
2. Show that $dxdy = (a^2 \cos^2 \theta + 2a^2 \sin^2 \theta) dad\omega$.
3. Draw and shade the region in the xy plane bounded by $2 \le a \le 3$ and $0 \le \omega \le 2\pi$. You should have a region between two ellipses.
4. Use a double integral to compute the area of the region above.

You'll need the fact that $\int_0^{2\pi} \cos^2 \omega d\omega = \int_0^{2\pi} \sin^2 \omega d\omega = \pi$, something we'll see repeatedly.

4.4 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 5

Functions

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Describe uses for, and construct graphs of, space curves and parametric surfaces. Find derivatives of space curves, and use this to find velocity, acceleration, and find equations of tangent lines.
2. Describe uses for, and construct graphs of, functions of several variables. For functions of the form $z = f(x, y)$, this includes both 3D surface plots and 2D level curve plots. For functions of the form $w = f(x, y, z)$, construct plots of level surfaces.
3. Describe uses for, and construct graphs of, vector fields and transformations. Develop the formulas for cylindrical and spherical coordinates.
4. If you are given a description of a vector field, curve, or surface (instead of a function or parametrization), explain how to obtain a function for the vector field, or a parametrization for the curve or surface.

You'll have a chance to teach your examples to your peers prior to the exam.

Most of the work in this chapter requires graphing. You'll find many links throughout this chapter that point to SageMath and/or WolframAlpha plots (see sagemath.org for more information about SageMath). Alternatively, you can use this Mathematica Technology introduction to have technology create plots for you. Please use technology to check how you are doing.

5.1 Function Terminology

A function is a set of instructions involving two sets (called the domain and codomain). A function assigns to each element of the domain D exactly one element in the codomain R. We'll often refer to the codomain R as the target space. We'll write

$$f : D \to R$$

when we want to remind ourselves of the domain and target space. In this class, we will study what happens when the domain and target space are subsets of $\mathbb{R}^n$ (Euclidean n-space). In particular, we will study functions of the form

$$f : \mathbb{R}^n \to \mathbb{R}^m,$$

when m and n are 3 or less. The value of n is the dimension of the input vector (or number of inputs). The number m is the dimension of the output vector (or number of outputs). Our goal is to understand uses for each type of function, and be able to construct graphs to represent the function.

We will focus most of our time this semester on two- and three-dimensional problems. However, many problems in the real world require a higher number of dimensions. When you hear the word "dimension", it does not always represent a physical dimension, such as length, width, or height. If a quantity depends on 30 different measurements, then the problem involves 30 dimensions. As a quick illustration, the formula for the distance between two points depends on 6 numbers, so distance is really a 6-dimensional problem. As another example, if a piece of equipment has a color, temperature, age, and cost, we can think of that piece of equipment being represented by a point in four-dimensional space (where the coordinate axes represent color, temperature, age, and cost).

Problem 5.1 A pebble falls from a 64 ft tall building. Its height (in ft) above the ground t seconds after it drops is given by the function $y = f(t) = 64 - 16t^2$. What are n and m when we write this function in the form $f\colon \mathbb{R}^n \to \mathbb{R}^m$? Construct a graph of this function. How many dimensions do you need to graph this function?

See Sage or Wolfram Alpha. Follow the links to Sage or Wolfram Alpha in all the problems below to see how to get the computer to graph the function.

5.2 Parametric Curves: $\vec{f}\colon \mathbb{R} \to \mathbb{R}^m$

Problem 5.2 A horse runs around an elliptical track. Its position at time t is given by the function $\vec{r}(t) = (2\cos t, 3\sin t)$. We could alternatively write this as $x = 2\cos t, y = 3\sin t$.

See Sage or Wolfram Alpha. See also Chapter 3 of this problem set. There's a lot more practice of this idea in 11.1. You'll also find more practice in 13.1: 1-8.

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. Construct a graph of this function.
3. Next to a few points on your graph, include the time t at which the horse is at this point on the graph. Include an arrow for the horse's direction.
4. How many dimensions do you need to graph this function?

Notice in the problem above that we placed a vector symbol above the function name, as in $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$. When the target space (codomain) is 2-dimensional or larger, we place a vector above the function name to remind us that the output is more than just a number.

Problem 5.3 Consider the pebble from problem 5.1. The pebble's height was given by $y = 64 - 16t^2$. The pebble also has some horizontal velocity (it's moving at 3 ft/s to the right). If we let the origin be the base of the 64 ft building, then the position of the pebble at time t is given by $\vec{r}(t) = (3t, 64 - 16t^2)$.

See Sage or Wolfram Alpha. The text has more practice in 13.1: 1-8.

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. At what time does the pebble hit the ground (the height reaches zero)? Construct a graph of the pebble's path from when it leaves the top of the building till when it hits the ground.
3. Find the pebble's velocity and acceleration vectors at $t = 1$? Draw these vectors on your graph with their base at the pebble's position at $t = 1$.

See Section 3.2.1 and Definition 3.5.

4. At what speed is the pebble moving when it hits the ground?

In the next problem, we keep the input as just a single number t, but the output is now a vector in $\mathbb{R}^3$.

Problem 5.4 A jet begins spiraling upwards to gain height. The position of the jet after t seconds is modeled by the equation $\vec{r}(t) = (2\cos t, 2\sin t, t)$. We could alternatively write this as $x = 2\cos t$, $y = 2\sin t$, $z = t$.

See Sage or Wolfram Alpha. The text has more practice in 13.1: 9-14.

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. Construct a graph of this function by picking several values of t and plotting the resulting points $(2\cos t, 2\sin t, t)$.
3. Next to a few points on your graph, include the time t at which the jet is at this point on the graph. Include an arrow for the jet's direction.
4. How many dimensions do you need to graph this function?

In all the problems above, you should have noticed that in order to draw a function (provided you include arrows for direction, or use an animation to represent "time"), you can determine how many dimensions you need to graph a function by just summing the dimensions of the domain and codomain. This is true in general.

Problem 5.5 Use the same set up as problem 5.4, namely

$$\vec{r}(t) = (2\cos t, 2\sin t, t).$$

See Section 3.2.1 and Definition 3.5.
The text has more practice in 13.1: 19-22.

You'll need a graph of this function to complete this problem.

1. Find the first and second derivative of $\vec{r}(t)$.
2. Compute the velocity and acceleration vectors at $t = \pi/2$. Place these vectors on your graph with their tails at the point corresponding to $t = \pi/2$.
3. Give an equation of the tangent line to this curve at $t = \pi/2$.

5.3 Parametric Surfaces: $\vec{f}\colon \mathbb{R}^2 \to \mathbb{R}^3$

We now increase the number of inputs from 1 to 2. This will allow us to graph many space curves at the same time.

Problem 5.6 The jet from problem 5.4 is actually accompanied by several jets flying side by side. As all the jets fly, they leave a smoke trail behind them (it's an air show). The smoke from each jet spreads outwards to mix together, so that it looks like the jets are leaving wide sheet of smoke behind them as they spiral upwards. The position of two of the many other jets is given by $\vec{r}_3(t) = (3\cos t, 3\sin t, t)$ and $\vec{r}_4(t) = (4\cos t, 4\sin t, t)$. A function which represents the smoke stream from these jets is $\vec{r}(a,t) = (a\cos t, a\sin t, t)$ for $0 \leq t \leq 4\pi$ and $2 \leq a \leq 4$.

More practice in 16.5: 1-16.

1. What are n and m when we write the function $\vec{r}(a,t) = (a\cos t, a\sin t, t)$ in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?

2. Start by graphing the position of the three jets $\vec{r}(2,t) = (2\cos t, 2\sin t, t)$, $\vec{r}(3,t) = (3\cos t, 3\sin t, t)$ and $\vec{r}(4,t) = (4\cos t, 4\sin t, t)$.

3. Let $t = 0$ and graph the curve $r(a, 0) = (a, 0, 0)$ for $a \in [2, 4]$, which represents the segment along which the smoke has spread. Then repeat this for $t = \pi/2, \pi, 3\pi/2$.

4. Describe the resulting surface, and make sure you check your answer with technology (use the links to the side). See Sage or Wolfram Alpha.

We call the surface you drew above a parametric surface. The vector equation describing the smoke screen is a parametrization of this surface.

Definition 5.1: Parametric Surface, Parametrization of a surface. A parametrization of a surface is a collection of three equations to tell us the position

$$x = x(u, v), y = y(u, v), z = z(u, v)$$

of a point (x, y, z) on the surface. We call u and v parameters, and these parameters give us a two dimensional pair (u, v), the input, needed to obtain a specific location (x, y, z), the output, on the surface. We can also write a parametrization in vector form as

$$\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

We'll often give bounds on the parameters u and v, which help us describe specific portions of the surface. A parametric surface is a surface together with a parametrization.

We draw parametric surfaces by joining together many parametric space curves, as done in the previous problem. Just pick one variable, hold it constant, and draw the resulting space curve. Repeat this several times, and you'll have a 3D surface plot. Most of 3D computer animation is done using parametric surfaces. Woody's entire body in *Toy Story* is a collection of parametric surfaces. Car companies create computer models of vehicles using parametric surfaces, and then use those parametric surfaces to study collisions. Often the mathematics behind these models is hidden in the software program, but parametric surfaces are at the heart of just about every 3D computer model.

Problem 5.7 Consider the parametric surface $\vec{r}(u, v) = (u\cos v, u\sin v, u^2)$ for $0 \leq u \leq 3$ and $0 \leq v \leq 2\pi$. Construct a graph of this function. Remember, to do so we just let u equal a constant (such as 1, 2, 3) and then graph the resulting space curve where we let v vary. After doing this for several values of u, swap and let v equal a constant (such as 0, $\pi/2$, etc.) and graph the resulting space curve as u varies. [Hint: Did you get a satellite dish? Use the software links to the right to make sure you did this right.] See Sage or Wolfram Alpha.

5.4 Functions of Several Variables: $f\colon \mathbb{R}^n \to \mathbb{R}$

In this section we'll focus on functions where the output is a single number. These functions take the form $f\colon \mathbb{R}^2 \to \mathbb{R}^1$ and $f\colon \mathbb{R}^3 \to \mathbb{R}^1$. In the next problem, you should notice that the input is a vector (x, y) and the output is a number z. There are two common ways we graph functions of this type. The next two problems show you how.

Problem 5.8 A computer chip has been disconnected from electricity and sitting in cold storage for quite some time. The chip is connected to power, and a few moments later the temperature (in Celsius) at various points (x, y) on the chip is measured. From these measurements, statistics is used to create a temperature function $z = f(x, y)$ to model the temperature at any point on the chip. Suppose that this chip's temperature function is given by the equation $z = f(x, y) = 9 - x^2 - y^2$. We'll be creating a 3D model of this function in this problem, so you'll want to place all your graphs on the same x, y, z axes.

See Sage or Wolfram Alpha.

1. What is the temperature at $(0, 0)$, $(1, 2)$, and $(-4, 3)$?

See 14.1: 1-4.

2. If you let $y = 0$, construct a graph of the temperature $z = f(x, 0) = 9 - x^2 - 0^2$, or just $z = 9 - x^2$. In the xz plane (where $y = 0$) draw this upside down parabola.
3. Now let $x = 0$. Draw the resulting parabola in the yz plane.
4. Now let $z = 0$. Draw the resulting curve in the xy plane.
5. Once you've drawn a curve in each of the three coordinate planes, it's useful to pick an input variable (either x or y) and let it equal various constants. So now let $x = 1$ and draw the resulting parabola in the plane $x = 1$. Then repeat this for $x = 2$.
6. Describe the shape. Add any extra features to your graph to convey the 3D image you are constructing.

See 14.1: 37-48.

Problem 5.9 We'll be using the same function $z = f(x, y) = 9 - x^2 - y^2$ as the previous problem. However, this time we'll construct a graph of the function by only studying places where the temperature is constant. We'll create a graph in 2D of the surface (similar to a topographical map).

See Sage or Wolfram Alpha.

1. Which points in the plane have zero temperature? Just let $z = 0$ in $z = 9 - x^2 - y^2$. Plot the curve corresponding to these points in the xy-plane with the same temperature, and write $z = 0$ next to this curve. We call this curve a level curve. As long as you stay on this curve, your temperature will remain level; it will not increase nor decrease.

See 14.1: 13-16 and 31-36, 37-48.

Because the function here represents temperature, we can also call this curve an isotherm. If the function represented pressure, we'd call it an isobar. There are many names given to level curves. We'll use the words "level curve" throughout the semester rather than isotherm, isobar, isocline, etc.

2. Which points in the plane have temperature $z = 5$? Add this level curve to your 2D plot and write $z = 5$ next to it.
3. Repeat the above for $z = 8$, $z = 9$, and $z = 1$. What's wrong with letting $z = 10$?
4. Using your 2D plot, construct a 3D image of the function by lifting each level curve to its corresponding height.

Definition 5.2. A level curve of a function $z = f(x, y)$ is a curve in the xy-plane found by setting the output z equal to a constant. Symbolically, a level curve of $f(x, y)$ is the curve $c = f(x, y)$ for some constant c. A 2D plot consisting of several level curves is called a contour plot of $z = f(x, y)$.

Problem 5.10 Consider the function $f(x, y) = x - y^2$.

See Sage or Wolfram Alpha.More practice is in 14.1: 37-48.

1. Construct a 3D surface plot of f. [So just graph in 3D the curves given by $x = 0$ and $y = 0$ and then try setting x or y equal to some other constants, like $x = 1$, $x = 2$, $y = 1$, $y = 2$, etc.]

2. Construct a contour plot of f. [So just graph in 2D the curves given by setting z equal to a few constants, like $z = 0$, $z = 1$, $z = -4$, etc.]

3. Which level curve passes through the point $(2, 2)$? Draw this level curve on your contour plot.

See 14.1: 49-52.

Notice that when we graphed the previous two functions (of the form $z = f(x, y)$) we could either construct a 3D surface plot, or we could reduce the dimension by 1 and construct a 2D contour plot by letting the output z equal various constants. The next function is of the form $w = f(x, y, z)$, so it has 3 inputs and 1 output. We could write $f: \mathbb{R}^3 \to \mathbb{R}^1$. We would need 4 dimensions to graph this function, but graphing in 4D is not an easy task. Instead, we'll reduce the dimension and create plots in 3D to describe the level surfaces of the function.

Problem 5.11 Suppose that an explosion occurs at the origin $(0, 0, 0)$. Heat from the explosion starts to radiate outwards. Suppose that a few moments after the explosion, the temperature at any point in space is given by $w = T(x, y, z) = 100 - x^2 - y^2 - z^2$.

See Sage. Wolfram Alpha currently does not support drawing level surfaces. You could also use Mathematica or Wolfram Demonstrations.
You can access more problems on drawing level surfaces in 12.6:1-44 or 14.1:53-60.

1. Which points in space have a temperature of 99? To answer this, replace $T(x, y, z)$ by 99 to get $99 = 100 - x^2 - y^2 - z^2$. Use algebra to simplify this to $x^2 + y^2 + z^2 = 1$. Draw this object.

2. Which points in space have a temperature of 96? of 84? Draw the surfaces.

3. What is your temperature at $(3, 0, -4)$? Draw the level surface that passes through $(3, 0, -4)$.

4. The 4 surfaces you drew above are called level surfaces. If you walk along a level surface, what happens to your temperature?

5. As you move outwards, away from the origin, what happens to your temperature?

Problem 5.12 Consider the function $w = f(x, y, z) = x^2 + z^2$. This function has an input y, but notice that changing the input y does not change the output of the function.

See Sage.

1. Draw a graph of the level surface $w = 4$. [When $y = 0$ you can draw one curve. When $y = 1$, you should draw the same curve. When $y = 2$, again you draw the same curve. This kind of graph is called a cylinder, and is important in manufacturing where you extrude an object through a hole.]

2. Graph the surface $9 = x^2 + z^2$ (so the level surface $w = 9$).

3. Graph the surface $16 = x^2 + z^2$.

The examples I give you for functions of the form $w = f(x, y, z)$ we can draw by using our knowledge about conic sections. We can graph ellipses and hyperbolas if there are only two variables. So the key idea is to set one of the variables equal to a constant and then graph the resulting curve. Repeat this with a few variables and a few constants, and you'll know what the surface is. Sometimes when you set a specific variable equal to a constant, you'll get an ellipse. If this occurs, try setting that variable equal to other constants, as ellipses are generally the easiest curves to draw. If setting a variable equal to a contant gives you a hyperbola, try picking a different variable to set equal to a constant. It gets really messy to graph several hyperbolas on the same 3D axes by hand.

Problem 5.13 Consider the function $w = f(x, y, z) = x^2 - y^2 + z^2$.

See Sage. Remember you can find more practice in 12.6:1-44 or 14.1: 53-64.
We'll have a few people present this problem.

1. Draw a graph of the level surface $w = 1$. [You need to graph $1 = x^2 - y^2 + z^2$. Let $x = 0$ and draw the resulting curve. Then let $y = 0$ and draw the resulting curve. Let either x or y equal some more constants (whichever gave you an ellipse), and then draw the resulting ellipses.]
2. Graph the level surface $w = 4$. [Divide both sides by 4 (to get a 1 on the left) and then repeat the previous part.]
3. Graph the level surface $w = -1$. [Try dividing both sides by a number to get a 1 on the left. If $y = 0$ doesn't help, try $y = 1$ or $y = 2$.]
4. Graph the level surface that passes through the point $(3, 5, 4)$. [Hint: what is $f(3, 5, 4)$?]

5.4.1 Vector Fields and Transformations: $\vec{f}\colon \mathbb{R}^n \to \mathbb{R}^n$

We've covered the following types of functions in the problems above.

- $y = f(x)$ or $f\colon \mathbb{R} \to \mathbb{R}$ (functions of a single variable)
- $\vec{r}(t) = (x, y)$ or $f\colon \mathbb{R} \to \mathbb{R}^2$ (parametric curves)
- $\vec{r}(t) = (x, y, z)$ or $f\colon \mathbb{R} \to \mathbb{R}^3$ (space curves)
- $\vec{r}(u, v) = (x, y, z)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ (parametric surfaces)
- $z = f(x, y)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}$ (functions of two variables)
- $z = f(x, y, z)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}$ (functions of three variables)

In this class, we will ignore functions of the form $\vec{f}\colon \mathbb{R}^3 \to \mathbb{R}^2$, though one way to view these is to just create functions of the form $f_i\colon \mathbb{R}^3 \to \mathbb{R}$.

The only examples that remain are functions where the dimension of the input matches the dimension of the output. In our previous chapters we've look at two examples of this form, namely vector fields and coordinate transformations (change-of-coordinates). Let's finish this section by revisiting these two types of functions, namely

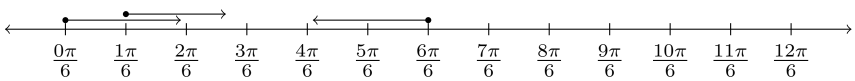
- $\vec{F}(x) = (M) = M\mathbf{i}$ or $f\colon \mathbb{R} \to \mathbb{R}$ (vector fields along a line)
- $\vec{F}(x, y) = (M, N)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (vector fields in the plane)
- $\vec{F}(x, y, z) = (M, N, P)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ (vector fields in space)
- $T(u) = x$ or $f\colon \mathbb{R} \to \mathbb{R}$ (1D change-of-coordinates)
- $\vec{T}(u, v) = (x, y)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (2D change-of-coordinates)
- $\vec{T}(u, v, w) = (x, y, z)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ (3D change-of-coordinates)

The difference between vector fields and transformations has to do with how we apply the function. Let's examine this difference first by considering one dimension.

Problem 5.14 Consider the function $f(x) = \cos x$. We can view this function as the change-of-coordinates $T(x) = \cos x$, or the vector field $F(x) = \cos x\mathbf{i}$.

1. Compute the definite integral $\int_0^{\pi/2} (\sin x)e^{\cos x}\, dx$. Any time you use substitution to compute integral, you're using a change-of-coordinates.

2. Let's now make a vector field plot of $F(x) = \cos x\mathbf{i}$. Let x be a multiple of $\frac{\pi}{6}$ on the number line below. For each such x, draw the vector $\vec{F}(x)$ so that the tail starts at x. I've already added the vectors for $x = 0$, $\frac{\pi}{6}$, and π, I'll let you add the other 10. You'll want to use a calculator to get an appropriate scale.



   You might want to offset these horizontal arrows a bit so they don't overlap each other. This graph is a one-dimensional vector field.

3. You've used one-dimensional vector fields quite a bit in your past without knowing it. First, find a function $g(x)$ so that $f(x)$ is the derivative of $g(x)$. Construct a plot of $g(x)$ in the xy-plane, and then add to this 2D graph the 1D vectors from your vector field plot. What connections can you make between your one-dimensional vector field plot and your two-dimensional graph? How does the length of the vector correspond to the shape of the curve? What does the direction of the arrow tell you about the 2D graph?

The previous chapter focused quite a bit on how to work with a two-dimensional change-of-coordinates. In particular, we've already seen examples of coordinate transformations with polar coordinates. In three dimensions, some common coordinate systems are cylindrical and spherical coordinates. The equations for these coordinate systems are shown in the table below.

Cylindrical Coordinates	Spherical Coordinates
$x = r\cos\theta$ $y = r\sin\theta$ $z = z$	$x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$

The next two problems have you develop these equations, similar to the first few problems in the previous chapter.

Problem 5.15 Let $P = (x, y, z)$ be a point in space. This point lies on a cylinder of radius r, where the cylinder has the z axis as its axis of symmetry. The height of the point is z units up from the xy plane. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the ray $\vec{OQ}$ and the x-axis is θ. See Figure 5.1 for a picture. Use the graph and the information above to explain why the equations for cylindrical coordinates are See page 893.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Problem 5.16 Let $P = (x, y, z)$ be a point in space. This point lies on a sphere of radius ρ ("rho"), where the sphere's center is at the origin $O = (0, 0, 0)$. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the See page 897.

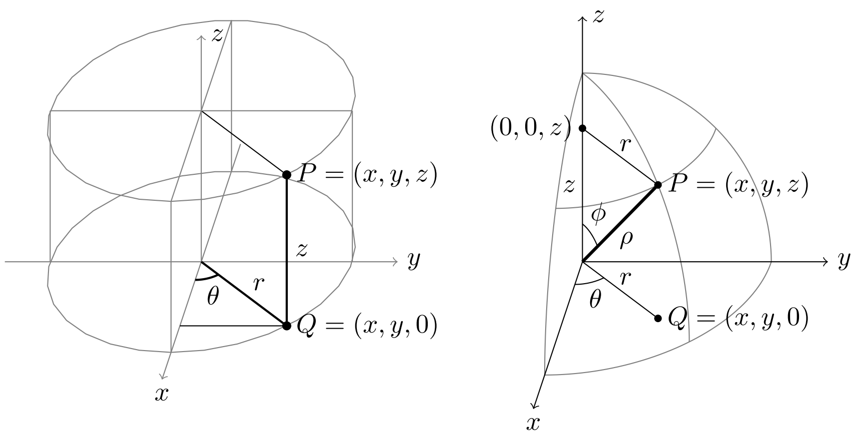


Figure 5.1: Cylindrical and spherical coordinates.

ray $\vec{OQ}$ and the x-axis is θ, and we call the azimuth angle. The angle between the ray $\vec{OP}$ and the z axis is ϕ ("phi"), and we call call the inclination angle, polar angle, or zenith angle. See Figure 5.1 for a picture. Use this information to develop the equations for spherical coordinates, in other words explain why

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

There is some disagreement between different scientific fields about the notation for spherical coordinates. In some fields (like physics), ϕ represents the azimuth angle and θ represents the inclination angle, swapped from what we see here. In some fields, like geography, instead of the inclination angle, the *elevation* angle is given — the angle from the xy-plane (lines of latitude are from the elevation angle). Additionally, sometimes the coordinates are written in a different order. You should always check the notation for spherical coordinates before communicating to others with them. As long as you have an agreed upon convention, it doesn't really matter how you denote them.

See the Wikipedia or MathWorld for a discussion of conventions in different disciplines.

Problem 5.17 Consider the spherical coordinates transformation

$$\vec{T}(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi).$$

Graphing this transformation requires $3+3 = 6$ dimensions. In this problem we'll construct parts of this graph by graphing various surfaces. We did something similar for the polar coordinate transformation in problem 4.5.

1. Let $\rho = 2$ and graph the resulting surface. What do you get if $\rho = 3$? See Sage or Wolfram Alpha.
2. Let $\phi = \pi/4$ and graph the resulting surface. What do you get if $\phi = \pi/2$? See Sage or Wolfram Alpha.
3. Let $\theta = \pi/4$ and graph the resulting surface. What do you get if $\theta = \pi/2$?

Let's now turn our focus to vector fields.

Problem 5.18 Consider the vector field $\vec{F}(x, y) = (2x + y, x + 2y)$. In this problem, you will construct a graph of this vector field by hand. We did something quite similar in Problem 2.10 on page 12.

See Sage or Wolfram Alpha. The computer will shrink the largest vector down in size so it does not overlap any of the others, and then reduce the size of all the vectors accordingly. See 16.2: 39-44 for more practice.

1. Compute $\vec{F}(1, 0)$. Then draw the vector $\vec{F}(1, 0)$ with its base at $(1, 0)$.

2. Compute $\vec{F}(1,1)$. Then draw the vector $\vec{F}(1,1)$ with its base at $(1,1)$.
3. Repeat the above process for the points $(0,1)$, $(-1,1)$, $(-1,0)$, $(-1,-1)$, $(0,-1)$, and $(1,-1)$. Remember, at each point draw a vector. When you finish, check your answer with software.

Problem 5.19: Spin field Consider the vector field $\vec{F}(x,y) = (-y,x)$. Construct a graph of this vector field. Remember, the key to plotting a vector field is "at the point (x,y), draw the vector $\vec{F}(x,y)$ with its base at (x,y)." Plot at least 8 vectors (a few in each quadrant), so we can see what this field is doing.

Use the links above to see the computer plot this. See 16.2: 39-44 for more practice.

Drawing 3D vector fields by hand can be tough, luckily Sage and Mathematica can help us visualize 3D vector fields. The sage example show a 3D visualization of the vector field $\vec{F}(x,y,z) = (y,z,x)$.

5.5 Constructing Functions

We now know how to draw a vector field provided someone tells us the equation. What we really need is to do the reverse. If we see vectors (forces, velocities, etc.) acting on something, how do we obtain an equation of the vector field? The spin field from the previous problem is directly related to the field you would need to understand the forces at play on a merry-go-round or carousel. The following problem will help you develop the gravitational vector field.

Problem 5.20: Radial fields Do the following:

Use Sage to plot your vector fields. See 16.2: 39-44 for more practice.

1. Let $P = (x,y,z)$ be a point in space. At the point P, let $\vec{F}(x,y,z)$ be the vector which points from P to the origin. Give a formula for $\vec{F}(x,y,z)$.
2. Give an equation of the vector field where at each point P in the plane, the vector $\vec{F}_2(P)$ is a unit vector that points towards the origin.
3. Give an equation of the vector field where at each point P in the plane, the vector $\vec{F}_3(P)$ is a vector of length 7 that points towards the origin.
4. Give an equation of the vector field where at each point P in the plane, the vector $\vec{G}(P)$ points towards the origin, and has a magnitude equal to $1/d^2$ where d is the distance to the origin.

If someone gives us parametric equations for a curve in the plane, we know how to draw the curve. Again, what we really need is the ability to go backwards. How do we obtain parametric equations of a curve that we can see? In problem 5.2, we were given the parametric equation for the path of a horse, namely $x = 2\cos t, y = 3\sin t$ or $\vec{r}(t) = (2\cos t, 3\sin t)$. From those equations, we drew the path of the horse, and could have written a Cartesian equation for the path. How do we work this in reverse, namely if we had only been given the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, could we have obtained parametric equations $\vec{r}(t) = (x(t), y(t))$ for the curve?

Problem 5.21 Give a parametrization of the top half (so $y \geq 0$) of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. You can write your parametrization in the vector form $\vec{r}(t) = (?, ?)$, or in the parametric form $x = ?,\ y = ?$, and you'll need to give bounds for t of the form $? \leq t \leq ?$ so that we only obtain the top half. [Hint: Read the paragraph above, and/or review Problem 5.2.]

Use Sage or Wolfram Alpha to visualize your parameterizations.

Problem 5.22 Give a parametrization of the straight line from $(a,0)$ to $(0,b)$. You can write your parametrization in the vector form $\vec{r}(t) = (?,?)$, or in the parametric form $x =?,\ y =?$. Remember to include bounds for t. [Hint: Review 2.9 and 3.14.]

We often use t as the parameter when writing equations for planar and space curves, because we'll often use the curve to describe the motion of an object as time elapses. You are welcome to use whatever variable you want for your parameter, such as x, y, z, θ, r, etc.

Problem 5.23 Give a parametrization ($\vec{r}(?) = (?,?)$) of the parabola $y = x^2$ from $(-1,1)$ to $(2,4)$. Remember the bounds for your parameter.

Problem 5.24 Give a parametrization of the function $y = f(x)$ for $x \in [a,b]$. You can write your parametrization in the vector form $\vec{r}(?) = (?,?)$, or in the parametric form $x =?,\ y =?$. Include bounds for your parameter.

If someone gives us parametric equations for a surface, we can draw the surface. This is what we did in problems 5.6 and 5.7. How do we work backwards and obtain parametric equations for a given surface? This requires that we write an equation for x, y, and z in terms of two parameters, i.e. input variables (see 5.6 and 5.7 for examples). Using function notation, we need a function of the form $\vec{r}\colon \mathbb{R}^2 \to \mathbb{R}^3$.

Problem 5.25 Consider the surface $z = 9 - x^2 - y^2$ plotted in problem 5.8.

Use Sage or Wolfram Alpha to plot your parametrization. See 16.5: 1-16 for more practice.

1. Give a parametrization $\vec{r}\colon \mathbb{R}^2 \to \mathbb{R}^3$ of the surface. In other words, you'll need to give equations

$$x =?, y =?, z =? \qquad \text{or} \qquad \vec{r}(?,?) = (?,?,?).$$

   [Hint: You can use the parameters x and y to help you out. Then you just have $x = x$, $y = y$, and $z =?$. This should be quite fast.]

2. What bounds must you place on x and y to obtain the portion of the surface above the plane $z = 0$?

3. If $z = f(x,y)$ is any surface, give a parametrization of the surface (i.e., $x =?, y =?, z =?$ or $\vec{r}(?,?) = (?,?,?)$.)

When a surface has a lot of symmetry, we can often use an appropriate coordinate transformation $\vec{T}\colon \mathbb{R}^3 \to \mathbb{R}^3$ to obtain a parametrization of a surface. Note that the coordinate transformation has three inputs and three inputs, whereas the parametric surface has only two inputs. All we have to do is remove one input variables by expressing it in terms of the others, and the function instantly describes a surface. We did this already in problem 5.17, where obtained a 6 dimensional graph to represent spherical coordinates.

Problem 5.26 Again consider the surface $z = 9 - x^2 - y^2$.

Use Sage or Wolfram Alpha to plot your parametrization with your bounds (see 5.25 for examples). See 16.5: 1-16 for more practice.

1. Using cylindrical coordinates, so $\vec{T}(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$, obtain a parametrization $\vec{r}(r, \theta) = (?, ?, ?)$ of the surface using the two parameters r and θ. So you'll need to give equations

$$x = ?, y = ?, z = ? \qquad \text{or} \qquad \vec{r}(r, \theta) = (?, ?, ?).$$

   [Hint: We already know $x = r\cos\theta$ and $y = ?$ from cylindrical coordinates. The equation $z = 9 - x^2 - y^2$, when written in terms of r and θ, should give you the last equation for your parametrization.

2. What bounds must you place on r and θ to obtain the portion of the surface above the plane $z = 0$? Make sure you use technology to graph your parametric equations and verify that your bounds are correct.

Problem 5.27 Recall the spherical coordinate transformation

We did very similar things in problem 5.17. See 16.5: 1-16 for more practice.

$$\vec{T}(\rho, \phi, \theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi).$$

This is a function of the form $\vec{T}: \mathbb{R}^3 \to \mathbb{R}^3$. If we hold one of the three inputs constant, then we have a function of the form $\vec{r}: \mathbb{R}^2 \to \mathbb{R}^3$, which is a parametric surface.

Use Sage or Wolfram Alpha to plot each parametrization (see 5.25 for examples).

1. Give a parametrization of the sphere of radius 2, using ϕ and θ as your parameters.
2. What bounds should you place on ϕ and θ if you want to hit each point on the sphere exactly once?
3. What bounds should you place on ϕ and θ if you only want the portion of the sphere above the plane $z = 1$?

Sometimes you'll have to invent your own coordinate system when constructing parametric equations for a surface. If you notice that there are lots of circles parallel to one of the coordinate planes, try using a modified version of cylindrical coordinates. Instead of circles in the xy plane ($x = r\cos\theta, y = r\sin\theta, z = z$), maybe you need circles in the yz-plane ($x = x, y = r\sin\theta, z = r\sin\theta$) or the xz plane. Just look for lots of symmetry, and then construct your parametrization accordingly.

Problem 5.28 Find parametric equations for the surface $x^2 + z^2 = 9$. [Hint: read the paragraph above.]

Use Sage or Wolfram Alpha to plot each parametrization (see 5.25 for examples).

1. What bounds should you use to obtain the portion of the surface between $y = -2$ and $y = 3$?
2. What bounds should you use to obtain the portion of the surface above $z = 0$?
3. What bounds should you use to obtain the portion of the surface with $x \geq 0$ and $y \in [2, 5]$?

We'll finish the chapter with a few review problems.

Problem 5.29 Construct a graph of the surface $z = x^2 - y^2$. Do so in 2 ways. (1) Construct a 3D surface plot. (2) Construct a contour plot, which is a graph with several level curves. Which level curve passes through the point $(3, 4)$? Use Wolfram Alpha to know if you're right.

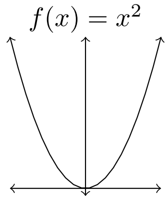
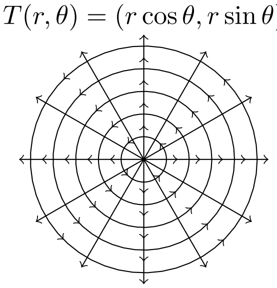
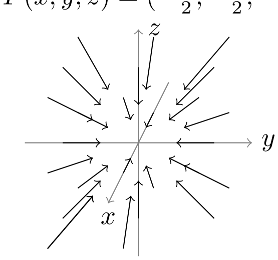
Problem 5.30 Construct a plot of the vector field

$$\vec{F}(x,y) = (x+y, -x+1)$$

by graphing the field at many integer points around the origin (I generally like to get the 8 integer points around the origin, and then a few more). Then explain how to modify your graph to obtain a plot of the vector field

$$\hat{F}(x,y) = \frac{(x+y, -x+1)}{\sqrt{(x+y)^2+(1-x)^2}}.$$

Problem 5.31 For this problem, construct a grid like the one below that contains examples of the different types of functions and how we graph them. Once we know the dimensions of the domain and codomain, there are specific ways we graph the function. In each cell, I've given you the function form. Your job is to select a function that fits this form, and then appropriately graph it. I've filled in a few for you. Feel free to use examples from earlier in this chapter.

$f:\mathbb{R}\to\mathbb{R}$ $f(x)=x^2$ 	$f:\mathbb{R}^2\to\mathbb{R}$ $f(x,y)=$ Surface Plot | Contour Plot	$f:\mathbb{R}^3\to\mathbb{R}$ $f(x,y,z)=$ 4D Plot Skip | 3D Contour Plot
$\vec{r}:\mathbb{R}\to\mathbb{R}^2$	$\vec{T}:\mathbb{R}^2\to\mathbb{R}^2$ $T(r,\theta)=(r\cos\theta, r\sin\theta)$  | $\vec{F}:\mathbb{R}^2\to\mathbb{R}^2$	$\vec{f}:\mathbb{R}^3\to\mathbb{R}^2$ Skip (Math 316)
$\vec{r}:\mathbb{R}\to\mathbb{R}^3$	$\vec{r}:\mathbb{R}^2\to\mathbb{R}^3$	$\vec{T}:\mathbb{R}^3\to\mathbb{R}^3$ | $\vec{F}:\mathbb{R}^3\to\mathbb{R}^3$ $\vec{F}(x,y,z)=(-\frac{x}{2},-\frac{y}{2},-\frac{z}{2})$ 

5.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 6

Differentials and the Derivative

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Find limits, and be able to explain when a function does not have a limit by considering different approaches.
2. Compute partial derivatives. Explain how to obtain the total derivative from the partial derivatives (using a matrix).
3. Find equations of tangent lines and tangent planes to surfaces.
4. Find derivatives of composite functions, using the chain rule (matrix multiplication).

You'll have a chance to teach your examples to your peers prior to the exam.

6.1 Limits

In the previous chapter, we learned how to describe lots of different functions. In first-semester calculus, after reviewing functions, you learned how to compute limits of functions, and then used those ideas to develop the derivative of a function. The exact same process is used to develop calculus in high dimensions. One hurdle that will prevent us from developing calculus this way in high dimensions is the epsilon-delta definition of a limit. We'll review it briefly. Those of you who pursue further mathematical study will spend much more time on this topic in future courses.

In first-semester calculus, you learned how to compute limits of functions. Here's the formal epsilon-delta definition of a limit.

Definition 6.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We write $\lim_{x\to c} f(x) = L$ if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|f(x) - L| < \epsilon$.

We're looking at this formal definition here because we can compare it with the formal definition of limits in higher dimensions. The only difference is that we just put vector symbols above the input x and the output $f(x)$.

Definition 6.2. Let $\vec{f}: \mathbb{R}^n \to \mathbb{R}^m$ be a function. We write $\lim_{\vec{x}\to\vec{c}} \vec{f}(\vec{x}) = \vec{L}$ if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |\vec{x} - \vec{c}| < \delta$ implies $|\vec{f}(\vec{x}) - \vec{L}| < \epsilon$.

We'll find that throughout this course, the key difference between first-semester calculus and multivariate calculus is that we replace the input x and output y of functions with the vectors $\vec{x}$ and $\vec{y}$.

Problem 6.1 For the function $f(x, y) = z$, we can write f in the vector notation $\vec{y} = \vec{f}(\vec{x})$ if we let $\vec{x} = (x, y)$ and $\vec{y} = (z)$. Notice that $\vec{x}$ is a vector of inputs, and $\vec{y}$ is a vector of outputs. For each of the functions below, state what $\vec{x}$ and $\vec{y}$ should be so that the function can be written in the form $\vec{y} = \vec{f}(\vec{x})$.

The point to this problem is to help you learn to recognize the dimensions of the domain and codomain of the function. If we write $\vec{f}: \mathbb{R}^n \to \mathbb{R}^m$, then $\vec{x}$ is a vector in $\mathbb{R}^n$ with n components, and $\vec{y}$ is a vector in $\mathbb{R}^m$ with m components.

1. $f(x, y, z) = w$
2. $\vec{r}(t) = (x, y, z)$
3. $\vec{r}(u, v) = (x, y, z)$
4. $\vec{F}(x, y) = (M, N)$
5. $\vec{F}(\rho, \phi, \theta) = (x, y, z)$

You learned to work with limits in first-semester calculus without needing the formal definitions above. Many of those techniques apply in higher dimensions. The following problem has you review some of these technique, and apply them in higher dimensions.

Problem 6.2 Do these problems without using L'Hopital's rule.

See 14.2: 1-30 for more practice.

1. Compute $\lim_{x\to 2} x^2 - 3x + 5$ and then $\lim_{(x,y)\to(2,1)} 9 - x^2 - y^2$.
2. Compute $\lim_{x\to 3} \dfrac{x^2 - 9}{x - 3}$ and then $\lim_{(x,y)\to(4,4)} \dfrac{x - y}{x^2 - y^2}$.
3. Explain why $\lim_{x\to 0} \dfrac{x}{|x|}$ does not exist. [Hint: graph the function.]

In first semester calculus, we can show that a limit does or does not exist by considering what happens from the left, and comparing it to what happens on the right. You probably used the following theorem extensively.

> If $y = f(x)$ is a function defined on some open interval containing c, then $\lim_{x\to c} f(x)$ exists if and only if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$.

A limit exists precisely when the limits from every direction exists, and all directional limits are equal. In first-semester calculus, this required that you check two directions (left and right). This theorem generalizes to higher dimensions, but it becomes much more difficult to apply.

Example 6.3. Consider the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Our goal is to determine if the function has a limit at the origin $(0, 0)$. We can approach the origin along many different lines.

One line through the origin is the line $y = 2x$. If we stay on this line, then we can replace each y with $2x$ and then compute

$$\lim_{\substack{(x,y)\to(0,0)\\ y=2x}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-(2x)^2}{x^2+(2x)^2} = \lim_{x\to 0} \frac{-3x^2}{5x^2} = \lim_{x\to 0} \frac{-3}{5} = \frac{-3}{5}.$$

This means that if we approach the origin along the line $y = 2x$, we will have a height of $-3/5$ when we arrive at the origin.

If the function $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ has a limit at the origin, the previous problem suggests that limit will be $-3/5$.

Problem 6.3 Please read the previous example. Recall that we are looking for the limit of the function $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ at the origin (0,0). Our goal is to determine if the function has a limit at the origin $(0,0)$.

You may want to look at a graph in Sage or Wolfram Alpha (try using the "contour lines" option). As you compute each limit, make sure you understand what that limit means in the graph.

1. In the xy-plane, how many lines pass through the origin $(0,0)$? Give an equation a line other than $y = 2x$ that passes through the origin. Then compute
$$\lim_{\substack{(x,y)\to(0,0)\\ \text{your line}}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-(?)^2}{x^2+(?)^2} = \dots.$$
2. Give another equation a line that passes through the origin. Then compute
$$\lim_{\substack{(x,y)\to(0,0)\\ \text{your line}}} \frac{x^2-y^2}{x^2+y^2}.$$
3. Does this function have a limit at $(0,0)$? Explain.

See 14.2: 41-50 for more practice.

The theorem from first-semester calculus generalizes as follows.

> If $\vec{y} = \vec{f}(\vec{x})$ is a function defined on some open region containing $\vec{c}$, then $\lim_{\vec{x}\to\vec{c}} \vec{f}(\vec{x})$ exists if and only if the limit exists along every possible approach to $\vec{c}$ and all these limits are equal.

There's a fundamental problem with using this theorem to check if a limit exists. Once the domain is 2-dimensional or higher, there are infinitely many ways to approach a point. There is no longer just a left and right side. To prove a limit exists, you must check infinitely many cases — that takes a really long time. The real power to this theorem is it allows us to show that a limit does not exist. All we have to do is find two approaches with different limits.

Problem 6.4 Consider the function $f(x,y) = \dfrac{xy}{x^2+y^2}$. Does this function have a limit at $(0,0)$? Examine the function at $(0,0)$ by considering the limit as you approach the origin along several lines.

See Sage.

See 14.2: 41-50 for more practice.

In all the examples above, we considered approaching a point by traveling along a line. Even if a function has a consistent limit along EVERY line, that is not enough to guarantee the function has a limit. The theorem requires EVERY approach, which includes parabolic approaches, spiraling approaches, and more.

For our purposes, checking along straight lines will do. If you are interested in seeing an example of a function $f(x,y)$ so that the limit at $(0,0)$ along every straight line $y = mx$ exists and equals 0, but the function has no limit at $(0,0)$, then please ask. Alternately, if this interests you, try coming up with an example yourself, and then come show me when you get it. This is a fun challenge.

Problem: Challenge Give an example of a function $f(x,y)$ so that the limit at $(0,0)$ along every straight line $y = mx$ exists and equals 0. However, show that the function has no limit at $(0,0)$ by considering an approach that is not a straight line.

6.2 Differentials

Let's recall the definition of a differential. If $y = f(x)$ is a function, then we say the differential dy is the expression $dy = f'(x)dx$. We can also write this in the form $dy = \frac{dy}{dx}dx$.

Observation 6.4. Here's the key. Think of differential notation $dy = f'(x)dx$ in the following way:

> A small change in the output y equals the derivative multiplied by a small change in the input x. To get dy, we just need the derivative times dx.

To get the derivative in all dimensions, we just substitute in vectors to obtain the differential notation $d\vec{y} = f'(\vec{x})d\vec{x}$. The derivative is precisely the thing that tells us how to get $d\vec{y}$ from $d\vec{x}$. We'll quickly see that the derivative is a matrix, and the columns of that matrix we'll call partial derivatives. We'll start using the notation Df instead of f'.

Let's examine some problems you have seen before.

Problem 6.5 The volume of a right circular cylinder is $V(r,h) = \pi r^2 h$. Imagine that each of V, r, and h depends on t (we might be collecting rain water in a can, or crushing a cylindrical concentrated juice can, etc.).

See 3.10 for more practice.

1. Compute $\frac{dV}{dt}$ in terms of both $\frac{dr}{dt}$ and $\frac{dh}{dt}$. Then times both sides by dt to obtain the differential dV in terms of the differentials dr and dh. Write your answer in the form
$$dV = (?)dr + (?)dh.$$
2. Show that we can write dV as the matrix product
$$dV = \begin{bmatrix} 2\pi rh & \pi r^2 \end{bmatrix} \begin{bmatrix} dr \\ dh \end{bmatrix}.$$
3. If h is constant, what is $\frac{dV}{dr}$? Similarly, if r is constant, what is $\frac{dV}{dh}$?

The matrix $\begin{bmatrix} 2\pi rh & \pi r^2 \end{bmatrix}$ is the derivative of V. The columns we call the partial derivatives. The partial derivatives make up the whole.

Problem 6.6 The volume of a box is $V(x,y,z) = xyz$. Imagine that each variable depends on t.

1. Compute $\frac{dV}{dt}$ in terms of $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$. The write the differential dV in the form
$$dV = (?)dx + (?)dy + (?)dz.$$

2. Show that we can write dV as the matrix product (fill in the blanks)
$$dV = \begin{bmatrix} yz & ? & ? \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}.$$

The matrix $\begin{bmatrix} yz & ? & ? \end{bmatrix}$ is the derivative. The columns we call the partial derivatives. The partial derivatives make up the whole.

3. If both y and z remain constant, what is $\frac{dV}{dx}$? Similarly, if both x and z remain constant, what is $\frac{dV}{dy}$?

Problem 6.7 In this problem we'll use differential notation to approximate an increase in volume related to the previous two problems.

1. We showed that a change in the volume of a cylinder is approximately
$$dV = \begin{bmatrix} 2\pi rh & \pi r^2 \end{bmatrix} \begin{bmatrix} dr \\ dh \end{bmatrix}.$$
If we know that $r = 3$ and $h = 4$, and we know that r could increase by about .1 and h could increase by about .2, then by about how much will V increase by?

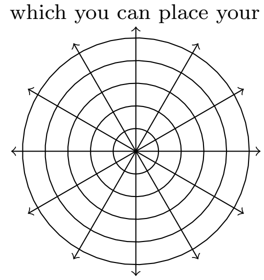
Make sure you ask me in class to show you physically exactly how you can see these differential formulas.

2. The volume of a box is given by $V = xyz$. We know the differential of the volume is $dV = \begin{bmatrix} yz & xz & xy \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$. If the current measurements are $x = 2$, $y = 3$, and $z = 5$, and we know that $dx = .01$, $dy = .02$, and $dz = .03$, then by about how much will the volume increase.

When the output of the function is a vector, we can draw vectors to help us understand geometrically what differentials tell us.

Problem 6.8 Consider the change of coordinates $\vec{T}(r, \theta) = (r\cos\theta, r\sin\theta)$, where $x = r\cos\theta$, $y = r\sin\theta$.

Here is a sketch of the polar coordinate transformation on which you can place your vectors.



1. Compute the differentials dx and dy. Write you answer as both
$$d\vec{T} = \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} dr + \begin{pmatrix} ? \\ ? \end{pmatrix} d\theta \quad \text{and} \quad d\vec{T} = \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}.$$

2. Consider the polar point $(r, \theta) = (4, \pi/2)$. Compute $T(4, \pi/2)$ (the Cartesian coordinate) and also compute the differentials above at $(4, \pi/2)$. Each column of the matrix above is an important vector (we'll call them partial derivatives). With their tail at $T(4, \pi/2)$, draw both vectors.

3. Imagine you are standing at the polar point $(4, \pi/2)$. If someone says, "Hey you, keep your angle constant, but increase your radius," then which direction would you move? What if someone said, "Hey you, keep your radius constant, but increase your angle"?

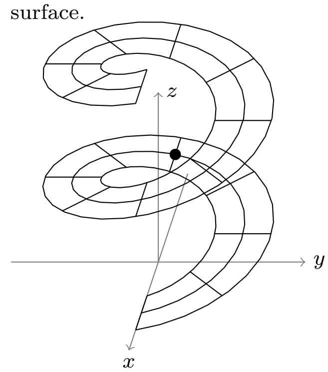
4. Now change the polar point to $(r, \theta) = (2, 3\pi/4)$. Without doing any computations, repeat part 2 (at the point draw both partial derivatives). Explain.

If your answers to the 2nd and 3rd part above were the same, then you're doing this correctly. The columns of the matrix above, when viewed as vectors, tell us precisely about motion. The next problem reinforces this concept. First though, we need a review of how to obtain equations of lines in 3D.

Review If you know that a line passes through the point $(1, 2, 3)$ and is parallel to the vector $(4, 5, 6)$, give a vector equation, and parametric equations, of the line. See [1] for an answer.

Problem 6.9 Consider the parametric surface $\vec{r}(a, t) = (a \cos t, a \sin t, t)$ for $2 \le a \le 4$ and $0 \le t \le 4\pi$. We encountered this parametric surface in chapter 5 when we considered a smoke screen left by multiple jets.

Here's a rough sketch of the surface.



1. Compute the differential $d\vec{r}$ which is the same as finding dx, dy, and dz. Write your answer in both vector and matrix forms

$$d\vec{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} da + \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} dt \quad \text{and} \quad d\vec{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} da \\ dt \end{pmatrix}.$$

2. Suppose an object is on this surface at the point $\vec{r}(3, \pi) = (-3, 0, \pi)$ (the dot on the graph to the right). Evaluate the matrix above at this point. Each column of the matrix above is an important vector, called a partial derivative. Draw both vectors with their tail at the point $\vec{r}(3, \pi)$.

3. If you were standing at $\vec{r}(3, \pi)$ and someone told you, "Hey you, hold t constant and increase a," then in which direction would you move? If they said, "Hey you, hold a constant and increase t," then in which direction would you move?

4. Give vector equations for two tangent lines to the surface at $\vec{r}(3, \pi)$.

   [Hint: You've got the point as $\vec{r}(3, \pi)$, and you've got two different direction vectors as the columns of the matrix. Use the ideas from chapter 2 to get an equation of a line, or see the review problem above.]

In the previous problem, you should have noticed that the columns of your matrix are tangent vectors to the surface. Because we have two tangent vectors to the surface, we should be able to use them to construct a normal vector to the surface, and from that we can get the equation of a tangent plane.

Review If you know that a plane passes through the point $(1, 2, 3)$ and has normal vector $(4, 5, 6)$, then give an equation of the plane. See [2] for an answer.

Problem 6.10 Consider again the parametric surface

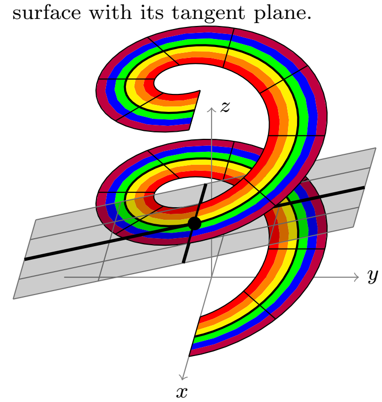
$$\vec{r}(a, t) = (a \cos t, a \sin t, t)$$

for $2 \leq a \leq 4$ and $0 \leq t \leq 4\pi$. We'd like to obtain an equation of the tangent plane to this surface at the point $\vec{r}(3, 2\pi)$. Once you have a point on the plane, and a normal vector to the surface, we can use the concepts in chapter 2 to get an equation of the plane. Give an equation of the tangent plane.

[Hint: To get the point, what is $\vec{r}(3, 2\pi)$? The columns of the matrix we obtain, when computing the differential $d\vec{r}$, give us two tangent vectors. How do we obtain a vector orthogonal to both these vectors?]

[Here's an alternate version of this problem, for Mario Kart fans. Mario and Luigi are booking it up rainbow road. About half way up, there is a glitch in the computer game and the road temporarily disappears. Instead of following the road, they instead are stuck on an infinite plane that meets the road tangentially where the glitch occurred. Give an equation of this plane.]

Here's a rough sketch of the surface with its tangent plane.



Let's end this section with one final problem related to functions of the form $z = f(x, y)$. In general, we can compute the change in a function $f(x, y)$ if we know how much x and y will change. This gives us a physical change in height, namely dz.

Problem 6.11 Consider the function $f(x, y) = x^2y + 3x + 4\sin(5y)$.

1. Assume that both x and y depend on t, and then use implicit differentiation to obtain a formula for $\frac{df}{dt}$ in terms of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
2. Solve for df, and write your answer as both
$$df = (?)dx + (?)dy \quad \text{and} \quad df = \begin{bmatrix} ? & x^2 + 20\cos(5y) \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}.$$
3. If you hold y constant, then what is $\frac{df}{dx}$? Similarly, if you hold x constant, then what is $\frac{df}{dy}$?

6.3 Derivatives and Partial Derivatives

The previous problems are precisely the content to this chapter. We just need to add some vocabulary to make it easier to talk about what we did. Let's introduce the vocabulary in terms of the problem above, and then make a formal definition.

- We can find the differential of a function by using implicit differentiation and assuming each variable in the problem depends on some common variable. We can write the differential as the product of a matrix and a vector containing the differentials of the inputs.
- The derivative is a matrix, such as
$$\begin{aligned} Df(x, y) &= \begin{bmatrix} 2xy + 3 & x^2 + 20\cos(5y) \end{bmatrix} \\ \text{or} \quad D\vec{T}(r, \theta) &= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\ \text{or} \quad D\vec{r}(a, t) &= \begin{bmatrix} \cos t & -a\sin t \\ \sin t & a\cos t \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

[1]A vector equation is $\vec{r}(t) = (4, 5, 6)t + (1, 2, 3)$ or $\vec{r}(t) = (4t + 1, 5t + 2, 6t + 3)$. Parametric equations for this line are $x = 4t + 1$, $y = 5t + 2$, and $z = 6t + 3$.

[2]An equation of the plane is $4(x - 1) + 5(y - 2) + 6(y - 3) = 0$. If (x, y, z) is any point in the plane, then the vector $(x - 1, y - 2, z - 3)$ is a vector in the plane, and hence orthogonal to $(4, 5, 6)$. The dot product of these two vectors should be equal to zero, which is why the plane's equation is $(4, 5, 6) \cdot (x - 1, y - 2, z - 3) = 0$.

Each column of this matrix we call a partial derivative. Some people call this matrix the "total" derivative, as it's made up of several parts.

- The first column of this matrix is just part of the whole derivative. We can get the first column by holding every variable, except the first, constant, and then differentiating with respect to the first variable. For the first matrix above, we call this the partial of f with respect to x, and use the notation $\frac{\partial f}{\partial x} = 2xy + 3$, or equivalently $f_x = 2xy + 3$. The first column of the second matrix is $\frac{\partial \vec{T}}{dr}$ or $D_r\vec{T}$, and the first column of the third matrix is $\frac{\partial \vec{r}}{da}$ or $\vec{r}_a$.
- The second column of the derivative is the partial of f with respect to the second variable. We can get the second column by holding the other variables constant, and then differentiating with respect to the second. For the first matrix, we use the notation $\frac{\partial f}{\partial x} = 2xy + 3$, or $f_x = 2xy + 3$. The second column of the second matrix we'd name $\frac{\partial \vec{T}}{d\theta}$ or $D_\theta\vec{T}$, and the second column of the third matrix is $\frac{\partial \vec{r}}{dt}$.
- Remember, the derivative of f is a matrix. The columns of the matrix are the partial derivatives with respect to an input variable. The differential is the product of the derivative and a vector of differentials.

Definition 6.5: Derivatives and Partial Derivatives. Let f be a function.

- The partial derivative of f with respect to x is the regular derivative of f, provided we hold every every input variable constant except x. We'll use the notations
$$\frac{\partial f}{\partial x}, \quad \frac{\partial}{\partial x}[f], \quad f_x, \quad \text{and } D_x f$$
to mean the partial of f with respect to x.
- The partial derivative of f with respect to y, written $\dfrac{\partial f}{\partial y}$ or f_y, is the regular derivative of f, provided we hold every input variable constant except y. A similar definition holds for partial derivatives with respect to any variable.
- The derivative of f is a matrix. The columns of the derivative are the partial derivatives. When there's more than one input variable, we'll use Df rather than f' to talk about derivatives. The order of the columns must match the order you list the variables in the function. If the function is $f(x, y)$, then the derivative is $Df(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$. If the function is $V(x, y, z)$, then the derivative is $DV(x, y, z) = \begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{bmatrix}$.

It's time to practice these new words on some problems. Remember, we're doing the exact same thing as before the definitions above. Now we just have some vocabulary which makes it much easier to talk about differentiation.

Problem 6.12 Compute the partial derivatives, the derivative, and the differential, as requested below.

See 14.3: 1-40 for more practice. I strongly suggest you practice a lot of this type of problem until you can compute partial derivatives with ease.

1. For $f(x, y) = x^2+2xy+3y^2$, compute $\dfrac{\partial f}{\partial x}$ and f_y. Then state the derivative $Df(x, y)$ and the differential df.
2. For $f(x, y, z) = x^2y^3z^4$, compute all three of f_x, $\dfrac{\partial f}{\partial y}$, and $D_z f$. Then state the derivative $Df(x, y, z)$ and the differential df.

Remember, the partial derivative of a function with respect to x is just the regular derivative with respect to x, provided you hold all other variables constant. We put the partials into the columns of a matrix to obtain the (total) derivative.

Please take a moment and practice computing partial and total derivatives. Your textbook has lots of examples to help you with partial derivatives in section 14.3. However, the textbook leaves out the actual derivative. The exercise below has 6 problems, with solutions, that you can use as extra practice for total derivatives. Complete the exercise below before moving on.

Exercise For each function below, compute the total derivative.

1. $f(x,y) = 9 - x^2 + 3y^2$
2. $\vec{r}(t) = (t, \cos t, \sin t)$
3. $f(x,y,z) = xy^2z^3$
4. $\vec{r}(u,v) = (u^2, v^2, u - v)$
5. $\vec{F}(x,y) = (-y + 3x, x + 4y)$
6. $\vec{T}(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$.

See [3] for answers.

Problem 6.13 Compute the requested partial and total derivatives.

1. Consider the parametric surface $\vec{r}(u,v) = (u, v, v\cos(uv))$. Compute both $\dfrac{\partial \vec{r}}{\partial u}$ and $\dfrac{\partial \vec{r}}{\partial v}$. Then state $D\vec{r}(u,v)$ and the differential $d\vec{r}$. If you end up with a 3 by 2 matrix for the derivative, you did this correctly.
2. Consider the vector field $\vec{F}(x,y) = (-y, xe^{3y})$. Compute both $\dfrac{\partial \vec{F}}{\partial x}$ and $\dfrac{\partial \vec{F}}{\partial y}$. Then state $D\vec{F}(x,y)$ and the differential $d\vec{F}$.

As you completed the problems above, did you notice any connections between the size of the matrix and the size of the input and output vectors? Make sure you ask in class about this. We'll make a connection.

We've now seen that the derivative of $z = f(x,y)$ is a matrix $Df(x,y) = \begin{bmatrix} f_x & f_y \end{bmatrix}$. This is a function itself that has inputs x and y, and outputs f_x and f_y. This means it has 2 inputs and 2 outputs, so it's a vector field. What does the vector field tell us about the original function?

Problem 6.14 Consider the function $f(x,y) = y - x^2$.

1. In the xy plane, please draw several level curves of f (maybe $z = 0$, $z = 1$, $z = -4$, etc.) Write the height on each curve (so you're making a topographical map).

[3]The derivatives of each function are shown below.

1. $Df(x,y) = \begin{bmatrix} -2x & 6y \end{bmatrix}$
2. $D\vec{r}(t) = \begin{bmatrix} 1 \\ -\sin t \\ \cos t \end{bmatrix}$
3. $Df(x,y,z) = \begin{bmatrix} y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{bmatrix}$
4. $D\vec{r}(u,v) = \begin{bmatrix} 2u & 0 \\ 0 & 2v \\ 1 & -1 \end{bmatrix}$
5. $D\vec{F}(x,y) = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}$
6. $D\vec{T}(r,\theta,z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. Compute the derivative of f. (Remember this is now a vector field.)
3. Pick several points in the xy plane that lie on the level curves you already drew. At these points, add the vector given by the derivative. (So at (0,0), you'll need to draw the vector (0,1). At (1,1), you'll need to draw the vector (-2,1).) Add 8 vectors to your picture, and then share with the class any observations you make.

We'll come back to this problem more in chapter 9 as we discuss optimization. There are lots of connections between the derivative and level curves.

Since a partial derivative is a function, we can take partial derivatives of that function as well. If we want to first compute a partial with respect to x, and then with respect to y, we would write one of

$$f_{xy} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}f = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}.$$

The shorthand notation f_{xy} is easiest to write. In upper-level courses, we will use subscripts to mean other things. At that point, we'll have to use the fractional partial notation to avoid confusion.

Problem 6.15 Consider the functions $f(x, y, z) = xy^2z^3$ and $g(x, y) = x\cos(xy)$.

1. Compute f_x, f_{xy}, f_{xyy}, and $\frac{\partial^2 f}{\partial z^2}$.
2. Compute g_x and g_{xy}, and then compute g_y and g_{yx}.

Problem 6.16: Mixed Partials Agree Complete the following:

1. Let $f(x, y) = 3xy^3 + e^x$. Compute the four second partials
$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and } \frac{\partial^2 f}{\partial x \partial y}.$$
2. For $f(x, y) = x^2\sin(y) + y^3$, compute both f_{xy} and f_{yx}.
3. Make a conjecture about a relationship between f_{xy} and f_{yx}. Then use your conjecture to quickly compute f_{xy} if
$$f(x, y) = 3xy^2 + \tan^2(\cos(x))(x^{49} + x)^{1000}.$$

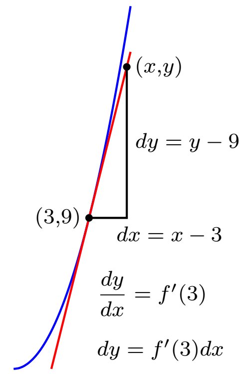
6.4 Tangent Planes

We can obtain most of the results in multivariate calculus by replacing the x and y in $dy = f'dx$ with $\vec{x}$ and $\vec{y}$. As an example, we can use differential notation to find an equation of the tangent plane to a function of the form $z = f(x, y)$. Let's first review how to do it for functions of the form $y = f(x)$, and then generalize.

Example 6.6: Tangent Lines. Consider the function $y = f(x) = x^2$.

1. The derivative is $f'(x) = 2x$. When $x = 3$ this means the derivative is $f'(3) = 6$ and the output y is $y = f(3) = 9$.

2. We know the tangent line passes through the point $P = (3, 9)$. We let $Q = (x, y)$ be any other point on the tangent line, and then a vector between these points is $\vec{PQ} = (x, y) - (3, 9) = (x - 3, y - 9)$. This vector tells us that when our change in x is $dx = x - 3$, then the change in y is $dy = y - 9$.



3. Differential notation states that a change in the output dy equals the derivative times a change in the input dx. In symbols, we have the equation $dy = f'(3)dx$. We then replace dx, dy, and $f'(3)$ with what we know they equal from the parts above to obtain

$$\underbrace{y - 9}_{dy} = \underbrace{6}_{f'(3)}\underbrace{(x - 3)}_{dx}.$$

This is an equation of the tangent line.

In first semester calculus, differential notation is $dy = f'dx$. At $x = c$, the line passes through the point $P = (c, f(c))$. If $Q = (x, y)$ is any other point on the line, then the vector $\vec{PQ} = (x - c, y - f(c))$ tells us that when $dx = x - c$ we have $dy = y - f(c)$. Substitution give us an equation for the tangent line tangent line as

$$\underbrace{y - f(c)}_{dy} = f'(c)\underbrace{(x - c)}_{dx}.$$

This equation tells us that a change in the output $(y - f(c))$ equals the derivative times a change in the input $(x - c)$. We now repeat this for the next problem, where the output is z and input is (x, y), which means differential notation says

$$dz = Df\begin{bmatrix} dx \\ dy \end{bmatrix}.$$

Problem 6.17 Consider the function $z = f(x, y) = 9 - x^2 - y^2$. If you haven't yet, read the example above.

See Sage for a picture.
See 14.6: 9-12 for more practice.

1. Compute the derivative $Df(x, y)$ and differential df. Then at $(x, y) = (2, 1)$, evaluate the derivative $Df(2, 1)$ and the output $z = f(2, 1)$.

2. One point on the tangent plane to the surface at $(2, 1)$ is the point $P = (2, 1, f(2, 1))$. Let $Q = (x, y, z)$ be another point on this plane. Use the vector $\vec{PQ}$ obtain dz when $dx = x - 2$ and $dy = y - 1$.

3. We'd like an equation of the tangent plane to $f(x, y)$ when $x = 2$ and $y = 1$. Differential notation tells us that

$$dz = Df(2, 1)\begin{bmatrix} dx \\ dy \end{bmatrix} \quad \text{or} \quad z - ? = \begin{bmatrix} -4 & ? \end{bmatrix}\begin{bmatrix} x - 2 \\ ? \end{bmatrix}$$

Fill in the blanks and compute this matrix product. When you are done you should have an equation of a plane. If you subtract dz from both sides, a normal vector for your plane should be $(-4, -2, -1)$.

The first semester calculus tangent line equation, with differential notation, generalizes immediately to the tangent plane equation for functions of the form $z = f(x, y)$. Let's try this on another problem.

Problem 6.18 Let $f(x, y) = x^2 + 4xy + y^2$. Give an equation of the tangent plane at $(3, -1)$, and then state a normal vector to this plane. [Hint: find $Df(x, y)$, $Df(3, -1)$, df, dx, dy, and dz.]

See Sage.
See 14.6: 9-12 for more practice.

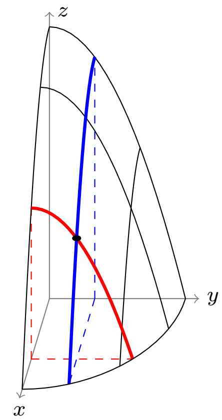
Let's now return to the function $z = 9 - x^2 - y^2$, and show how parametric surfaces can add more light to unlocking the derivative and its geometric meaning. With a parametrization, partial derivatives are vectors, instead of just numbers. Once we have vectors, we can describe motion. This makes it easier to visualize.

Problem 6.19 Let $z = f(x, y) = 9 - x^2 - y^2$. We can parameterize this function by writing $x = x, y = y, z = 9 - x^2 - y^2$, or in vector notation

$$\vec{r}(x, y) = (x, y, f(x, y)).$$

1. Compute $\dfrac{\partial \vec{r}}{\partial x}$ and $\dfrac{\partial \vec{r}}{\partial y}$ and then evaluate these partials at $(x, y) = (2, 1)$. The surface is drawn to the right, where $x = 2$ is highlighted in red and $y = 1$ is highlighted in blue. Based at the point $(2, 1, 4)$, draw both of these partial derivatives (they are vectors).
2. You should see that the partial derivatives above are tangent vectors to the surface. Cross them to obtain a normal vector to the tangent plane.
3. Give an equation of the tangent plane to the surface at $(2, 1, 4)$.

See 16.5: 27-30 for more practice. Here's a picture of the surface on which you can draw your partial derivatives.

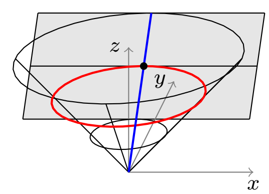


The next problem generalizes the tangent plane and normal vector calculations above to work for any parametric surface $\vec{r}(u, v)$.

Problem 6.20 Let $\vec{r}(u, v) = (u \cos v, u \sin v, u)$, a parametrization of a cone.

1. Give vector equations of two tangent lines to the surface at $\vec{r}(2, \pi/2)$ (so $u = 2$ and $v = \pi/2$).
2. Give a normal vector to the surface at $\vec{r}(2, \pi/2)$ and an equation of the tangent plane at $\vec{r}(2, \pi/2)$.

See 16.5: 27-30 for more practice.



We now have two different ways to compute tangent planes. One way generalizes differential notation $dy = f' dx$ to $dz = Df \begin{bmatrix} dx \\ dy \end{bmatrix}$ and then uses matrix multiplication. This way will extend to tangent objects in EVERY dimension. It's the key idea needed to work on really large problems. The other way requires that we parametrize the surface $z = f(x, y)$ as $\vec{r}(x, y) = (x, y, f(x, y))$ and then use the cross product on the partial derivatives to obtain a normal vector. The next problem has you give a general formula for a tangent plane. To tackle this problem, you'll need to make sure you can use symbolic notation. The review problem should help with this.

Review Joe wants to to find the tangent line to $y = x^3$ at $x = 2$. He knows the derivative is $y = 3x^2$, and when $x = 2$ the curve passes through 8. So he writes an equation of the tangent line as $y - 8 = 3x^2(x - 2)$. What's wrong? What part of the general formula $y - f(c) = f'(c)(x - c)$ did Joe forget? See [4] for an answer.

---

[4] Joe forgot to replace x with 2 in the derivative. The equation should be $y - 8 = 12(x - 2)$. The notation $f'(c)$ is the part he forgot. He used $f'(x) = 3x^2$ instead of $f'(2) = 12$.

Problem 6.21: Tangent Plane General Formula First read the review problem above, and its solution. Now, consider the function $z = f(x, y)$. Prove that an equation of the tangent plane to f at $(x, y) = (a, b)$ is given by

$$z - f(a,b) = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-a).$$

Then give an equation of the tangent plane to $f(x, y) = x^2 + 3xy$ at $(3, -1)$. [Hint: Use either differential notation or a parametrization. Try both ways.]

Let's finish this section with a third way to obtain an equation of the tangent plane. This final approach requires that you think geometrically about the connection between derivatives and slope. For example, the derivative $\frac{dy}{dx}$ gives us a slope, so if x changes by 1 unit, then y should change by $\frac{dy}{dx}$ units. As a vector in the plane, this means the derivative $\frac{dy}{dx}$ gives us a vector $(1, \frac{dy}{dx})$ that is tangent to the curve.

Problem 6.22 Consider the generic function $z = f(x, y)$. We'd like to obtain a generic equation of the tangent plane to $z = f(x, y)$ at $x = (a, b)$.

1. The partial derivative $\frac{\partial f}{\partial x}$ tells us the slope of the surface when y is held constant. This gives us a ratio between changes in x and changes in z. Use this ratio to explain why the vector $(1, 0, f_x(a, b))$ must be tangent to the surface at (a, b).
2. Obtain a tangent vector to the surface from the partial derivative $\frac{\partial z}{\partial y}$.
3. The two vectors above lie on the tangent plane to the surface. Use them to obtain a normal vector to the surface.
4. Use the normal vector you just computed to give an equation of the tangent plane to $z = f(x, y)$ at $x = (a, b)$.

6.5 The Chain Rule (substitution)

Suppose we know that the temperature at points in the plane is given by some function $T = f(x, y)$. We also know that an object is traveling around the plane following the curve $\vec{r}(t) = (x(t), y(t))$. As the object moves around, it encounters different temperatures. One function f tells us the temperature based on position. The other function $\vec{r}$ tells position based on time. Combining these two functions together (function composition $f(\vec{r}(t))$ we can compute the temperature based on time. These functions are like a chain of events. Changing t causes position to change, which in turn causes the temperature to change. This might cause something else to change. The chain rule helps us see how to compute the derivative of a function that is composed of several smaller pieces.

Problem 6.23 The temperature at points in the plane is $f(x, y) = 3x+4y+7$. On object follows the straight line path $\vec{r}(t) = (2t + 1, -t + 6)$.

1. Compute the composite function $f(\vec{r}(t))$ which gives the temperature encountered by the object at time t. Then compute the derivative of the temperature with respect to time, namely $\frac{df}{dt}$.

2. Compute the derivatives $Df(x,y)$ and $D\vec{r}(t)$, and then state the differentials df and $d\vec{r}$. You should obtain

$$df = \begin{bmatrix} ? & ? \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \quad \text{and} \quad d\vec{r} = \begin{bmatrix} ? \\ ? \end{bmatrix} \begin{bmatrix} dt \end{bmatrix}.$$

3. Recall that $\vec{r} = (x,y)$, which means that $d\vec{r} = \begin{bmatrix} dx \\ dy \end{bmatrix}$. Use this fact in your differential df to replace $\begin{bmatrix} dx \\ dy \end{bmatrix}$ with what it equals in terms of dt. Then compute any matrix products that arise.

This section focuses on using the chain rule to compute derivatives of composite functions. Just as before, we'll find that the first semester calculus chain rule will generalize to all dimensions, if we replace f' with the matrix Df. Let's recall the chain rule from first-semester calculus.

Theorem (The Chain Rule)**.** *Let x be a real number and f and g be functions of a single real variable. Suppose f is differentiable at $g(x)$ and g is differentiable at x. The derivative of $f \circ g$ at x is*

$$(f \circ g)'(x) = \frac{d}{dx}(f \circ g)(x) = f'(g(x)) \cdot g'(x).$$

Let's practice using the notation of the chain rule.

Problem 6.24 Suppose we know that $f'(x) = \dfrac{\sin(x)}{2x^2+3}$ and $g(x) = \sqrt{x^2+1}$. Notice we don't know $f(x)$.

Not knowing a function f is actually quite common in real life. We can often measure how something changes (a derivative) without knowing the function itself.

1. State $f'(x)$ and $g'(x)$.
2. State $f'(g(x))$, and explain the difference between $f'(x)$ and $f'(g(x))$.
3. Use the chain rule to compute $(f \circ g)'(x)$.

We can also view the chain rule using differentials.

Example 6.7. Suppose that $y = f(u)$ and that $u = g(x)$. This means we have the differentials

$$dy = f'(u)du \quad \text{and} \quad du = g'(x)dx.$$

Simple substitution tells us that

$$dy = f'(\underbrace{g(x)}_{u})\,\underbrace{g'(x)dx}_{du}.$$

This means instantly that the derivative of y with respect to x must be the product $f'(g(x))g'(x)$.

Problem 6.25 Suppose that $f(x,y) = 3xy^2 + \sin x$ and $\vec{r}(t) = (e^{2t}, t^3)$, which is just shorthand for saying $x(t) = e^{2t}$ and $y(t) = t^3$.

1. Compute the derivative $Df(x,y)$. Also state the differential df in terms of x, y, dx, and dy.
2. Compute the derivative $D\vec{r}(t)$. Then state the differential $d\vec{r}$ in terms of t and dt.

3. Recall that $\vec{r} = (x, y)$, which means that $d\vec{r} = \begin{bmatrix} dx \\ dy \end{bmatrix}$. At this point you should have

$$df = \begin{bmatrix} ? & ? \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \quad \text{and} \quad d\vec{r} = \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix} \begin{bmatrix} dt \end{bmatrix}.$$

Use substitution (as done in the example above) to write the differential df in terms of t and dt. Then state the derivative df/dt.

We now generalize to higher dimensions. If I want to write $\vec{f}(\vec{g}(\vec{x}))$, then $\vec{x}$ must be a vector in the domain of g. After computing $\vec{g}(\vec{x})$, we must get a vector that is in the domain of f. Since the chain rule in first semester calculus states $(f(g(x))' = f'(g(x))g'(x)$, then in high dimension it should state $D(f(g(x)) = Df(g(x))Dg(x)$, the product of two matrices.

Problem 6.26 In problem 6.5, we showed that for a circular cylinder with volume $V = \pi r^2 h$, the derivative is

$$DV(r, h) = \begin{bmatrix} 2\pi rh & \pi r^2 \end{bmatrix}.$$

Suppose that the radius and height are both changing with respect to time, where $r = 3t$ and $h = t^2$. We'll write this parametrically as $(r, h)(t) = (3t, t^2)$.

1. In $V = \pi r^2 h$, replace r and h with what they are in terms of t. Then compute $\frac{dV}{dt}$.

2. We know $DV(r, h) = \begin{bmatrix} 2\pi rh & \pi r^2 \end{bmatrix}$ and $D(r, h)(t) = \begin{bmatrix} 3 \\ 2t \end{bmatrix}$. In first semester calculus, the chain rule was the product of derivatives. Compute the matrix product
$$DV((r, h)(t)) \cdot D(r, h)(t)$$
and verify that you get $\frac{dV}{dt}$.

3. To get the correct answer to the previous part, you had to replace r and h with what they equaled in terms of t. What part of the notation $\frac{dV}{dt} = DV((r, h)(t)) \cdot D(r, h)(t)$ tells you to replace r and h with what they equal in terms of t?

Let's look at some physical examples involving motion and temperature, and try to connect what we know should happen to what the chain rule states.

Problem 6.27 Consider $f(x, y) = 9 - x^2 - y^2$ and $\vec{r}(t) = (2\cos t, 3\sin t)$. Imagine the following scenario. A horse runs around outside in the cold. The horse's position at time t is given parametrically by the elliptical path $\vec{r}(t)$. The function $T = f(x, y)$ gives the temperature of the air at any point (x, y).

1. At time $t = 0$, what is the horse's position $\vec{r}(0)$, and what is the temperature $f(\vec{r}(0))$ at that position? Find the temperatures at $t = \pi/2$, $t = \pi$, and $t = 3\pi/2$ as well.

2. In the plane, draw the path of the horse for $t \in [0, 2\pi]$. Then, on the same 2D graph, include a contour plot of the temperature function f. Make sure you include the level curves that pass through the points in part 1, and write the temperature on each level curve you draw.

If you end up with an ellipse and several concentric circles, then you've done this right.

3. As the horse runs around, the temperature of the air around the horse is constantly changing. At which t does the temperature around the horse reach a maximum? A minimum? Explain, using your graph.

This idea leads to an optimization technique, Lagrange multipliers, later in the semester.

4. As the horse moves past the point at $t = \pi/4$, is the temperature of the surrounding air increasing or decreasing? In other words, is $\frac{df}{dt}$ positive or negative? Use your graph to explain.

Problem 6.28 Consider again $f(x, y) = 9 - x^2 - y^2$ and $\vec{r}(t) = (2\cos t, 3\sin t)$, which means $x = 2\cos t$ and $y = 3\sin t$.

1. At the point $\vec{r}(t)$, we'd like a formula for the temperature $f(\vec{r}(t))$. What is the temperature of the horse at any time t? [In $f(x, y)$, replace x and y with what they are in terms of t.]
2. Compute df/dt (the derivative as you did in first-semester calculus).
3. Construct a graph of $f(t)$ (use software to draw this if you like). From your graph, at what time values do the maxima and minima occur?
4. What is $\frac{df}{dt}$ at $t = \pi/4$?

Problem 6.29 Consider again $f(x, y) = 9 - x^2 - y^2$ and $\vec{r}(t) = (2\cos t, 3\sin t)$.

1. Compute both $Df(x, y)$ and $D\vec{r}(t)$ as matrices. One should have two columns. The other should have one column (but two rows).
2. The temperature at any time t we can write symbolically as $f(r(t))$. First semester calculus suggests the derivative of f with respect to t should be the product $(f(\vec{r}(t)))' = f'(\vec{r}(t))\vec{r}'(t)$. Write this using D notation instead of prime notation.
3. Compute the matrix product $DfD\vec{r}$, and then substitute $x = 2\cos t$ and $y = 3\sin t$.
4. What is the change in temperature with respect to time at $t = \pi/4$?

The previous three problems all focused on exactly the same concept, but in a slightly different way. The first looked at the concept graphically, showing what it means to write $(f \circ \vec{r})(t) = f(\vec{r}(t))$. The second reduced the problem to first-semester calculus. The third tackled the problem by considering matrix derivatives. In all three cases, we wanted to understand the following problem.

> If $z = f(x, y)$ is a function of x and y, and both x and y are functions of t ($\vec{r}(t) = (x(t), y(t))$), then how do we discover how quickly f changes as we change t. In other words, what is the derivative of f with respect to t. Notationally, we seek $\frac{df}{dt}$ which we formally write as $\frac{d}{dt}[f(x(t), y(t))]$ or $\frac{d}{dt}[f(\vec{r}(t))]$.

To answer this problem, we use the chain rule, which is just matrix multiplication.

Theorem 6.8 (The Chain Rule). *Let $\vec{x}$ be a vector and $\vec{f}$ and $\vec{g}$ be functions so that the composition $\vec{f}(\vec{g}(\vec{x}))$ makes sense (we can use the output of g as an input to f). Suppose $\vec{f}$ is differentiable at $\vec{g}(\vec{x})$ and that $\vec{g}$ is differentiable at $\vec{x}$. Then the derivative of $\vec{f} \circ \vec{g}$ at $\vec{x}$ is*

$$D(\vec{f} \circ \vec{g})(\vec{x}) = D\vec{f}(\vec{g}(\vec{x})) \cdot D\vec{g}(\vec{x}).$$

This is exactly the same as the chain rule in first-semester calculus. The only difference is that now we have vectors above every variable and function, and we replaced the one-by-one matrices f' and g' with potentially larger matrices Df and Dg. If we write everything in vector notation, the chain rule in all dimensions is the EXACT same as the chain rule in one dimension.

Problem 6.30 Suppose that $f(x, y) = x^2 + xy$ and that $x = 2t + 3$ and $y = 3t^2 + 4$.

See 14.4: 1-6 for more practice. Don't use the formulas in the chapter, rather practice using matrix multiplication. The formulas are just a way of writing matrix multiplication without writing down the matrices, and only work for functions from $\mathbb{R}^n \to \mathbb{R}$. Our matrix multiplication method works for any function from $\mathbb{R}^n \to \mathbb{R}^m$.

1. Rewrite the parametric equations $x = 2t + 3$ and $y = 3t^2 + 4$ in vector form, so we can apply the chain rule. This means you need to create a function $\vec{r}(t) = (\underline{\qquad\qquad}, \underline{\qquad\qquad})$.
2. Compute the derivatives $Df(x, y)$ and $D\vec{r}(t)$, and then multiply the matrices together to obtain $\frac{df}{dt}$. How can you make your answer only depend on t (not x or y)?
3. The chain rule states that $D(f \circ \vec{r})(t) = Df(\vec{r}(t))D\vec{r}(t)$. Explain why we write $Df(\vec{r}(t))$ instead of $Df(x, y)$.

If you'd like to make sure you are correct, try the following. Replace x and y in $f = x^2 + xy$ with what they are in terms of t, and then just use first-semester calculus to find df/dt. Is it the same?

Problem 6.31 Suppose $f(x, y, z) = x + 2y + 3z^2$ and $x = u + v$, $y = 2u - 3v$, and $z = uv$. Our goal is to find how much f changes if we were to change u (so $\partial f/\partial u$) or if we were to change v (so $\partial f/\partial v$). Try doing this problem without looking at the steps below, but instead try to follow the patterns in the previous problem on your own.

See 14.4: 7-12 for more practice.

1. Rewrite the equations for x, y, and z in vector form $\vec{r}(u, v) = (x, y, z)$.
2. Compute the derivatives $Df(x, y, z)$ and $D\vec{r}(u, v)$, and then multiply them together. Notice that since this composite function has 2 inputs, namely u and v, we should expect to get two columns when we are done.
3. What are $\partial f/\partial u$ and $\partial f/\partial v$? [Hint: remember the columns are the partial derivatives.]

Problem 6.32 Let $\vec{F}(s, t) = (2s + t, 3s - 4t, t)$ and $s = 3pq$ and $t = 2p + q^2$. This means that changing p and/or q should cause $\vec{F}$ to change. Our goal is to find $\partial\vec{F}/\partial p$ and $\partial\vec{F}/\partial q$. Note that since $\vec{F}$ is a vector-valued function, the two partial derivatives should be vectors. Try doing this problem without looking at the steps below, but instead try to follow the patterns in the previous problems.

1. Rewrite the parametric equations for s and t in vector form.

2. Compute $D\vec{F}(s,t)$ and the derivative of your vector function from part 1, and then multiply them together to find the derivative of $\vec{F}$ with respect to p and q. How many columns should we expect to have when we are done multiplying matrices?

3. What are $\partial\vec{F}/\partial p$ and $\partial\vec{F}/\partial q$?

Review Suppose $f(x,y) = x^2 + 3xy$ and $(x,y) = \vec{r}(t) = (3t, t^2)$. Compute both $Df(x,y)$ and $D\vec{r}(t)$. Then explain how you got your answer by writing what you did in terms of partial derivatives and regular derivatives. See [5] for an answer.

Problem 6.33: General Chain Rule Formulas Complete the following:

See 14.4: 13-24 for more practice. Practice these problems by using matrix multiplication. The examples problems in the text use a "branch diagram," which is just a way to express matrix multiplication without having to introduce matrices.

1. Suppose that $w = f(x,y,z)$ and that x,y,z are all function of one variable t (so $x = g(t), y = h(t), z = k(t)$). Use the chain rule with matrix multiplication to explain why

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt}.$$

which is equivalent to writing

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

[Hint: Rewrite the parametric equations for x, y, and z in vector form $\vec{r}(t) = (x,y,z)$ and compute $Dw(x,y,z)$ and $D\vec{r}(t)$.]

2. Suppose that $R = f(V,T,n,P)$, and that V,T,n,P are all functions of x. Give a formula (similar to the above) for $\dfrac{dR}{dx}$.

Problem 6.34 Suppose $z = f(s,t)$ and s and t are functions of u, v and w. Use the chain rule to give a general formula for $\partial z/\partial u$, $\partial z/\partial v$, and $\partial z/\partial w$.

Make sure you practice problems 14.4: 13-24. Use matrix multiplication, rather than the "branch diagram" referenced in the text.

Review If $w = f(x,y,z)$ and x,y,z are functions of u and v, obtain formulas for $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$. See [6] for an answer.

---

[5] We have $Df(x,y) = \begin{bmatrix} 2x+3y & 3y\end{bmatrix}$ and $D\vec{r}(t) = \begin{bmatrix} 3 \\ 2t\end{bmatrix}$. We just computed f_x and f_y, and dx/dt and dy/dt, which gave us $Df(x,y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y\end{bmatrix}$ and $D\vec{r}(t) = \begin{bmatrix} dx/dt \\ dy/dt\end{bmatrix}$.

[6] We have $Df(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{bmatrix}$. The parametrization $\vec{r}(u,v) = (x,y,z)$ has derivative $D\vec{r} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{bmatrix}$. The product is $D(f(\vec{r}(u,v))) = \begin{bmatrix} \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} & \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}\end{bmatrix}$. The first column is $\frac{\partial f}{\partial u}$, and the second column is $\frac{\partial f}{\partial v}$.

You've now got the key ideas needed to use the chain rule in all dimensions. The chain rule shows up many places in upper-level math, physics, and engineering courses as the key tool needed to develop new formulas. The following problem will show you one such use, namely how you can use the general chain rule to get an extremely quick way to perform implicit differentiation from first-semester calculus.

Problem 6.35 Suppose $z = f(x, y)$. If z is held constant, this produces a level curve. As an example, if $f(x, y) = x^2 + 3xy - y^3$ then $5 = x^2 + 3xy - y^3$ is a level curve. Our goal in this problem is to find dy/dx in terms of partial derivatives of f.

See 14.4: 25-32 to practice using the formula you developed. To practice the idea developed in this problem, show that if $w = F(x, y, z)$ is held constant at $w = c$ and we assume that $z = f(x, y)$ depends on x and y, then $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$. This is done on page 798 at the bottom.

1. Suppose $x = x$ and $y = y(x)$, so y is a function of x. We can write this in parametric form as $\vec{r}(x) = (x, y(x))$. We now have $z = f(x, y)$ and $\vec{r}(x) = (x, y(x))$. Compute both $Df(x, y)$ and $D\vec{r}(x)$ symbolically. Don't use the function $f(x, y) = x^2 + 3xy - y^3$ until you get to part 4 below.
2. Use the chain rule to compute $D(f(\vec{r}(x)))$. What is dz/dx (i.e., df/dx)?
3. Since z is held constant, we know that $dz/dx = 0$. Use this fact, together with part 2 to explain why $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$.
4. For the curve $5 = x^2 + 3xy - y^3$, use this formula to compute dy/dx.

Problem: Optional Suppose $\vec{F}(u, v) = (3u - v, u + 2v, 3v)$, $\vec{G}(x, y, z) = (x^2 + z, 4y - x)$, and $\vec{r}(t) = (t^3, 2t + 1, 1 - t)$. We want to examine $\vec{F}(\vec{G}(\vec{r}(t)))$. This means that $\vec{F} \circ \vec{G} \circ \vec{r}$ is a function from $\mathbb{R}^n \to \mathbb{R}^m$ for what n and m? Similar to first-semester calculus, since we have several functions nested inside of each other, we'll just need to apply the chain rule twice. Our goal is to find $d\vec{F}/dt$. Try to do this problem without looking at the steps below.

1. Compute $D\vec{F}(u, v)$, $D\vec{G}(x, y, z)$, and $D\vec{r}(t)$.
2. Use the chain rule (matrix multiplication) to find the derivative of $\vec{F}$ with respect to t. What size of matrix should we expect for the derivative? See [7] for an answer.

---

[7] The requested derivatives are

$$D\vec{F}(u,v) = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}, D\vec{G}(x,y,z) = \begin{bmatrix} 2x & 0 & 1 \\ -1 & 4 & 0 \end{bmatrix}, D\vec{r}(t) = \begin{bmatrix} 3t^2 \\ 2 \\ -1 \end{bmatrix}.$$

The product of these matrices is

$$\begin{aligned} \frac{d\vec{F}}{dt} &= D(\vec{F}(\vec{G}(\vec{r}(t)))) = D\vec{F}(\vec{G}(\vec{r}(t)))D\vec{G}(\vec{r}(t))D\vec{r}(t) \\ &= \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2x & 0 & 1 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 3t^2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 18xt^2 + 3t^2 - 11 \\ 6xt^2 - 6t^2 + 15 \\ 24 - 9t^2 \end{bmatrix} \\ &= \begin{bmatrix} 18(t^3)t^2 + 3t^2 - 11 \\ 6(t^3)t^2 - 6t^2 + 15 \\ 24 - 9t^2 \end{bmatrix}. \end{aligned}$$

The final step comes from noting that $x = t^3$, $y = 2t + 1$, and $z = 1 - t$, so we replace x with t^3 so that all variables are in terms of t.

6.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 7

Kinematics - The Geometry of Motion

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Develop formulas for the velocity and position of a projectile, given the acceleration. Then show how to find the range, maximum height, and flight time of the projectile.

2. Decompose acceleration as the sum of a vector parallel to the velocity plus a vector orthogonal to the velocity, obtaining the formula $\vec{a} = a_T\vec{T} + a_N\vec{N}$. Use this decomposition to find the tangential and normal components of acceleration and obtain the TNB frame.

3. Use the product rule to (1) prove that a vector valued function of constant length is orthogonal to its derivative, and (2) prove that $\vec{a} = \frac{d|\vec{v}|}{dt}\vec{T} + \kappa|\vec{v}|^2\vec{N}$.

4. Compute and describe geometrically the concepts of curvature κ, radius of curvature ρ, and center of curvature. Show how to obtain several different formulas for computing the curvature.

You'll have a chance to teach your examples to your peers prior to the exam.

I have created a YouTube play list to go along with this section. There are 11 videos, each 4-6 minutes long.

- YouTube play list for 07 - Kinematics - The Geometry Of Motion.
- A PDF copy of the finished product (so you can follow along on paper).

7.1 Projectile Motion

Have you ever dropped a rock from the top of a waterfall, or skipped a rock across a lake. This section explores some simple connections between position, velocity, and acceleration. If we wanted to send a rocket to space, or shoot a missile across an ocean, the same principles will apply. If we know how much thrust a rocket provides (the acceleration), we can determine the velocity of our rocket at any time along its path. This means we can predict the flight path of the rocket? To make a good flight plan, we'd need to know how to determine position and velocity from acceleration. That's the content of this first section.

Review If $y'(t) = 3t^2 + 12e^{2t}$ (the velocity) and $y(0) = 2$ (the initial height), then what is $y(t)$? See footnote [1] for an answer.

To solve the next problem, we need to remember that acceleration is the derivative of velocity, and that velocity is the derivative of position We've already seen that these facts hold true for vector-valued functions as well, which means

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t), \quad \vec{v}(t) = \frac{d}{dt}\vec{r}(t).$$

Problem 7.1 Consider a rocket in space (so we can neglect air resistance and gravity). The rocket's boosters apply the acceleration $\vec{a}(t) = (2t, 8e^{-t})$ m/s^2. The rocket's initial velocity is $\vec{v}(0) = (4, 5)$ m/s and the initial position is $\vec{r}(0) = (1, 16)$ m. Use this information to determine the position of the rocket after t seconds.

[Hint: Do the review problem if you're stuck. Then for this problem, integrate each component of acceleration to get the velocity, but don't forget that there are two arbitrary constants introduced when you integrate. You can use the initial velocity to figure out what value these constants should be. Then repeat the process to find position from velocity.]

Suppose we fire a projectile (like a pumpkin) from a cannon. The projectile leaves the cannon with an initial speed v_0, at an angle of α above the x-axis. All the motion in this problem occurs within a plane, and we'll use x and y to represent motion in that plane. Our goal is to find the velocity $\vec{v}(t)$ and position $\vec{r}(t)$ of the projectile at any time t. We'll need some assumptions prior to solving.

There is actually an annual pumpkin tossing competition on the east coast. See www.punkinchunkin.com for more details.

- We assume the only force acting on the object is the force due to gravity. In particular we will neglect air resistance.
- We'll assume the projectile is shot over a small enough distance that we can assume gravity only pulls the object straight down.
- Most branches of science use the letter g to represent the magnitude of the vertical component of acceleration, so we can write the acceleration of the projectile as

$$\vec{a}(t) = (0, -g) \qquad \text{or} \qquad \vec{a}(t) = 0\mathbf{i} - g\mathbf{j}.$$

- We'll use the approximations $g \approx 9.8$ m/s^2 or $g \approx 32$ ft/s^2.

You've probably heard before that when you throw a baseball to a friend, the path of the baseball is parabolic. The next problem proves this. If you feel shaky on getting a Cartesian equation from a parametrization, please tackle this review problem, otherwise, jump straight to the problem.

Review The function $\vec{r}(t) = (2t + 3, 4t^2 + 7t + 5)$ is a parametrization of a plane curve. Give a Cartesian equation of the curve. See [2] for an answer.

Problem 7.2 Suppose we fire a projectile from the point (x_0, y_0) with an initial velocity of $\vec{v}(0) = (v_{x_0}, v_{y_0})$. Assume that gravity is the only force acting on the object so that $\vec{a}(t) = (0, -g)$.

Click here to watch a YouTube video.

You can practice finding position from velocity and acceleration with problems 13.2: 11-18, and especially 13.2: 29.

[1] Integrate to get $y(t) = t^3 + 6e^{2t} + C$. Since $y(0) = 2$, we know $2 = 0 + 6(1) + C$, which gives $C = -4$. So the height is $y(t) = t^3 + 6e^{2t} - 4$.

[2] Solve the first equation involving x to obtain $t = \frac{x-3}{2}$, a Cartesian equation is $y = 4\left(\frac{x-3}{2}\right)^2 + 7\left(\frac{x-3}{2}\right) + 5$.

1. Show that the velocity at any time t is $\vec{v}(t) = (c_1, -gt + c_2)$. Show how to use the initial velocity to state the constants c_1 and c_2.
2. Show that the position at any time t is $\vec{r}(t) = (v_{x_0}t + c_3, -\frac{1}{2}gt^2 + v_{y_0}t + c_4)$ and find the constants c_3 and c_4.
3. Finish by eliminating the parameter t to give a Cartesian equation of the projectile's path. This will prove that the path of the particle is parabolic.

If a projectile starts at (x_0, y_0), we can move the origin to this point. As long as we are not trying to gauge the location of two projectiles simultaneously, we could always make the origin $(0,0)$ our starting point. We make the following definitions for a projectile that starts at $(0,0)$ and hits the ground at $(R,0)$.

- The range R is the horizontal distance traveled by the projectile.
- The flight time is the amount of time the projectile is in the air. It is the time t at which $\vec{r}(t) = (R, 0)$.
- The maximum height is the largest y value obtained by the projectile.

The next problem has you develop formulas for the range, flight time, and maximum height.

Problem 7.3 Answer the following questions. Assume we fire a projectile from the origin, which means the acceleration, velocity, and position are

Watch a YouTube video.

$$\vec{a}(t) = (0, -g), \quad \vec{v}(t) = (v_{x_0}, -gt + v_{y_0}), \quad \vec{r}(t) = (v_{x_0}t, -\frac{1}{2}gt^2 + v_{y_0}t).$$

1. Find the time to max height and the flight time.
2. Show that the maximum height is $y_{\max} = \dfrac{v_{y_0}^2}{2g}$ and then show that the range is $R = \dfrac{2v_{x_0}v_{y_0}}{g}$.
3. If the initial speed is v_0, with a firing angle of α above the horizontal, rewrite v_{x_0} and v_{y_0} in terms of v_0 and α, and then state the range in terms of v_0 and α.

The next problem comes from your text. (See section 13.2.) Try it without reading the text. It's a fun application of the ideas above.

Problem 7.4 An archer stands at ground level and shoots an arrow at an object which is 90 feet away in the horizontal direction and 74 ft above ground. The arrow leaves the bow at about 6 ft above ground level (not the origin). The archer wants the arrow to hit the target at the peak of its parabolic path. For the purposes of this problem, Let $g = 32\text{ft/s}^2$. What initial speed v_0 and firing angle α are needed to achieve this result? [Hint: This is much easier to solve if you first find v_{x_0} and v_{y_0}, the horizontal and vertical components of the velocity. You may want to move the origin as well, so that you can use the formulas above.]

This problem was created around the opening ceremony of the Barcelona Spain Olympics. Antonio Rebollo was the archer, but he didn't try to hit the flame at the peak of the flight. You can watch a YouTube video of the opening ceremony by following the link.
See 13.2: 19-28 for more practice.

7.2 Components Of Acceleration

Problem 7.5 Consider the two planar curves

$$\vec{r}_1(t) = (t, 2t) \quad \text{and} \quad \vec{r}_2(t) = (\cos t, \sin t).$$

When a curve travels through the xy plane, we call it a planar curve. When a curve travels through xyz space, we call it a space curve.

1. For each curve, draw the curve in the xy plane and compute the velocity $\vec{v}(t) = \frac{d\vec{r}}{dt}$ and acceleration $\vec{a}(t) = \frac{d\vec{v}}{dt}$.
2. For each curve, compute the speed $|\vec{v}(t)|$ and show that the speed is constant.
3. Use the web to google "define accelerate." You should see there are two definitions of the word accelerate. If you are driving in a car, and the speedometer remains constant, are you accelerating? Explain why the answer to this question depends on which definition you use for the word "accelerate." In our class, we will always use the scientific definition.

It's quite common that the scientific definition and colloquial definition of a word mean different things.

While driving a car, we have two main ways to cause a change in our velocity. The gas pedal and brakes allow us to change our speed. This acceleration acts parallel to our direction of motion. The steering wheel allows us to turn, and this creates an acceleration that points in the direction we are turning. This means that the total acceleration (change in velocity) we feel is formed as the sum of a part acting in the direction of motion and a part acting in the direction we are turning. The next few problems develop this idea.

If we want to design a roller coaster, build an F15 fighter plane, send a satellite in orbit, or construct anything that doesn't move in a straight line, we need to understand how acceleration causes us to leave a straight path. We may still be speeding up or slowing down (tangential acceleration), but now we'll have a component that veers us off the straight path. We'll call this normal acceleration, and it's orthogonal to the velocity.

Back in the vector chapter, we practiced writing a force $\vec{F}$ as the sum of the component parallel to a displacement $\vec{d}$ and the component orthogonal to $\vec{d}$. We could write this as $\vec{F} = \vec{F}_{||\vec{d}} + \vec{F}_{\perp\vec{d}}$. The parallel part came from a projection. The orthogonal part came from vector subtraction. If you've forgotten how to do this, please do this review problem.

Review Consider the force vector $\vec{F} = (0, -10)$, and displacement vector $\vec{(2, -1)}$. Compute the projection of $\vec{F}$ onto $\vec{d}$, and then write $\vec{F}$ as the sum of a vector parallel to $\vec{d}$ and a vector orthogonal to $\vec{d}$. See [3].

Problem 7.6 Consider the vectors $\vec{F} = (-2, 10)$ and $\vec{d} = (3, 4)$. We'd like to write $\vec{F}$ as the sum of a vector parallel to $\vec{d}$ plus a vector orthogonal to $\vec{d}$.

1. Compute $\vec{F}_{\|\vec{d}}$, the vector component of $\vec{F}$ that is parallel to $\vec{d}$.
2. Compute $\vec{F}_{\perp\vec{d}}$, the vector component of $\vec{F}$ that is orthogonal to $\vec{d}$.

[3] The projection is $\text{proj}_{\vec{d}}\vec{F} = \frac{10}{5}(2, -1) = (4, -2)$. This is the parallel component $\vec{F}_{||\vec{d}} = (4, -2)$. To get the orthogonal component, we know that $\vec{F} = \vec{F}_{||\vec{d}} + \vec{F}_{\perp\vec{d}}$. Vector subtraction gives $\vec{F}_{\perp\vec{d}} = \vec{F} - \vec{F}_{||\vec{d}} = (0, -10) - (4, -2) = (-4, -8)$. We now have

$$\vec{F} = \vec{F}_{\|\vec{d}} + \vec{F}_{\perp\vec{d}} = (4, -2) + (-4, -8).$$

3. Show that $\vec{F} = \vec{F}_{\|\vec{d}} + \vec{F}_{\perp\vec{d}}$. We call this decomposing a vector into the sum of a vector parallel to $\vec{d}$ and a vector orthogonal to $\vec{d}$.

Definition 7.1: Orthogonal Decomposition. Suppose $\vec{F}$ and $\vec{d}$ are vectors. We decompose a vector into the sum of a vector parallel to $\vec{d}$ and a vector orthogonal to $\vec{d}$ when we express $\vec{F}$ in the form

$$\vec{F} = \vec{F}_{\|\vec{d}} + \vec{F}_{\perp\vec{d}}$$

where $\vec{F}_{\|\vec{d}}$ is parallel to $\vec{d}$ and $\vec{F}_{\perp\vec{d}}$ is orthogonal to $\vec{d}$.

- We call $\vec{F}_{\|\vec{d}}$ the vector component of $\vec{F}$ parallel to $\vec{d}$.
- We call $\vec{F}_{\perp\vec{d}}$ the vector component of $\vec{F}$ orthogonal to $\vec{d}$.

Review Write $\vec{v} = (1, 2, 3)$ as the product of a magnitude and a unit vector. See [4] for an answer.

Problem 7.7 Consider the vectors $\vec{a} = (10, 5, 1)$ and $\vec{v} = (1, 2, -2)$.

1. Decompose $\vec{a}$ into the sum of a vector parallel to $\vec{v}$ and a vector orthogonal to $\vec{v}$. The definition is above if you didn't read it.
2. Compute the magnitude of each vector component. We call these magnitudes the scalar components of $\vec{a}$.
3. Write both of the vectors in your decomposition as the product of a magnitude and a unit vector (see the review problem if needed). Your work should show that we can write $\vec{a}$ in the form

$$\vec{a} = \underbrace{6}_{\text{mag}} \underbrace{\frac{(1, 2, -2)}{3}}_{\text{unit vector}} + \underbrace{\sqrt{90}}_{\text{mag}} \underbrace{\frac{(?, ?, ?)}{?}}_{\text{unit vector}} .$$

Definition 7.2: Tangential And Normal Components Of Acceleration, Unit Tangent Vector, Principle Unit Normal Vector. Let $\vec{v}(t)$ and $\vec{a}(t)$ be the velocity and acceleration of an object in motion. Suppose that

$$\vec{a} = \vec{a}_{\|\vec{v}} + \vec{a}_{\perp\vec{v}},$$

is a decomposition of $\vec{a}$ in to a vector parallel to the velocity and a vector orthogonal to the velocity. Letting $\vec{T}$ be the unit vector in the direction of the velocity, and writing each vector as a scalar times a unit vector gives us

$$\vec{a} = a_T\vec{T} + a_N\vec{N} \quad \text{where} \quad \vec{a}_{\|\vec{v}} = a_T\vec{T} \quad \text{and} \quad \vec{a}_{\perp\vec{v}} = a_N\vec{N}.$$

Note that a_T and a_N are scalars, and $\vec{T}$ and $\vec{N}$ are unit vectors.

[4] The magnitude of $\vec{v}$ is $|\vec{v}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$. A unit vector in the direction of $\vec{v}$ is simply $\frac{\vec{v}}{|\vec{v}|}$. Hence we have

$$(1, 2, 3) = \underbrace{\sqrt{14}}_{\text{mag}} \underbrace{\frac{(1, 2, 3)}{\sqrt{14}}}_{\text{unit vector}} .$$

- The unit tangent vector $\vec{T}$ is the unit vector pointing in the direction of $\vec{v}$.
- The principle unit normal vector $\vec{N}$ is the unit vector pointing in the same direction as $\vec{a}_{\perp\vec{v}}$. If $\vec{a}_{\perp\vec{v}}$ is the zero vector, then we don't define $\vec{N}$.
- The tangential component of acceleration, a_T, is the scalar needed so that $a_T\vec{T} = \vec{a}_{\|\vec{v}}$. Note that $a_T = \pm|\vec{a}_{\|\vec{v}}|$. It's positive if $\vec{a}_{\|\vec{v}}$ and $\vec{v}$ point in the same direction.
- The normal component of acceleration, a_N, is the magnitude of $\vec{a}_{\perp\vec{v}}$.

In summary, when we have decomposed $\vec{a}$ as $\vec{a} = \vec{a}_{\|\vec{v}} + \vec{a}_{\perp\vec{v}}$, then we have

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}, \quad \vec{N} = \frac{\vec{a}_{\perp\vec{v}}}{|\vec{a}_{\perp\vec{v}}|}, \quad a_T = \pm|\vec{a}_{\|\vec{v}}| = \frac{\vec{a}\cdot\vec{v}}{|\vec{v}|}, \quad a_N = |\vec{a}_{\perp\vec{v}}|.$$

Problem 7.8 Let's practice using the definitions above. For example, if we know $\vec{a}$ and $\vec{v}$, then we should be able to find a_T, a_N, $\vec{T}$ and $\vec{N}$. Similarly, if we know a_T, a_N, $\vec{T}$, and $\vec{N}$, then we should be able to state $\vec{a}$.

1. Suppose we know that $\vec{v} = (6, 3, 0)$ and $\vec{a} = (3, 4, 3)$. Use this to compute a_T, $\vec{T}$, a_N, and $\vec{N}$.
2. Suppose we know $a_T = 2$, $\vec{T} = \dfrac{(-2, 1)}{\sqrt{5}}$, $a_N = 3$, and $\vec{N} = \dfrac{(1, 2)}{\sqrt{5}}$. State $\vec{a}$.
3. Explain in general why we must always have $|\vec{a}|^2 = a_T^2 + a_N^2$.

Problem 7.9 Suppose an object moves through the plane following the curve $\vec{r}(t) = (t, 9 - t^2)$. We know that the velocity and acceleration are

$$\vec{v}(t) = (1, -2t) \quad \text{and} \quad \vec{a}(t) = (0, -2).$$

At time $t = 2$, the velocity and acceleration are $\vec{v}(2) = (1, -4)$ and $\vec{a}(2) = (0, -2)$. Compute $a_T(2)$, $\vec{T}(2)$, $a_N(2)$, and $\vec{N}(2)$, and then use your answer to write $\vec{a}(2)$ in the form $\vec{a} = a_T\vec{T} + a_N\vec{N}$.

Look at the two problems above. Do you see any connection between $\vec{T}$ and $\vec{N}$? What's similar, and what's different?

Problem 7.10 Suppose you have already computed the unit tangent vector for a curve in the plane and found at a specific time it equals $\vec{T} = (a, b)$.

See 13.4: 7-8 for more practice, and perhaps a hint.

1. State a nonzero vector that is orthogonal to (a, b). (Guess one, and use the dot product to check. If you're struggling because of the variables, find a nonzero vector that is orthogonal to $(2, 3)$.)
2. Let $\vec{r}(t) = (t, t^2)$. We then have $\frac{d\vec{r}}{dt} = (1, 2t)$ and $\vec{T}(t) = \frac{(1,2t)}{\sqrt{1+4t^2}}$. Without any more computations, guess the principle unit normal vector $\vec{N}(t)$?
3. Draw a picture of the curve. At $t = 1$ add to your picture the tangent vector $(1, 2)/\sqrt{5}$ and your guessed normal vector. (If your guess was off by a sign, tell us how to modify your guess.)

Observation 7.3. From the previous problems, we learn the following fact. If the tangent vector to a planar curve is $\vec{T}(t) = (a(t), b(t))$, then the principle unit normal vector is either $\vec{N}(t) = (-b(t), a(t))$ or $\vec{N}(t) = (b(t), -a(t))$. You just interchange the components, and then negate one of them. To determine which one to negate, draw a picture.

Problem 7.11 Consider the curve $\vec{r}(t) = (t^2, t)$. Compute $\vec{T}(t)$. Reverse the order and negate one of the component to find $\vec{N}(t)$. To know if you guessed the write component to negate, draw the curve and on your graph include these vectors at $t = 1$.

Problem 7.12 Consider the curve $y = \sin x$, parametrized by $r(t) = (t, \sin t)$. We know that $\vec{T}(t) = \dfrac{(1, \cos t)}{\sqrt{1 + \cos^2 t}}$.

See 13.4: 1-4 for more practice. Use the previous problems.

1. Draw the curve from $-\pi$ to π. Then on your graph draw $\vec{T}$ and $\vec{N}$ at $t = \pi/2$, $\pi/4$, $-\pi/4$.
2. What is $\vec{T}(t)$ at each of $t = \pi/2$, $\pi/4$, $-\pi/4$?
3. Show how to get $\vec{N}(t)$ at each of $t = \pi/2$, $\pi/4$, $-\pi/4$?

The previous three problems all involved planar curves. Finding the principle unit normal vector for planar curves is a simple as swapping the components of the unit tangent vector and negating one. However, for space curves this is no longer true. When working with space curves, we call the plane containing $\vec{T}$ and $\vec{N}$ the osculating plane, and its normal vector $\vec{B} = \vec{T} \times \vec{N}$ we call the binormal vector. This gives us the TNB frame for describing motion.

Definition 7.4: Binormal Vector And The TNB Frame. For a space curve $\vec{r}(t)$, with unit tangent vector $\vec{T}(t)$ and principle unit normal vector $\vec{N}(t)$, the binormal vector is the cross product

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t).$$

Together, the three vectors $\vec{T}$, $\vec{N}$, and $\vec{B}$, we call the TNB frame.

Let's carry through a single problem with a space curve where we compute $\vec{T}$, $\vec{N}$, and $\vec{B}$.

Problem 7.13 Consider the helix $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$. Find the unit tangent vector $\vec{T}(t)$, principle unit normal vector $\vec{N}(t)$, and the binormal vector $\vec{B}(t)$. You should find along the way that the speed is constant, namely $|\vec{v}(t)| = 5$ and that the tangential component of acceleration is $a_T = 0$.

See 13.4: 9-16 and 13.5: 9-16 (the relevant parts) for more practice.

In the previous problem, the speed (magnitude of the velocity) is constant. This led to a quick decomposition of $\vec{a}$, since we find that $\vec{a} = 0\vec{T} + a_N\vec{N}$, which means $\frac{d\vec{v}}{dt}$ and $\vec{v}$ are orthogonal. This leads to the following generalization.

Theorem 7.5. *If a vector valued function $\vec{v}(t)$ has constant length, then the vector $\vec{v}(t)$ and the derivative $\dfrac{d\vec{v}(t)}{dt}$ are always orthogonal, i.e. we always have*

$$\vec{v}(t) \cdot \frac{d\vec{v}(t)}{dt} = 0.$$

A vector valued function of constant length is always orthogonal to its derivative.

Problem 7.14: Proof of Theorem 7.5 Prove the theorem above. Here are some hints [as an alternative to watching the YouTube video].

Follow this link and watch a YouTube Video.

- We know that $\vec{v}(t)$ has constant length, so we write $|\vec{v}(t)| = c$ for some constant c.
- We need to get from a magnitude to a dot product. Look in your text for a way to relate magnitude to the dot product. See problem 2.17.
- After writing $|\vec{v}(t)| = c$ in terms of a dot product (squaring both sides may help), take the derivative of both sides. Apply the product rule to the dot product.
- Watch the YouTube video linked to on the right.

Once the speed is no longer constant, things get a lot messier. Ask me in class to show you what happens with the computations when you consider something like $\vec{r}(t) = (t, t^2, t^3)$. Things get ugly really fast, but luckily software can automate this. Luckily, when you're working with a planar curve the computations are quite simple (swap coordinates of the tangent vector and change a sign).

The next problem uses the theorem above to give an alternative way to compute $\vec{N}$ that does not require using an orthogonal decomposition, namely it shows that we can compute $\vec{N}$ from $\frac{d\vec{T}}{dt}$.

Problem 7.15 Let $\vec{r}$ be a parametrization of a curve.

Watch a YouTube Video.

1. How long is the *unit* tangent vector $\vec{T}(t)$?
2. Why is $\vec{T}$ always orthogonal to $\dfrac{d\vec{T}}{dt}$?
3. Suppose you've computed $\dfrac{d\vec{T}}{dt} = (t, -3, 2t)$. State $\vec{N}(t)$.

From the problem above, we see that $\vec{N}$ is the unit vector in the same direction as $\dfrac{d\vec{T}}{dt}$. Our new formula for $\vec{N}$ is

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}.$$

Review Find the area of a parallelogram with corners $(0, 0, 0)$, $(1, 0, 2)$, and $(1, 1, 3)$, and $(2, 1, 5)$. See [5].

Problem 7.16 Answer the following questions (this will review your knowledge of the dot and cross products).

1. What is $\vec{T} \cdot \vec{N}$? Explain. Then explain why $\vec{T} \cdot \vec{B} = 0$ and $\vec{N} \cdot \vec{B} = 0$.

[5] We find area of a parallelogram by the vectors that form the edges. Since one point is the origin (subtracting it from the 2nd and 3rd points won't change anything), we compute the cross product $(1, 0, 2) \times (1, 1, 3) = (-2, -1, 1)$. The magnitude of the cross product is the area $A = \sqrt{4 + 1 + 1} = \sqrt{6}$.

2. We know that $\vec{B} = \vec{T} \times \vec{N}$. Why is $\vec{B}$ is a unit vector? [What's the connection between the cross product and area. How long are $\vec{T}$ and $\vec{N}$?]

3. We defined $\vec{B} = \vec{T} \times \vec{N}$. This means that $\vec{N} \times \vec{T} = -\vec{B}$. Does $\vec{T} \times \vec{B}$ equal $\vec{N}$ or $-\vec{N}$? Explain.

We've now developed the TNB frame for an object in motion. Engineers will see this again when they study dynamics. Mathematicians who study differential geometry will use these ideas as well. Any time you want to analyze the forces acting on a moving object, the TNB frame may save the day. Chemists will encounter the TNB frame briefly when they study P-chem and the motion of subatomic particles.

7.3 Curvature, Derivatives With Respect To Length

So far we've developed a way to compute the TNB frame for an object in motion, as well as compute and the tangential component and normal components of acceleration. Most of the computations we've done involved vectors, namely $\vec{r}, \vec{v}, \vec{a}$, $\vec{T}$, $\vec{N}$, and $\vec{B}$. We've seen that the scalars a_T and a_N fit into the Pythagorean theorem to give us $|\vec{a}| = a_T^2 + a_N^2$. In this section we will analyze few more scalars to see how distance s and speed v, together with a scalar κ called curvature, give us excellent ways to analyze and understand a_T and a_N.

We'll need to be able to compute the distance, written $s(t)$, an object travels. We can use the vector $\vec{r}(t)$ to give us displacement, and we use arc length to get us distance. We did this in chapters 3 and 4 for curves in the plane. The next problem has you generalize this for space curves. We can always parameterize a space curve with $\vec{r}(t) = (x, y, z)$ (one input, 3 outputs).

Review A horse runs once around an elliptical track, which is parametrized by $\vec{r}(t) = (3\cos t, 4\sin t)$. Set up, do not solve, an integral formula that tells us the distance the horse traveled. What's the displacement? See [6] for an answer.

Problem 7.17 A space ship travels through the galaxy. Let $\vec{r}(t) = (x, y, z)$ be the position of the space ship at time t, with the earth at the origin $(0, 0, 0)$.

Watch a YouTube video.

Technically, we should write $\vec{r}(t) = (x(t), y(t), z(t))$. However, we already know that x, y, and z depend on t, hence we'll just leave the dependence on t off.

1. What are the velocity $\vec{v}(t)$ and speed $v(t)$ of the space ship at time t? Your answers should involve derivatives (such as $\frac{dx}{dt}$).

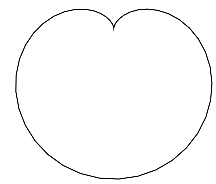
2. As the ship travels from time $t = a$ to time $t = b$, explain why the distance traveled (the arc length of the path followed) is

$$s = \int_a^b |\vec{v}(t)|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

In all our work that follows, we want to consider space curves that have nice smooth paths. What does this mean? We want to be able to compute tangent vectors at any point, so we will first require that a parametrization $\vec{r}$ be differentiable. However, the next problem shows this isn't enough.

[6] The velocity is $\vec{v}(t) = (3\sin t, -4\cos t)$. The speed is $v(t) = \sqrt{9\sin^2 t + 16\cos^2 t}$. The distance traveled is the arc length $s = \int_0^{2\pi} \left(\sqrt{9\sin^2 t + 16\cos^2 t}\right) dt$. Since the horse's initial and final position are equal, the displacement is zero. Arc length does not equal displacement.

Problem 7.18 We've encountered the polar curve $r = 1 - \sin\theta$ before, shown to the right. We can switch from polar to Cartesian using the coordinate transformation $x = r\cos\theta$ and $y = r\sin\theta$, which gives a parametrization for this curve as



$$\vec{r}(t) = ((1 - \sin t)\cos t, (1 - \sin t)\sin t).$$

1. Compute $\dfrac{d\vec{r}}{dt}$ and show that the velocity at $t = \frac{\pi}{2}$ is the zero vector.
2. Explain why there is no tangent vector to this curve at $t = \frac{\pi}{2}$.

We'd like to avoid paths that contain a cusp, because at a cusp the direction of motion changes rather abruptly. This can happen physically, but it requires the speed of an object to reach zero, i.e. the object stops moving, and starts moving in a completely new direction. Whenever the speed of an object reaches zero, cusps can happen. In addition, if the speed of an object reaches zero, we can't divide by it in our work that follows. To avoid this complication, we now make a formal definition that requires the path is differentiable, and the speed is never zero.

Definition 7.6: Smooth Curves. Let $\vec{r}(t) = (x, y, z)$ for $a \leq t \leq b$ be a parametrization of a curve C. We say that $\vec{r}$ is a smooth parametrization of C if both $\vec{r}$ is differentiable and $\vec{v}(t) \neq \vec{0}$ for $a \leq t \leq b$. If $\vec{r}$ is a smooth parameterization for C, then we say that C a smooth curve.

Problem 7.19 Consider the helical space curve C with parameterization

Watch a YouTube Video.
See 13.3: 1-10 for more practice.

$$\vec{r}(t) = (\cos t, \sin t, t) \quad \text{for} \quad 0 \leq t < \infty.$$

1. Is C a smooth curve? (Check the definition above, and make sure both parts of the definition are satisfied.)
2. Find the length of this space curve for $t \in [0, 2\pi]$. Compute any integrals.
3. Find the displacement from $t = 0$ to $t = 2\pi$. You should have a vector.
4. Now find the length of the space curve from $t = 0$ to time $t = t$.

In the previous problem, you developed a formula for the length of a curve from time $t = 0$ to any time $t = t$. Given any parametrization, we now have a function $s(t)$ that tells us how far we have traveled after t seconds.

Definition 7.7: Arc Length Parameter, Distance Traveled. Let $\vec{r}(t) = (x, y, z)$ for $a \leq t$ be a parametrization of a smooth space curve. The distance traveled from $t = a$ to $t = t$ is the quantity $s(t) = \displaystyle\int_a^t \left|\frac{d\vec{r}(\tau)}{d\tau}\right| d\tau$. We call $s(t)$ the arc length parameter. We'll often begin with $a = 0$, so we have

$$s(t) = \int_0^t \left|\frac{d\vec{r}(\tau)}{d\tau}\right| d\tau.$$

If we know how much time has elapsed, we can now compute the distance s traveled from that time. Before moving on, let's examine the derivative of $s(t)$. We'll see that $\frac{ds}{dt}$ is a quantity we already know. To do this, you'll need to make sure you review the fundamental theorem of calculus.

Problem 7.20 Let $\vec{r}(t) = (x, y, z)$ be a parametrization of a smooth space curve. Let $s(t) = \int_0^t \left|\frac{d\vec{r}(\tau)}{d\tau}\right| d\tau$, the arc length parameter.

You can remember $\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|$ as follows. We use the differential ds to represents a change in distance, and dt represents a change in time. So the speed of an object is the change in distance ds over the change in time dt.

1. Explain why $\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|$.
2. Explain why $s(t)$ is an increasing function.

[Hint: For the first part, look up the fundamental theorem of calculus in your calculus text. In Thomas's calculus, the 12th edition, you'll want page 327. To answer why is s increasing, why do we know that $\frac{ds}{dt}$ is always positive? Review the definition of "smooth" given above.]

Because $s(t)$ is an increasing function, it has an inverse. This means that if we know how far we've traveled, then we can find the time it took to arrive there. So we have t as a function of s, namely $t(s)$, and we can compute the derivative dt/ds. This allows us to compute derivatives of $\vec{r}$ with respect to s. When we take a derivative with respect to s, we ask how much a curve changes as we increase length, rather than increasing time. Computing a derivative with respect to length will allow us to study the geometry of a curve without reference to the speed at which we travel along a curve.

Problem 7.21 Suppose that $\vec{r}$ is a function of t, and t is an invertible function of s. Assume as well that all functions are differentiable.

1. What rule tells us that $\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds}$ as well as $\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$.
2. Use one of the formulas above to prove that $\frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt}$.

Problem 7.22 Suppose an object moves along the space curve given by $\vec{r}(t) = (a\cos t, a\sin t, bt)$.

See 13.3: 1-10 for more practice.

1. State the velocity $\vec{v}(t)$, and the speed $v(t)$. What is the quantity ds/dt?
2. State the unit tangent vector $\vec{T}(t)$.
3. Compute $\frac{d\vec{r}}{ds}$, the derivative of $\vec{r}$ with respect to arc length. Leave your answer in terms of t. [Hint: Use the formula from the problem above.]

Your work above proves that $\vec{T} = \frac{d\vec{r}}{ds}$. As we progress in this chapter, we'll be computing more derivatives with respect to s, instead of t. Did you notice in the previous problem that to compute a derivative with respect to s, you just compute the regular derivative with respect to t, and then divide by the speed. This is precisely how we will compute all derivatives with respect to s, namely compute the derivative with respect to t and then divide by speed. Please do the following review problem to make sure you've got down what $\frac{d}{ds}$ means.

Review Suppose $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$. Compute $\vec{v}$, ds/dt, $d\vec{r}/ds$, $\vec{T}$, and $d\vec{T}/ds$. See [7] for an answer.

It's time to see how derivatives with respect to s relate to the decomposition

$$\vec{a} = a_T\vec{T} + a_N\vec{N}.$$

Problem 7.23 We know that $\vec{a} = \dfrac{d\vec{v}}{dt}$. We also know that $\vec{v} = |\vec{v}|\vec{T}$. Put together, this means that $\vec{a} = \dfrac{d}{dt}\left(|\vec{v}|\vec{T}\right)$.

1. What differentiation rule tells us that $\vec{a} = \dfrac{d|\vec{v}|}{dt}\vec{T} + |\vec{v}|\dfrac{d\vec{T}}{dt}$?
2. Show that $\vec{a} = \dfrac{d|\vec{v}|}{dt}\vec{T} + |\vec{v}|^2\dfrac{d\vec{T}}{ds}$.

The above computation looks an awful lot like $\vec{a} = a_T\vec{T} + a_N\vec{N}$. The computation above however has the vector $\frac{d\vec{T}}{ds}$ instead of $\vec{N}$. Let's analyze the vector $\dfrac{d\vec{T}}{ds}$ in some problems below.

Problem 7.24 Sammy sits on a merry go round. He sits 3 feet from the center of the merry ground, and lets his big sister spin him around. We can parameterize Sammy's path with the vector equation $\vec{r}(t) = (3\cos t, 3\sin t)$, where t is in seconds.

1. Draw the curve for $0 \leq t \leq 2\pi$. Compute $\dfrac{d\vec{r}}{ds}$ and find the magnitude $\left|\dfrac{d\vec{r}}{ds}\right|$.
2. Compute $\dfrac{d\vec{T}}{ds}$ and find the magnitude $\left|\dfrac{d\vec{T}}{ds}\right|$. How is the magnitude of $\dfrac{d\vec{T}}{ds}$ related to the curve?
3. Now let $\vec{r}(t) = (5\cos t, 5\sin t)$. Guess the quantity $\left|\dfrac{d\vec{T}}{ds}\right|$, and then perform the computations to verify that you are correct.
4. End by stating the relationship between $\left|\dfrac{d\vec{T}}{ds}\right|$ and the radius of the circle parametrized by $\vec{r}(t) = (a\cos t, a\sin t)$

When we compute the derivative of $\vec{T}$ with respect to s, we ask the question, "How much does the direction we are currently moving change as we increase length?" If we travel along a circle with a large radius, then $|\frac{d\vec{T}}{ds}|$ is small because

[7] We have $\vec{v} = \dfrac{d\vec{r}}{dt} = (-3\sin t, 3\cos t, 4)$. The speed is $\dfrac{ds}{dt} = |\vec{v}| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$. We then compute $\dfrac{d\vec{r}}{ds} = \dfrac{d\vec{r}/dt}{ds/dt} = \dfrac{1}{5}(-3\sin t, 3\cos t, 4)$, which equals $\vec{T}$. We finally compute

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt} = \left(\frac{1}{ds/dt}\right)\frac{d\vec{T}}{dt} = \left(\frac{1}{5}\right)\frac{1}{5}(-3\cos t, -3\sin t, 0) = \frac{1}{25}(-3\cos t, -3\sin t, 0).$$

increasing length a little will not cause much change in direction. However, if the radius of the circle is small, then increasing length a little bit can cause a rather large change in direction, hence $|\frac{d\vec{T}}{ds}|$ is quite large. The previous problem shows that $|\frac{d\vec{T}}{ds}|$ is the reciprocal of the radius of the circle. For a general curve, the quantity $|\frac{d\vec{T}}{ds}|$ helps us understand the bends in a curve. Large values of $|\frac{d\vec{T}}{ds}|$ result in a bend with a tiny radius, and small values of $|\frac{d\vec{T}}{ds}|$ result in bends with a large radius. Because $\dfrac{d\vec{T}}{ds}$ and $\left|\dfrac{d\vec{T}}{ds}\right|$ show up a lot, let's give them a definition.

Definition 7.8: Curvature κ and The Curvature Vector $\vec{\kappa}$. Let $\vec{r}(t)$ be a smooth curve (so that $\vec{v}$ is never zero).

- The vector $\vec{\kappa} = \dfrac{d\vec{T}}{ds}$ we'll call the curvature vector. It measures how quickly the unit tangent vector changes as we increase length (not time).
- The number $\kappa = \left|\dfrac{d\vec{T}}{ds}\right|$ we'll call the curvature.

Problem 7.25 Suppose $\vec{r}(t) = (4\cos t, 4\sin t, 3t)$. Compute the curvature vector $\vec{\kappa}$ and show that the curvature is $\kappa = \frac{4}{25}$.

For any curve, we can approximate how rapidly the curve turns at a point by drawing a circle that best approximates the curve (kind of like a Taylor polynomial, only now we'll use a circle.) We want the circle to meet the curve $\vec{r}$ tangentially, and we want the curvature of the circle to match the curvature of the curve. Since the curvature of the circle must match the curvature of the curve, we know the radius of the circle and the curvature are inversely related. This gives the following definition.

Definition 7.9: Radius of Curvature ρ. When the curvature κ of a smooth curve is nonzero, we'll define the radius of curvature, written ρ, to be the reciprocal $\rho = \dfrac{1}{\kappa}$. The curvature and radius of curvature are inversely related.

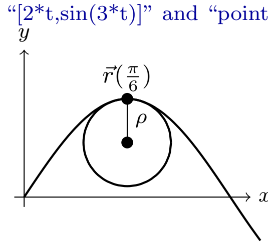
Watch a YouTube Video.

Problem 7.26 Consider the curve $\vec{r}(t) = (2t, \sin 3t)$. We can compute $\vec{v}(t) = (2, 3\cos t)$ and $\vec{T}(t) = \frac{(2,3\cos 3t)}{\sqrt{4+9\cos^2(3t)}}$. Using the quotient rule, we find that

$$\frac{d\vec{T}}{dt} = \frac{\sqrt{4+9\cos^2(3t)}(?,?) - (2,3\cos 3t)(?)}{4+9\cos^2(3t)}.$$

1. Fill in the blanks in the quotient rule above.
2. Compute $\frac{dT}{dt}(\frac{\pi}{6})$. Then find $\vec{\kappa}$, κ, and ρ at $t = \pi/6$.
3. The curve, together with the circle of curvature at $t = \pi/6$ is drawn to the right. Give the coordinates of the center of curvature (the center of the circle of curvature) at $t = \pi/6$. The picture is to scale, so you should find that the radius is a little less than 1/2.

Use this Sage link to check your work, and see if your picture is correct. You'll have to type the appropriate function in, so use "[2*t,sin(3*t)]" and "point=pi/6."



To find the center of curvature in the previous problem, it was simple because the normal vector pointed straight down. In general, to find the center of curvature, we need to know our location (a point on the circle), the direction to the center (the normal vector), and the radius of the circle (ρ). Try the following review problem before finding the center of curvature for a space curve in the next problem.

Review If you are standing at $(2, 1, -3)$ and move 6 units in the direction of the unit vector $(1/3, 2/3, -2/3)$, where are you? See [8] for an answer.

Problem 7.27 Consider the helix $\vec{r}(t) = (t, \sin t, \cos t)$. Find the curvature and radius of curvature at $t = \pi/2$. Then draw the curve for $0 \leq t \leq 2\pi$. On your graph, at $t = \pi/2$ draw the circle of curvature. And then compute the center of curvature at $t = \pi/2$. Guess the center of curvature at $t = \pi$?

[Hint: If you're struggling with how to get from the curve to the center of curvature, please do the review problem above. Don't forget to use the Sage link to the right. It gives you the pictures and answer so you can check your work.]

Use this Sage link to check your work, and see if your picture is correct. You'll have to type the appropriate function in, so use "[t,sin(t),cos(t)]" and "point=pi/2."

When a civil engineering team builds a road, they have to pay attention to the curvature of the road. If the curvature of the road is too large, accidents will happen and the civil engineering team will be liable. How do they make sure the curvature never gets to large? They use the circle of curvature. When they want to cause a road to turn, they'll find the center of curvature, send a surveyor out to the center, and then have the surveyor make sure that the road follows the circle of curvature for a short distance. They actually pace out the circle of curvature and then build the road along this circle for a hundred feet or so. Then, they recompute the radius of curvature (if they need the direction to change again), and pace out another circle. In this way, they can guarantee that the curvature never gets too large.

In the beginning of the chapter, we showed that

$$\vec{a}(t) = a_T\vec{T} + a_N\vec{N}.$$

Using the product rule on $\vec{a} = \frac{d}{dt}(|\vec{v}|\vec{T})$, we've also shown that

$$\vec{a} = \frac{d|\vec{v}(t)|}{dt}\vec{T} + |\vec{v}|^2\frac{d\vec{T}}{ds}.$$

Problem 7.28 Suppose an object is moving along a curve $\vec{r}(t)$ with velocity $\vec{v}(t)$. Prove that $a_T = \frac{d|\vec{v}(t)|}{dt}$ and then prove that $a_N = \kappa|\vec{v}|^2 = \frac{|\vec{v}|^2}{\rho}$.

[Note: You'll need to (1) explain why $\frac{d\vec{T}}{ds}$ and $\vec{N}$ point in the same direction, and then (2) explain why a_N is the magnitude of $|\vec{v}|^2\frac{d\vec{T}}{ds}$ and show this equals $\kappa|\vec{v}|^2$.]

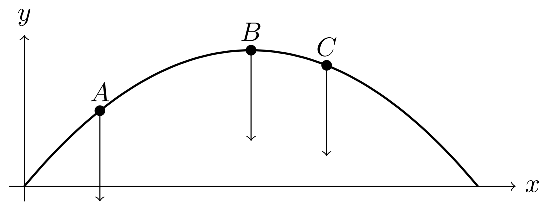
Engineers often use the formula $a_N = \frac{|\vec{v}|^2}{\rho}$, as ρ is a physical distance that they can measure.

Let's tackle two problems to help us see what the equations $a_T = \frac{d|\vec{v}(t)|}{dt}$ and $a_N = \kappa|\vec{v}|^2 = \frac{|\vec{v}|^2}{\rho}$ physically mean.

Problem 7.29 Consider the path an object follows after it has been fired from the origin, such as the path below.

See 13.5: 17-20 for more practice.

[8] We want to start at $(2, 1, -3)$ and move $6(1/3, 2/3, -2/3) = (2, 4, -4)$. We just add the vectors, giving $(4, 5, -7) = (2, 1, -3) + 6(1/3, 2/3, -2/3)$.



Recall that the acceleration $\vec{a}(t) = (0, -g)$ acts straight down for any time t.

1. At the three points A, B, and C, draw both $a_T\vec{T}$ and $a_N\vec{N}$. Use your pictures to determine if a_T is positive, negative, or zero at each point.
2. At each point, determine if the speed is increasing or decreasing.
3. What does the formula $a_T = \dfrac{d|\vec{v}(t)|}{dt}$ have to do with the above questions?

Problem 7.30 Imagine that you are riding as a passenger on a road and encounter a series of switchbacks (so the road starts to zigzag up the mountain). Right before each bend in the road, you see a yellow sign that tells you a U-turn is coming up, and that you should reduce your speed from 45 mi/hr to 15 mi/hr. Assume the largest curvature along the turn is κ. Recall that $a_N = \kappa|\vec{v}|^2$. The engineers of the road designed the road so that if you are moving at 15 mi/hr, then the normal acceleration will be at most A units.

1. Suppose that your driver (Ben) ignores the suggestion to slow down to 15 mi/hr. He keeps going 45 mi/hr through the turn. Had he slowed down, the max acceleration would be A. You're traveling 3 times faster than suggested. What will your maximum normal acceleration be? [It's more than $3A$.]
2. You yell at Ben to slow down (you don't want to die). So Ben decides to only slow to 30 mi/hr. He figures this means you'll only feel twice as much acceleration as A. Explain why this line of reasoning is flawed.
3. Ben gets frustrated by the fact that he has to slow down. He complains about the engineers who designed the road, and says, "they should have just built a larger corner so I could keep going 45." How much larger should the radius of the circle be so that you can travel 45 mi/hr instead of 15 mi/hr, and still feel the same acceleration A?
4. Which will cause the normal acceleration to decrease more, halving your speed or halving the curvature (doubling the radius)?

Problem 7.31 Give a complete proof, start to finish, that shows why

Watch a YouTube Video.

$$\vec{a}(t) = \frac{d}{dt}|\vec{v}|\vec{T} + \kappa|\vec{v}|^2\vec{N}.$$

We've done all the work in pieces in earlier problems, this problem just asks you to repeat the process as one entire whole.

Here's some hints.

- Rewrite the velocity $\vec{v}$ as a magnitude $|\vec{v}|$ times a direction $\vec{T}$, so $\vec{v} = |v|\vec{T}$.

- We know that $\vec{a}(t) = \frac{d}{dt}\vec{v}(t)$. Take the derivative of $\vec{v} = |\vec{v}|\vec{T}$ by using the product rule (on the scalar product $|\vec{v}|\vec{T}$).
- You should encounter the quantity $d\vec{T}/dt$. We know that $\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt}$. Use this to show that $\frac{d\vec{T}}{dt} = |\vec{v}|\kappa\vec{N}$?

Let's use the fact above to get a useful formula for the curvature.

Problem 7.32 Show that

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}.$$

[Hint: We know that $\vec{a} = a_T\vec{T} + a_N\vec{N} = \frac{d}{dt}|\vec{v}|\vec{T} + \kappa|\vec{v}|^2\vec{N}$. Cross both sides with $\vec{v}$. Why does $\vec{v} \times \vec{T} = \vec{0}$? Then take the magnitude of each side which should get you $|\vec{v} \times \vec{a}| = |\vec{v} \times |\vec{v}|^2\kappa\vec{N}|$. Why does $|\vec{v} \times |\vec{v}|^2\kappa\vec{N}| = |\vec{v}|^2\kappa|\vec{v} \times \vec{N}|$? You'll need to solve for κ and explain why $|\vec{v} \times \vec{N}| = |\vec{v}|$.]

We can use the above formula for curvature to get a quick way to compute the curvature of a function $y = f(x)$. If you use the previous problem, this formula falls out almost instantly. You'll see this formula in dynamics, and it shows up on the Fundamentals of Engineering exam (where you just have to use the formula, not prove where it comes from). This is a culminating idea from this chapter that you'll use again and again in engineering courses.

Problem 7.33 The function $y = f(x)$ can be given the parametrization $\vec{r}(x) = (x, f(x), 0)$, putting the curve in 3D so we can use the cross product. Use this parametrization (and the previous problem) to show that the curvature is See 13.4: 5.

$$\kappa(x) = \frac{|y''|}{(1 + (y')^2)^{3/2}},$$

and that the radius of curvature is

$$\rho(x) = \frac{(1 + (y')^2)^{3/2}}{|y''|}.$$

7.4 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 8

Line Integrals

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Describe how to integrate a function along a curve. Use line integrals to find the area of a sheet of metal with height $z = f(x, y)$ above a curve $\vec{r}(t) = (x, y)$ and the average value of a function along a curve.

2. Find the following geometric properties of a curve: centroid, mass, center of mass, inertia, and radii of gyration.

3. Compute the work (flow, circulation) and flux of a vector field along and across piecewise smooth curves.

4. Determine if a field is a gradient field (hence conservative), and use the fundamental theorem of line integrals to simplify work calculations.

You'll have a chance to teach your examples to your peers prior to the exam. Table 8.1 contains a summary of the key ideas for this chapter.

Surface Area	$\sigma = \int_C d\sigma = \int_C f ds = \int_a^b f \left\lvert \frac{d\vec{r}}{dt} \right\rvert dt$
Average Value	$\bar{f} = \frac{\int f ds}{\int ds}$
Work, Flow, Circulation	$W = \int_C d\text{Work} = \int_C (\vec{F} \cdot \vec{T}) ds = \int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy$
Flux	$\text{Flux} = \int_C d\text{Flux} = \int_C \vec{F} \cdot \vec{n} ds = \oint_C M dy - N dx$
Mass	$m = \int_C dm = \int_C \delta ds$
Centroid	$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\int x ds}{\int_C ds}, \frac{\int y ds}{\int_C ds}, \frac{\int z ds}{\int_C ds} \right)$
Center of Mass	$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\int x dm}{\int_C dm}, \frac{\int y dm}{\int_C dm}, \frac{\int z dm}{\int_C dm} \right)$
Fund. Thm of Line Int.	$f(B) - f(A) = \int_C \vec{\nabla} f \cdot d\vec{r}$

Table 8.1: A summary of the ideas in this unit.

I have created a YouTube playlist to go along with this chapter. Each video is about 4-6 minutes long.

- YouTube playlist for 08 - Line Integrals.
- A PDF copy of the finished product (so you can follow along on paper).

You'll also find the following links to Sage can help you speed up your time spent on homework. Thanks to Dr. Jason Grout at Drake university for contributing many of these (as well as being a constant help with editing, rewriting, and giving me great feedback). Thanks Jason.

- Sage Links
- Mathematica Notebook (If you have installed Mathematica)

If you would like homework problems from the text that line up with the ideas we are studying, please use the following tables.

Topic (11th ed.)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Line integrals	16.1	1-8, 9-22, 23-32			33-36
Work, Flow, Circulation, Flux	16.2	7-16, 25-28, 37-40	17-24, 29-30, 41-44	45-46	47-52
Gradient Fields	16.2	1-6			
Gradient Fields	14.5	1-8			
Potentials	16.3	1-12,13-24	25-33	34-38	

Topic (12th ed.)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Line integrals	16.1	1-8, 9-26, 33-42	27-32		43-46
Work, Flow, Circulation, Flux	16.2	7-12, 19-24, 31-36, 47-50	13-18, 25-30, 37-38,	51-54	55-60
Gradient Fields	16.2	1-6			
Gradient Fields	14.5	1-10			
Potentials	16.3	1-12,13-24	25-33	34-38	

8.1 Work, Flow, Circulation, and Flux

Now that we can describe motion, let's turn our attention to the work done by a vector field as we move through the field. Work is a transfer of energy.

- A tornado picks might pick up a couch, and applies forces to the couch as the couch swirls around the center. Work transfers the energy from the tornado to the couch, giving the couch it's kinetic energy.
- When an object falls, gravity does work on the object. The work done by gravity converts potential energy to kinetic energy.
- If we consider the flow of water down a river, it's gravity that gives the water its kinetic energy. We can place a hydro electric dam next to a river to capture a lot of this kinetic energy. Work transfers the kinetic energy of the river to rotational energy of the turbine, which eventually ends up as electrical energy available in our homes.

When we study work, we are really studying how energy is transferred. This is one of the key components of modern life.

Let's start with simple review. Recall that the work done by a vector field $\vec{F}$ through a displacement $\vec{d}$ is the dot product $\vec{F} \cdot \vec{d}$.

Review An object moves from $A = (6, 0)$ to $B = (0, 3)$. Along the way, it encounters the constant force $\vec{F} = (2, 5)$. How much work is done by $\vec{F}$ as the object moves from A to B? See [1].

[1] The displacement is $B - A = (-6, 3)$. The work is $\vec{F} \cdot \vec{d} = (2, 5) \cdot (-6, 3) = -12 + 15 = 3$.

Problem 8.1 An object moves from $A=(6,0)$ to $B=(0,3)$. A parametrization of the object's path is $\vec{r}(t)=(-6,3)t+(6,0)$ for $0\leq t\leq 1$.

1. For $0\leq t\leq .5$, the force encountered is $\vec{F}=(2,5)$. For $.5\leq t\leq 1$, the force encountered is $(2,6)$. How much work is done in the first half second? How much work is done in the last half second? How much total work is done?
2. If we encounter a constant force $\vec{F}$ over a small displacement $d\vec{r}$, explain why the work done is $dW=\vec{F}\cdot d\vec{r}=F\cdot\frac{d\vec{r}}{dt}dt$.
3. Suppose that the force constantly changes as we move along the curve. At t, we'll assume we encountered the force $F(t)=(2,5+2t)$, which we could think of as the wind blowing stronger and stronger to the north. Explain why the total work done by this force along the path is
$$W=\int\vec{F}\cdot d\vec{r}=\int_0^1(2,5+2t)\cdot(-6,3)dt.$$
Then compute this integral. It should be slightly larger than the first part.
4. (Optional) If you are familiar with the units of energy, complete the following. What are the units of $\vec{F}$, $d\vec{r}$, and dW.

You can visualize what's happening in this problem as follows. Attach a clothesline between the points (maybe representing two trees in your backyard). Put a cub scout space derby ship on the clothesline. Then the wind starts to blow. As the ship moves along the clothesline, the wind changes direction.

We know how to compute work when we move along a straight line. Prior to problem 2.21 on page 16, we made the following statements.

> If a force F acts through a displacement d, then the most basic definition of work is $W=Fd$, the product of the force and the displacement. This basic definition has a few assumptions.
>
> - The force F must act in the same direction as the displacement.
> - The force F must be constant throughout the displacement.
> - The displacement must be in a straight line.

We used the dot product to remove the first assumption, and we showed in problem 2.21 that the work is simply the dot product
$$W=\vec{F}\cdot\vec{r},$$
where $\vec{F}$ is a force acting through a displacement $\vec{r}$. The previous problem showed that we can remove the assumption that $\vec{F}$ is constant, by integrating to obtain
$$W=\int\vec{F}\cdot d\vec{r}=\int_a^b F\cdot\frac{d\vec{r}}{dt}dt,$$
provided we have a parametrization of $\vec{r}$ with $a\leq t\leq b$. The next problem gets rid of the assumption that $\vec{r}$ is a straight line.

Problem 8.2 Suppose that we move along the circle C parametrized by $\vec{r}(t)=(3\cos t,3\sin t)$. As we move along C, we encounter a rotational force $\vec{F}(x,y)=(-2y,2x)$.

Watch a YouTube video about work.

1. Draw C. Then at several points on the curve, draw the vector field $\vec{F}(x, y)$. For example, at the point $(3, 0)$ you should have the vector $\vec{F}(3, 0) = (-2(0), 2(3)) = (0, 6)$, a vector sticking straight up 6 units. Are we moving with the vector field, or against the vector field?

2. Explain why we can state that a little bit of work done over a small displacement is $dW = \vec{F} \cdot d\vec{r}$. Why does it not matter that $\vec{r}$ moves in a straight line?

3. Since a little work done by $\vec{F}$ along a small bit of C is $dW = \vec{F} \cdot d\vec{r}$, we know that the total work done is $\int dW = \int \vec{F} \cdot d\vec{r}$. This gives us the work as

$$W = \int_C (-2y, 2x) \cdot d\vec{r} = \int_0^{2\pi} (-2(3\sin t), 2(3\cos t)) \cdot (-3\sin t, 3\cos t)dt.$$

Complete the integral, showing that the work done by $\vec{F}$ along C is 36π.

We put the C under the integral $\int_C$ to remind us that we are integrating along the curve C. This means we need to get a parametrization of the curve C, and give bounds before we can integrate with respect to t.

Definition 8.1. The work done by a vector field $\vec{F}$, along a curve C with parametrization $\vec{r}(t)$ for $a \leq t \leq b$, is

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt.$$

If we let $\vec{F} = (M, N)$ and we let $\vec{r}(t) = (x, y)$, so that $d\vec{r} = (dx, dy)$, then we can write work in the differential form

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C (M, N) \cdot (dx, dy) = \int_C M dx + N dy.$$

Review Consider the curve $y = 3x^2 - 5x$ for $-2 \leq x \leq 1$. Give a parametrization of this curve. See [2].

Problem 8.3 Consider the parabolic curve $y = 4 - x^2$ for $-1 \leq x \leq 2$, and the vector field $\vec{F}(x, y) = (2x + y, -x)$.

Please use this Sage link to check your work.

1. Give a parametrization $\vec{r}(t)$ of the parabolic curve that starts at $(-1, 3)$ and ends at $(2, 0)$. See the review problem above if you need a hint.

2. Compute $d\vec{r}$ and state dx and dy. What are M and N in terms of t?

3. Compute the work done by $\vec{F}$ to move an object along the parabola from $(-1, 3)$ to $(2, 0)$ (i.e. compute $\int_C M dx + N dy$).

Click the link to check your answer with Sage.

4. How much work is done by $\vec{F}$ to move an object along the parabola from $(2, 0)$ to $(-1, 3)$. In other words, if you traverse along a path backwards, how much work is done?

---

[2] Whenever you have a function of the form $y = f(x)$, you can always use $x = t$ and $y = f(t)$ to parametrize the curve. So we can use $\vec{r}(t) = (t, 3t^2 - 5t)$ for $-2 \leq t \leq 1$ as a parametrization.

Problem 8.4 Again consider the vector field $\vec{F}(x,y) = (2x+y, -x)$. In the previous problem we considered how much work was done by $\vec{F}$ as an object moved along the the parabolic curve $y = 4 - x^2$ for $-1 \leq x \leq 2$. We now want to know how much work is done to move an object along a straight line from $(-1,3)$ to $(2,0)$.

1. Give a parametrization $\vec{r}(t)$ of the straight line curve that starts at $(-1,3)$ and ends at $(2,0)$. The opening problem of this chapter shows you the key to parameterizing a straight line segment. Make sure you give bounds for t.

2. Compute $d\vec{r}$ and state dx and dy. What are M and N in terms of t?

3. Compute the work done by $\vec{F}$ to move an object along the straight line path from $(-1,3)$ to $(2,0)$. Check your answer with Sage.

4. Optional (we'll discuss this in class if you don't have it). How much work does it take to go along the closed path that starts at $(2,0)$, follows the parabola $y = 4 - x^2$ to $(-3,1)$, and then returns to $(2,0)$ along a straight line. Show that this total work is $W = -9$.

When you enter your curve in Sage, remember to type the times symbol in "(3*t-1, ...)". Otherwise, you'll get an error.

The examples above showed us that we can compute work along any curve. All we have to do is parametrize the curve, take a derivative, and then compute $dW = \vec{F} \cdot d\vec{r}$. This gives us a little bit of work along a curve, and we sum up the little bits of work (integrate) to find the total work.

In the examples above, the vector fields represented forces. However, vector fields can represent much more than just forces. The vector field might represent the flow of water down a river, or the flow of air across an airplane wing. When we think of the vector field as a velocity field, then we mights ask the question, how much of the fluid flows along our curve. Alternately, we might ask how much of the fluid flows across our curve. These two problems lead to flow along a curve, and flux across a curve. Flow along a curve is directly related to the lift of an airplane wing (which occurs when the flow along the top of the wing is different than the flow below the wing). The flux across a curve will quickly take us to powering a wind mill as wind flows across the surface of a blade (once we hit 3D integrals).

Review If the unit tangent vector is $\vec{T} = \dfrac{(3,4)}{5}$, give two unit vectors that are orthogonal to $\vec{T}$. See [3].

Problem 8.5 We used the formulas

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C (M,N) \cdot (dx, dy)$$

to compute the work done by $\vec{F}$ along a curve C parametrized by $\vec{r} = (x,y)$.

1. Explain why $W = \int_C \vec{F} \cdot \vec{T} ds$. [Why does $\vec{T} ds = d\vec{r}$? Look up $\vec{T}$ in the last chapter.]

2. We know that $\vec{T} = \dfrac{(dx, dy)}{\sqrt{(dx)^2 + (dy)^2}}$. Suppose $\vec{n}$ is a unit normal vector to the curve. Give two options for $\vec{n}$. [Hint: Look at the review problem.]

Watch a YouTube video about flow and circulation.

Watch a YouTube video about flux.

[3] We just reverse the order and change a sign to get $\vec{N}_1 = \dfrac{(-4,3)}{5}$ and $\vec{N}_1 = \dfrac{(4,-3)}{5}$ as the orthogonal vectors.

3. We know $\vec{T}ds = (dx, dy)$. Why does $\vec{n}ds$ equal $(dy, -dx)$ or $(-dy, dx)$?
4. The integral $W = \int_C \vec{F} \cdot \vec{T}ds$ measures how much of the vector field flows along the curve. What does the integral $\Phi = \int_C \vec{F} \cdot \vec{n}ds$ measure?

Problem 8.6: Intro to Flux Consider the curve $\vec{r}(t) = (5\cos t, 5\sin t)$, and the vector field $\vec{F}(x, y) = (3x, 3y)$. This is a radial field that pushes things straight outwards (away from the origin).

1. Compute the work $W = \displaystyle\int_C (M, N) \cdot (dx, dy)$ and show that is equals zero. (Can you give a reason why it should be zero?)

   See Sage for the work calculation.

2. To get a normal vector, we could change (dx, dy) to $(dy, -dx)$ or to $(-dy, dx)$. Compute both $\displaystyle\int_C (M, N) \cdot (dy, -dx)$ and $\displaystyle\int_C (M, N) \cdot (-dy, dx)$. (They should differ by a sign.) Both integrals measure the flow of the field across the curve, instead of along the curve.
3. If we want flux to measure the flow of a vector field outwards across a curve, then the flux of this vector field should be positive. Which vector, $(dy, -dx)$ or $(-dy, dx)$, should we choose above for n.

   See Sage for the flux computation

4. (Challenge, we'll discuss in class.) Suppose $\vec{r}$ is a counterclockwise parametrization of a closed curve. The outward normal vector would always point to the right as you move along the curve. Prove that $(dy, -dx)$ always points to the right of the curve. [Hint: If you want a right pointing vector, what should $\vec{B} = \vec{T} \times \vec{n}$ always equal (either $(0, 0, 1)$ or $(0, 0, -1)$). Use the fact that $\vec{B} \times \vec{T} = \vec{n}$ to get $\vec{n}$.]

Definition 8.2: Flow, Circulation, and Flux. Suppose C is a smooth curve with parametrization $\vec{r}(t) = (x, y)$. Suppose that $\vec{F}(x, y)$ is a vector field that represents the velocity of some fluid (like water or air).

- Watch a YouTube video about flow and circulation.
- Watch a YouTube video about flux.

- We say that C is closed curve if C begins and ends at the same point.
- We say that C is a simple curve if C does not cross itself.
- The flow of $\vec{F}$ along C is the integral

$$\text{Flow} = \int_C (M, N) \cdot (dx, dy) = \int_C M dx + N dy.$$

- If C is a simple closed curve parametrized counter clockwise, then the flow of $\vec{F}$ along C is called circulation, and we write

$$\text{Circulation} = \oint_C M dx + N dy.$$

Any time you see a circle around an integral, it means that you're integrating along a closed curve.

- The flux of $\vec{F}$ across C is the flow of the fluid across the curve (an area/second). If C is a simple closed curve parametrized counter clockwise, then the outward flux is the integral

$$\text{Flux} = \Phi = \oint_C (M, N) \cdot (dy, -dx) = \oint_C M dy - N dx.$$

Problem 8.7 Consider the vector field $\vec{F}(x,y) = (2x+y, -x+2y)$. When you construct a plot of this vector field, you'll notice that it causes objects to spin outwards in the clockwise direction. Suppose an object moves counterclockwise around a circle C of radius 3 that is centered at the origin. (You'll need to parameterize the curve.)

If you haven't yet, please watch the YouTube videos for

- work,
- flow and circulation, and
- flux.

1. Should the circulation of $\vec{F}$ along C be positive or negative? Make a guess, and then compute the circulation $\oint_C Mdx + Ndy$. Whether your guess was right or wrong, explain why you made the guess.
2. Should the flux of $\vec{F}$ across C be positive or negative? Make a guess, and then actually compute the flux $\oint_C Mdy - Ndx$. Whether your guess was right or wrong, explain why you made the guess.
3. Please use this Sage link to check both computations.

We'll tackle more work, flow, circulation, and flux problems, as we proceed through this chapter.

8.2 Area and Average Value

In first semester calculus, we learned that the area under a function $f(x)$ above the x-axis is given by $A = \int_a^b f(x)dx$. The quantity $dA = f(x)dx$ represents a small bit of area whose length is dx and whose height is $f(x)$. To get the total area, we just added up the little bits of area, which is why

$$A = \int dA = \int_a^b f(x)dx.$$

Problem 8.8 Consider the surface in space that is below the function $f(x,y) = 9 - x^2 - y^2$ and above the curve C parametrized by $\vec{r}(t) = (2\cos t, 3\sin t)$ for $t \in [0, 2\pi]$. Think of this region as a metal plate that has been stood up with its base on C where the height above each spot is given by $z = f(x,y)$.

Watch a YouTube video. See Sage for a picture of this sheet.

1. Draw the curve C in the xy-plane. If we cut the curve up into lots of tiny bits, explain why the length of each bit is approximately
$$ds = \sqrt{(-2\sin t)^2 + (3\cos t)^2}dt.$$
See Problem 6.27
2. Explain why the area of the metal sheet that lies above C and under f is given by the integral
$$\sigma = \int_C f ds = \int_0^{2\pi} (9 - (2\cos t)^2 - (3\sin t)^2)\sqrt{(-2\sin t)^2 + (3\cos t)^2}dt.$$
You'll need to explain why a little bit of surface area equals $d\sigma = f ds$.
3. Find the surface area of the metal sheet. [Use technology to do this integral.]

This Sage worksheet will compute the integral.

Our results from the problem above suggest the following definition.

Definition 8.3: Line Integral. Let f be a function and let C be a piecewise smooth curve whose parametrization is $\vec{r}(t)$ for $t \in [a,b]$. We'll require that the composition $f(\vec{r}(t))$ be continuous for all $t \in [a,b]$. Then we define the line integral of f over C to be the integral

$$\int_C f ds = \int_a^b f(\vec{r}(t))\frac{ds}{dt}dt = \int_a^b f(\vec{r}(t))\left|\frac{d\vec{r}}{dt}\right|dt.$$

The line integral is also called the path integral, contour integral, or curve integral.

Notice that this definition suggests the following four steps. These four steps are the key to computing any line integral.

When we ask you to set up a line integral, it means that you should do steps 1–3, so that you get an integral with a single variable and with bounds that you could plug into a computer or complete by hand.

1. Start by getting a parametrization $\vec{r}(t)$ for $a \leq t \leq b$ of the curve C.
2. Find the speed by computing the velocity $\dfrac{d\vec{r}}{dt}$ and then the speed $\left|\dfrac{d\vec{r}}{dt}\right|$.
3. Multiply f by the speed, and replace each x, y, and/or z with what it equals in terms of t.
4. Integrate the product from the previous step. Practice doing this by hand on every problem, unless it specifically says to use technology. Some of the integrals are impossible to do by hand.

You should use the Sage line integral calculator to check all your answers.

Problem 8.9 Let $f(x, y, z) = x^2 + y^2 - 2z$ and let C be two coils of the helix $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$, starting at $t = 0$. Remember that the parameterization means $x = 3\cos t$, $y = 3\sin t$, and $z = 4t$. Compute $\int_C f ds$. [You will have to find the end bound yourself. How much time passes to go around two coils?]

See 16.1: 9-32. Some problems give you a parametrization, some expect you to come up with one on your own.

Check your answer with Sage.

Problem 8.10 Consider the function $f(x, y) = 3xy + 2$. Let C be a circle of radius 4 centered at the origin. Compute $\int_C f ds$. [You'll have to come up with your own parameterization.]

To practice matching parameterizations to curves, try 16.1:1-8.

Check your answer with Sage.

Problem 8.11 Let $f(x, y, z) = x^2 + 3yz$. Let C be the straight line segment from $(1, 0, 0)$ to $(0, 4, 5)$. Compute $\int_C f ds$.

If you've forgotten how to parametrize line segments, see 2.9.

Check your answer with Sage.

Problem 8.12 Let $f(x, y) = x^2 + y^2 - 25$. Let C be the portion of the parabola $y^2 = x$ between $(1, -1)$ and $(4, 2)$. We want to compute $\int_C f ds$.

See 5.21 if you forgot how to parametrize plane curves.

Check your answer with Sage.

1. Draw the curve C and the function $f(x, y)$ on the same 3D xyz axes.
2. Without computing the line integral $\int_C f ds$, determine if the integral should be positive or negative. Explain why this is so by looking at the values of $f(x, y)$ at points along the curve C. Is $f(x, y)$ positive, negative, or zero, at points along C?
3. Parametrize the curve and set up the line integral $\int_C f ds$. [Hint: if you let $y = t$, then $x =$? What bounds do you put on t?]
4. Use technology to compute $\int_C f ds$ to get a numeric answer. Was your answer the sign that you determined above?

Work, flow, circulation, and flux are all examples of line integrals. Remember that work, flow, and circulation are

$$W = \int_C (\vec{F} \cdot \vec{T}) ds = \int_C (M, N) \cdot (dx, dy) = \int_C M dx + N dy,$$

while the formula for flux is

$$\Phi = \int_C (\vec{F} \cdot \vec{n}) ds = \int_C (M, N) \cdot (dy, -dx) = \int_C M dy + N dx.$$

Do you see how these are both the line integral of a function $f = \vec{F} \cdot \vec{T}$ or $f = \vec{F} \cdot \vec{n}$ along a curve C. The function f inside the integrand does not have to represent the height of a sheet. We'll use it to represent lots of things. Let's practice two more work/flux problems, to sharpen our skills with these concepts.

Problem 8.13 Let $\vec{F} = (-y, x + y)$ and C be the triangle with vertices $(2, 0)$, $(0, 2)$, and $(0, 0)$.

Watch a YouTube video. Also, see Sage for a picture.

1. Look at a drawing of C and the vector field (see margin for the Sage link). We'll move along the triangle in a counter clockwise manner. Without doing any computations, for each side of the triangle make a guess to determine if the flow along that edge is positive, negative, or zero. Similarly, guess the sign of the flux along each edge. Explain.

2. Obtain three parameterizations for the edges of the triangle. One of the parameterizations is $\vec{r}(t) = (0, -2)t + (0, 2)$.

3. Now find the counterclockwise circulation (work) done by $\vec{F}$ along C. You'll have three separate calculations, one for each side. We'll do the flux computation in class. Check your work on each piece with the Sage calculator.

Problem 8.14 Consider the vector field $\vec{F} = (2x - y, x)$. Let C be the curve that starts at $(-2, 0)$, follows a straight line to $(1, 3)$, and then back to $(-2, 0)$ along the parabola $y = 4 - x^2$.

See Sage. Think of an airplane wing as you solve this problem.

1. Look at a drawing of C and the vector field (see margin for the Sage link). If we go counterclockwise around C, for each part of C, guess the signs of the counterclockwise circulation and the flux (positive, negative, zero).

2. Find the flux of $\vec{F}$ across C. There are two curves to parametrize. Make sure you traverse along the curves in the correct direction. [Hint: You should get integer values along both parts. Check your work with Sage, but make sure you show us how to do the integrals by hand.]

Ask me in class to change the vector fields above, and examine what happens with different vectors fields. In particular, it's possible to have any combination of values for circulation and flux. We'll be able to use technology to rapidly compute many values.

8.3 Average Value

The concept of averaging values together has many applications. In first-semester calculus, we saw how to generalize the concept of averaging numbers together to get an average value of a function. We'll review both of these concepts. Later, we'll generalize average value to calculate centroids and center of mass.

Problem 8.15 Suppose a class takes a test and there are three scores of 70, five scores of 85, one score of 90, and two scores of 95. We will calculate the average class score, $\bar{s}$, four different ways to emphasize four ways of thinking about the averages. We are emphasizing the pattern of the calculations in this problem, rather than the final answer, so it is important to write out each calculation completely in the form $\bar{s} = \underline{\qquad}$ before calculating the number $\bar{s}$.

1. Compute the average by adding 11 numbers together and dividing by the number of scores. Write down the whole computation before doing any arithmetic.

$\bar{s} = \frac{\sum \text{values}}{\text{number of values}}$

2. Compute the numerator of the fraction in the previous part by multiplying each score by how many times it occurs, rather than adding it in the sum that many times. Again, write down the calculation for $\bar{s}$ before doing any arithmetic.

$\bar{s} = \frac{\sum(\text{value}\cdot\text{weight})}{\sum \text{weight}}$

3. Compute $\bar{s}$ by splitting up the fraction in the previous part into the sum of four numbers. This is called a "weighted average" because we are multiplying each score value by a weight.

$\bar{s} = \sum(\text{value} \cdot (\% \text{ of stuff}))$

4. Another way of thinking about the average $\bar{s}$ is that $\bar{s}$ is the number so that if all 11 scores were the same value $\bar{s}$, you'd have the same sum of scores. Write this way of thinking about these computations by taking the formulas for $\bar{s}$ in the first two parts and multiplying both sides by the denominator.

$(\text{number of values})\bar{s} = \sum \text{values}$
$(\sum \text{weight})\bar{s} = \sum(\text{value} \cdot \text{weight})$

In the next problem, we generalize the above ways of thinking about averages from a discrete situation to a continuous situation. You did this in first-semester calculus when you did average value using integrals.

Problem 8.16 Suppose the price of a stock is \$10 for one day. Then the price of the stock jumps to \$20 for two days. Our goal is to determine the average price of the stock over the three days.

1. Why is the average stock price not \$15?

2. Let $f(t) = \begin{cases} 10 & 0 < t < 1 \\ 20 & 1 < t < 3 \end{cases}$, the price of the stock for the three-day period. Draw the function f, and find the area under f where $t \in [0, 3]$.

3. Now consider the constant function $y = \bar{f}$, where $\bar{f}$ is the average value of the function $f(t)$. The area under $\bar{f}$ over $[0, 3]$ is simply width times height, or $(3 - 0)\bar{f}$. What should $\bar{f}$ equal so that the area under $\bar{f}$ over $[0, 3]$ matches the area under f over $[0, 3]$.

4. We found a constant $\bar{f}$ so that the area under $\bar{f}$ matched the area under f. In other words, we solved the equation below for $\bar{f}$:

$$\int_a^b \bar{f} dx = \int_a^b f dx$$

Solve for $\bar{f}$ symbolically (without doing any of the integrals). This quantity we call the average value of f over $[a, b]$.

5. The formula for $\bar{f}$ in the previous part resembles at least one of the ways of calculating averages from Problem 8.15. Which ones and why?

Ask me in class about the "ant farm" approach to average value.

Problem 8.17 Consider the elliptical curve C given by the parametrization $\vec{r}(t) = (2\cos t, 3\sin t)$. Let f be the function $f(x, y) = 9 - x^2 - y^2$.

Watch a YouTube video.

1. Draw the surface f in 3D. Add to your drawing the curve C in the xy plane. Then draw the sheet whose area is given by the integral $\int_C f ds$.

2. What's the maximum height and minimum height of the sheet?

See problem 6.27.

3. We'd like to find a constant height $\bar{f}$ so that the area under f, above C, is the same as area under $\bar{f}$, above C. This height $\bar{f}$ is called the average value of f along C. Explain why the average value of f along C is

$$\bar{f} = \frac{\int_C f ds}{\int_C ds}.$$

Connect this formula with the ways of thinking about averages from Problem 8.15. [Hint: The area under $\bar{f}$ above C is $\int_C \bar{f} ds$. The area under f above C is $\int_C f ds$. Set them equal and solve for $\bar{f}$.]

Please read Isaiah 40:4 and Luke 3:5. These scriptures should help you remember how to find average value.

4. Use a computer to evaluate the integrals $\int_C f ds$ and $\int_C ds$, and then give an approximation to the average value of f along C. Is your average value between the maximum and minimum of f along C? Why should it be?

You can use the Sage line integral calculator.

Problem 8.18 The temperature $T(x, y, z)$ at points on a wire helix C given by $\vec{r}(t) = (\sin t, 2t, \cos t)$ is known to be $T(x, y, z) = x^2 + y + z^2$. What are the temperatures at $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$ and $t = 2\pi$? You should notice the temperature is constantly changing. Make a guess as to what the average temperature is (share with the class why you made the guess you made—it's OK if you're wrong). Then compute the average temperature of the wire using the integral formula from the previous problem. You can do all these computations by hand.

8.4 Physical Properties

A number of physical properties of real-world objects can be calculated using the concepts of averages and line integrals. We explore some of these in this section. Additionally, many of these concepts and calculations are used in statistics.

8.4.1 Centroids

Definition 8.4: Centroid. Let C be a curve. If we look at all of the x-coordinates of the points on C, the "center" x-coordinate, $\bar{x}$, is the average of all these x-coordinates. Likewise, we can talk about the averages of all of the y coordinates or z coordinates of points on the function ($\bar{y}$ or $\bar{z}$, respectively). The *centroid* of an object is the geometric center $(\bar{x}, \bar{y}, \bar{z})$, the point with coordinates that are the average x, y, and z coordinates.

Problem 8.19: Centroid Notice the word "average" in the definition of the centroid. Use the concept of average value to explain why the coordinates of the centroid are

Watch a YouTube video.

These are the formulas for the centroid.

$$\bar{x} = \frac{\int_C x ds}{\int_C ds}, \quad \bar{y} = \frac{\int_C y ds}{\int_C ds}, \quad \text{and} \quad \bar{z} = \frac{\int_C z ds}{\int_C ds}.$$

Notice that the denominator in each case is just the arc length $s = \int_C ds$.

Problem 8.20 Let C be the semicircular arc $\vec{r}(t) = (a \cos t, a \sin t)$ for $t \in [0, \pi]$. Without doing any computations, make an educated guess for the centroid $(\bar{x}, \bar{y})$ of this arc. Then compute the integrals given in problem 8.19 to find the actual centroid. Share with the class your guess, even if it was incorrect.

8.4.2 Mass and Center of Mass

Density is generally a mass per unit volume. However, when talking about a curve or wire, as in this chapter, it's simpler to let density be the mass per unit length. Sometimes an object is made out of a composite material, and the density of the object is different at different places in the object. For example, we might have a straight wire where one end is aluminum and the other end is copper. In the middle, the wire slowly transitions from being all aluminum to all copper. The centroid is the midpoint of the wire. However, since copper has a higher density than aluminum, the balance point (the center of mass) would not be at the midpoint of the wire, but would be closer to the denser and heavier copper end. In this section, we'll develop formulas for the mass and center of mass of such a wire. Such composite materials are engineered all the time (though probably not our example wire). In future mechanical engineering courses, you would learn how to determine the density δ (mass per unit length) at each point on such a composite wire.

Problem 8.21: Mass Suppose a wire C has the parameterization $\vec{r}(t)$ for $t \in [a,b]$. Suppose the wire's density (mass per unit length) at a point (x,y,z) on the wire is given by the function $\delta(x,y,z)$.

Watch a YouTube video.

1. Consider an extremely small portion of the curve at $t = t_0$ of length ds. Explain why the mass of the small portion of the curve is $dm = \delta(\vec{r}(t_0))ds$.

2. Explain why the mass m of the wire is given by the formulas below (explain why each equal sign is true):

$$m = \int_C dm = \int_C \delta ds = \int_a^b \delta(\vec{r}(t)) \left|\frac{d\vec{r}}{dt}\right| dt.$$

Problem 8.22 A wire lies along the straight segment from $(0,2,0)$ to $(1,1,3)$. The wire's density (mass per unit length) at a point (x,y,z) is $\delta(x,y,z) = x+y+z$.

1. Is the wire heavier at $(0,2,0)$ or at $(1,1,3)$?

2. What is the total mass of the wire? [You'll need to parameterize the line as your first step—see Problem 2.9 if you need a refresher.]

The center of mass of an object is the point where the object balances. In order to calculate the x-coordinate of the center of mass, we average the x-coordinates, but we weight each x-coordinate with its mass. Similarly, we can calculate the y and z coordinates of the center of mass.

Wikipedia has some interesting applications of center of mass.

The next problem helps us reason about the center of mass of a collection of objects. Calculating the center of mass of a collection of objects is important, for example, in astronomy when you want to calculate how two bodies orbit each other.

Problem 8.23 Suppose two objects are positioned at the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. Our goal in this problem is to understand the difference between the centroid and the center of mass.

1. Find the centroid of two objects.

2. Suppose both objects have the same mass of 2 kg. Find the center of mass.

3. If the mass of the object at point P_1 is 2 kg, and the mass of the object at point P_2 is 5 kg, will the center of mass be closer to P_1 or P_2? Give a physical reason for your answer before doing any computations. Then find the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the two points. [Hint: You should get $\bar{x} = \frac{2x_1+5x_2}{2+5}$.]

Problem 8.24 This problem reinforces what you just did with two points in the previous problem. However, it now involves two people on a seesaw. Ignore the mass of the seesaw in your work below (pretend it's an extremely light seesaw, so its mass is negligible compared to the masses of the people).

See Wikipedia for a seesaw picture.

1. My daughter and her friend are sitting on a seesaw. Both girls have the same mass of 30 kg. My wife stands about 1 m behind my daughter. We'll measure distance in this problem from my wife's perspective. We can think of my daughter as a point mass located at $(1\text{m}, 0)$ whose mass is 30 kg. Suppose her friend is located at $(5\text{m}, 0)$. Suppose the kids are sitting just right so that the seesaw is perfectly balanced. That means the the center of mass of the girls is precisely at the pivot point of the seesaw. Find the distance from my wife to the pivot point by finding the center of mass of the two girls.

2. My daughter's friend has to leave, so I plan to take her place on the seesaw. My mass is 100 kg. Her friend was sitting at the point $(5, 0)$. I would like to sit at the point $(a, 0)$ so that the seesaw is perfectly balanced. Without doing any computations, is $a > 5$ or $a < 5$? Explain.

3. Suppose I sit at the spot $(x, 0)$ (perhaps causing my daughter or I to have a highly unbalanced ride). Find the center of mass of the two points $(1, 0)$ and $(x, 0)$ whose masses are 30 and 100, respectively (units are meters and kilograms).

4. Where should I sit so that the seesaw is perfectly balanced (what is a)?

Problem 8.25: Center of mass In problem 8.23, we focused on a system with two points (x_1, y_1) and (x_2, y_2) with masses m_1 and m_2. The center of mass in the x direction is given by

Watch a YouTube video.

$$\bar{x} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2} = \frac{\sum_{i=1}^{2} x_i m_i}{\sum_{i=1}^{2} m_i}$$

1. If we consider a system with 3 points, what formula gives the center of mass in the x direction? What if there are 4 points, 5 points, or n points?

2. Suppose now that we have a wire located along a curve C. The density of the wire is known to be $\delta(x, y, z)$ (which could be different at different points on the curve). Imagine cutting the wire into a thousand or more tiny chunks. Each chunk would be centered at some point (x_i, y_i, z_i) and have length ds_i. Explain why the mass of each little chunk is $dm_i \approx \delta ds_i$.

3. Give a formula for the center of mass in the y direction of the thousands of points (x_i, y_i, z_i), each with mass dm_i. [This should almost be an exact copy of the first part.] Then explain why

$$\bar{y} = \frac{\int_C y dm}{\int_C dm} = \frac{\int_C y \delta ds}{\int_C \delta ds}.$$

Ask me in class to show you another way to obtain the formula for center of mass. It involves looking at masses weighted by their distance (called a moment of mass). Many of you will have already seen an idea similar to this in statics, but in that class you are talking about moments of force, not moments of mass.

For quick reference, the formulas for the centroid of a wire along C are

$$\bar{x} = \frac{\int_C x ds}{\int_C ds}, \quad \bar{y} = \frac{\int_C y ds}{\int_C ds}, \quad \text{and} \quad \bar{z} = \frac{\int_C z ds}{\int_C ds}. \quad \text{(Centroid)}$$

If the wire has density δ, then the formulas for the center of mass are

$$\bar{x} = \frac{\int_C x dm}{\int_C dm}, \quad \bar{y} = \frac{\int_C y dm}{\int_C dm}, \quad \text{and} \quad \bar{z} = \frac{\int_C z dm}{\int_C dm}, \quad \text{(Center of mass)}$$

where $dm = \delta ds$. Notice that the denominator in each case is just the mass $m = \int_C dm$.

The quantity $\int_C x dm$ is sometimes called the first moment of mass about the yz-plane (so $x = 0$). Notationally, some people write $M_{yz} = \int_C x ds$. Similarly, we could write $M_{xz} = \int_C y dm$ and $M_{xy} = \int_C z dm$. With this notation, we could write the center of mass formulas as

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right).$$

We'll often use the notation $(\bar{x}, \bar{y}, \bar{z})$ to talk about both the centroid and the center of mass. If no density is given in a problem, then $(\bar{x}, \bar{y}, \bar{z})$ is the centroid. If a density is provided, then $(\bar{x}, \bar{y}, \bar{z})$ refers to the center of mass. If the density is constant, it doesn't matter (the centroid and center of mass are the same, which is what the seesaw problem showed).

Problem 8.26 Suppose a wire with density $\delta(x, y) = x^2 + y$ lies along the curve C which is the upper half of a circle around the origin with radius 7.

1. Parametrize C (find $\vec{r}(t)$ and the domain for t).
2. Where is the wire heavier, at $(7, 0)$ or $(0, 7)$? [Compute δ at both.]
3. In problem 8.20, we showed that the centroid of the wire is $(\bar{x}, \bar{y}) = \left(0, \frac{2(7)}{\pi})\right)$. We now seek the center of mass. Before computing, will $\bar{x}$ change? Will $\bar{y}$ change? How will each change? Explain.
4. Set up the integrals needed to find the center of mass. Then use technology to compute the integrals. Give an exact answer (involving fractions), rather than a numerical approximation.
5. Change the radius from 7 to a, and guess what the center of mass will be. (This is why you need the exact answer above, not a numerical answer).

8.5 The Fundamental Theorem of Line Integrals

In this final section we'll return to the concept of work. Many vector fields are actually the derivative of a function. When this occurs, computing work along a curve is extremely easy. All you have to know is the endpoints of the curve, and the function f whose derivative gives you the vector field. This function is called a potential for a vector field. Once we are comfortable finding potentials, we'll show that the work done by such a vector field is the difference in the potential at the end points. This makes finding work extremely fast.

Definition 8.5: Gradients and Potentials. Let $\vec{F}$ be a vector field. A potential for the vector field is a function f whose derivative equals $\vec{F}$. So if $Df = \vec{F}$, then we say that f is a potential for $\vec{F}$. When we want to emphasize that the derivative of f is a vector field, we call Df the gradient of f and write $Df = \vec{\nabla} f$. If $\vec{F}$ has a potential, then we say that $\vec{F}$ is a gradient field.

Watch a YouTube Video.

The symbol $\vec{\nabla} f$ is read "the gradient of f" or "del f."

We'll quickly see that if a vector field has a potential, then the work done by the vector field is the difference in the potential. If you've ever dealt with kinetic and potential energy, then you hopefully recall that the change in kinetic energy is precisely the difference in potential energy. This is the reason we use the word "potential."

Problem 8.27 Let's practice finding gradients and potentials.

Watch a YouTube Video.

1. Let $f(x,y) = x^2 + 3xy + 2y^2$. Find the gradient of f, i.e. find $Df(x,y)$. Then compute $D^2 f(x,y)$ (you should get a square matrix). What are f_{xy} and f_{yx}?

2. Consider the vector field $\vec{F}(x,y) = (2x+y, x+4y)$. Find the derivative of $\vec{F}(x,y)$ (it should be a square matrix). Then find a function $f(x,y)$ whose gradient is $\vec{F}$ (i.e. $Df = \vec{F}$). What are f_{xy} and f_{yx}?

3. Consider the vector field $\vec{F}(x,y) = (2x+y, 3x+4y)$. Find the derivative of $\vec{F}$. Why is there no function $f(x,y)$ so that $Df(x,y) = \vec{F}(x,y)$? [Hint: what would f_{xy} and f_{yx} have to equal?]

See problem 6.16.

Based on your observations in the previous problem, we have the following key theorem.

Theorem 8.6. *Let $\vec{F}$ be a vector field that is everywhere continuously differentiable. Then $\vec{F}$ has a potential if and only if the derivative $D\vec{F}$ is a symmetric matrix. We say that a matrix is symmetric if interchanging the rows and columns results in the same matrix (so if you replace row 1 with column 1, and row 2 with column 2, etc., then you obtain the same matrix).*

Problem 8.28 For each of the following vector fields, find a potential, or explain why none exists.

If you haven't yet, please watch this YouTube video.

1. $\vec{F}(x,y) = (2x-y, 3x+2y)$

2. $\vec{F}(x,y) = (2x+4y, 4x+3y)$

3. $\vec{F}(x,y) = (2x+4xy, 2x^2+y)$

4. $\vec{F}(x,y,z) = (x+2y+3z, 2x+3y+4z, 2x+3y+4z)$

5. $\vec{F}(x,y,z) = (x+2y+3z, 2x+3y+4z, 3x+4y+5z)$

6. $\vec{F}(x,y,z) = (x+yz, xz+z, xy+y)$

7. $\vec{F}(x,y) = \left(\dfrac{x}{1+x^2} + \arctan(y), \dfrac{x}{1+y^2}\right)$

If a vector field has a potential, then there is an extremely simple way to compute work. To see this, we must first review the fundamental theorem of calculus. The second half of the fundamental theorem of calculus states,

> If f is continuous on $[a,b]$ and F is an anti-derivative of f, then $F(b) - F(a) = \int_a^b f(x)dx$.

If we replace f with f', then an anti-derivative of f' is f, and we can write, If f is continuously differentiable on $[a, b]$, then

$$f(b) - f(a) = \int_a^b f'(x)dx.$$

This last version is the version we now generalize.

Theorem 8.7 (The Fundamental Theorem of Line Integrals)**.** *Suppose f is a continuously differentiable function, defined along some open region containing the smooth curve C. Let $\vec{r}(t)$ be a parametrization of the curve C for $t \in [a, b]$. Then we have*

$$f(\vec{r}(b)) - f(\vec{r}(a)) = \int_a^b Df(\vec{r}(t))D\vec{r}(t)\ dt.$$

Watch a YouTube video.

Notice that if $\vec{F}$ is a vector field, and has a potential f, which means $\vec{F} = Df$, then we could rephrase this theorem as follows.

> Suppose $\vec{F}$ is a a vector field that is continuous along some open region containing the curve C. Suppose $\vec{F}$ has a potential f. Let A and B be the start and end points of the smooth curve C. Then the work done by $\vec{F}$ along C depends only on the start and end points, and is precisely
>
> $$f(B) - f(A) = \int_C \vec{F} \cdot d\vec{r} = \int_C Mdx + Ndy.$$
>
> The work done by $\vec{F}$ is the difference in a potential.

If you are familiar with kinetic energy, then you should notice a key idea here. Work is a transfer of energy. As an object falls, energy is transferred from potential energy to kinetic energy. The total kinetic energy at the end of a fall is precisely equal to the difference between the potential energy at the top of the fall and the potential energy at the bottom of the fall (neglecting air resistance). So work (the transfer of energy) is exactly the difference in potential energy.

Problem 8.29: Proof of Fundamental Theorem Suppose $f(x, y)$ is continuously differentiable, and suppose that $\vec{r}(t)$ for $t \in [a, b]$ is a parametrization of a smooth curve C. Prove that $f(\vec{r}(b)) - f(\vec{r}(a)) = \int_a^b Df(\vec{r}(t))D\vec{r}(t)\ dt$. [Let $g(t) = f(\vec{r}(t))$. Why does $g(b) - g(a) = \int_a^b g'(t)dt$? Use the chain rule (matrix form) to compute $g'(t)$. Then just substitute things back in.]

The proof of the fundamental theorem of line integrals is quite short. All you need is the fundamental theorem of calculus, together with the chain rule (6.8).

Problem 8.30 For each vector field and curve below, find the work done by $\vec{F}$ along C. In other words, compute the integral $\int_C Mdx + Ndy$ or $\int_C Mdx + Ndy + Pdz$.

Watch a YouTube video.

1. Let $\vec{F}(x, y) = (2x + y, x + 4y)$ and C be the parabolic path $y = 9 - x^2$ for x from -3 to 2. (See Sage for a picture.)
2. Let $\vec{F}(x, y, z) = (2x + yz, 2z + xz, 2y + xy)$ and C be the straight segment from $(2, -5, 0)$ to $(1, 2, 3)$. (See Sage for a picture.)

[Hint: If you parametrize the curve, then you've done the problem the HARD way. You don't need any parameterizations at all. Did you find a potential, and then plug in the end points?]

Problem 8.31 Let $\vec{F} = (x, z, y)$. Let C_1 be the curve which starts at $(1, 0, 0)$ and follows a helical path $(\cos t, \sin t, t)$ to $(1, 0, 2\pi)$. Let C_2 be the curve which starts at $(1, 0, 2\pi)$ and follows a straight line path to $(2, 4, 3)$. Let C_3 be any smooth curve that starts at $(2, 4, 3)$ and ends at $(0, 1, 2)$.

See Sage—C_1 and C_2 are in blue, and several possible C_3 are shown in red.

- Find the work done by $\vec{F}$ along each path C_1, C_2, C_3.
- Find the work done by $\vec{F}$ along the path C which follows C_1, then C_2, then C_3.
- If C is any path that can be broken up into finitely many smooth sub-paths, and C starts at $(1, 0, 0)$ and ends at $(0, 1, 2)$, what is the work done by $\vec{F}$ along C?

If you are parameterizing the curves, you're doing this the really hard way. Are you using the potential of the vector field?

In the problem above, the path we took to get from one point to another did not matter. The vector field had a potential, which meant that the work done did not depend on the path traveled.

Definition 8.8: Conservative Vector Field. We say that a vector field is conservative if the integral $\int_C \vec{F} \cdot d\vec{r}$ does not depend on the path C. We say that a curve C is piecewise smooth if it can be broken up into finitely many smooth curves.

Review Compute $\displaystyle\int \frac{x}{\sqrt{x^2+4}} dx$. See [4].

Problem 8.32 The gravitational vector field is directly related to the radial field $\vec{F} = \dfrac{(-x, -y, -z)}{(x^2 + y^2 + z^2)^{3/2}}$. Show that this vector field is conservative, by finding a potential for $\vec{F}$. Then compute the work done by an object that moves from $(1, 2, -2)$ to $(0, -3, 4)$ along ANY path that avoids the origin.

[See the review problem just before this if you're struggling with the integral.]

Problem 8.33 Suppose $\vec{F}$ is a gradient field. Let C be a piecewise smooth closed curve. Compute $\int_C \vec{F} \cdot d\vec{r}$ (you should get a number). Explain how you know your answer is correct.

8.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

[4] Let $u = x^2 + 4$, which means $du = 2xdx$ or $dx = \frac{du}{2x}$. This means

$$\int \frac{x}{\sqrt{x^2+4}} dx = \int \frac{x}{\sqrt{u}} \frac{du}{2x} = \frac{1}{2} \int u^{-1/2} du = \frac{1}{2} \frac{u^{1/2}}{1/2} = \sqrt{u} = \sqrt{x^2+4}.$$

Chapter 9

Optimization

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Explain the properties of the gradient, its relation to level curves and level surfaces, and how it can be used to find directional derivatives.

2. Find equations of tangent planes using the gradient and level surfaces. Use the derivative (tangent planes) to approximate functions, and use this in real world application problems.

3. Explain the second derivative test in terms of eigenvalues. Use the second derivative test to optimize functions of several variables.

4. Use Lagrange multipliers to optimize a function subject to constraints.

You'll have a chance to teach your examples to your peers prior to the exam. The following homework problems line up with the topics we will discuss in class.

Topic (11th ed.)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Directional Derivatives and the Gradient	14.5	1-22	23-32	33-36	
Tangent Planes and approximation	14.6	1-22	23-24, 47-58, 60-63	59	
2nd Derivative Test (use eigenvalues)	14.7	1-38	39-44, 49-52,	45-48, 53-64	65-70
Lagrange Multipliers	14.8	1-32	33-40	41-44	45-50

Topic (12th ed.)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Directional Derivatives and the Gradient	14.5	1-24	25-36	37-40	
Tangent Planes and approximation	14.6	1-22	23-24, 31-32, 49-62, 64-67	63	
2nd Derivative Test (use eigenvalues)	14.7	1-38	39-44, 49-60,	45-48, 61-68	69-74
Lagrange Multipliers	14.8	1-32	33-40	41-44	45-50

9.1 The Gradient

Recall from the previous unit that the derivative Df of a function $f : \mathbb{R}^n \to \mathbb{R}$ (one output dimension) is called the gradient of f, and written $\vec{\nabla} f$, when we want to emphasize that the derivative is a vector field.

Problem 9.1 Consider the functions $f(x, y) = 9 - x^2 - y^2$, $g(x, y) = 2x - y$, and $h(x, y) = \sin x \cos y$.

You'll want a computer to help you construct the graphs, particularly h. Please use the Mathematica introduction in Brainhoney. You could use Wolfram Alpha (use the links in the function chapter if you forgot how to graph).

1. Compute $\vec{\nabla} f(x, y)$. Then draw both $\vec{\nabla} f$ and several level curves of f on the same axes.
2. Compute $\vec{\nabla} g(x, y)$. Then draw both $\vec{\nabla} g$ and several level curves of g on the same axes.
3. Compute $\vec{\nabla} h(x, y)$. Then draw both $\vec{\nabla} h$ and several level curves of h on the same axes.
4. What relationships do you see between the gradient vector field and level curves?

See Sage. You can modify these commands to help in the plots below too.

When you present in class, be prepared to provide rough sketches of the level curves and gradients of each function.

The next few problems will focus on explaining why the relationships you saw are always true.

Problem 9.2 Suppose $\vec{r}(t)$ is a level curve of $f(x, y)$.

1. Suppose you know that at $t = 0$, the value of f at $\vec{r}(0)$ is 7. What is the value of f at $\vec{r}(1)$? [What does it mean to be on a level curve?]
2. As you move along the level curve $\vec{r}$, how much does f change? Use this to tell the class what $\frac{df}{dt}$ must equal.
3. At points along the level curve $\vec{r}$, we have the composite function $f(\vec{r}(t))$. Compute the derivative $\frac{df}{dt}$ using the chain rule.
4. Use your work from the previous parts to explain why the gradient always meets the level curve at a 90° angle. We say that the gradient is *normal* to level curves (i.e., a gradient vector is orthogonal to the tangent vector of the curve).

In the derivative chapter, we extended differential notation from $dy = f'dx$ to $d\vec{y} = D\vec{f}d\vec{x}$. The key idea is that a small change in the output variables is approximated by the product of the derivative and a small change in the input variables. As a quick refresher, if we have the function $z = f(x, y)$, then differential notation states that

$$dz = \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

where the derivative is $Df(x, y) = \begin{bmatrix} f_x & f_y \end{bmatrix}$.

Problem 9.3 Suppose the temperature at a point in the plane is given by the function $T(x, y) = x^2 - xy - y^2$ degrees Fahrenheit. A particle is at $P = (2, 3)$.

1. Use differentials to estimate the change in temperature if the particle moves 1 unit in the direction of $\vec{u} = (3, 4)$. [Hint: Find a unit vector in that direction.]
2. What is the actual change in temperature if the particle moves 1 unit in the direction of $\vec{u} = (3, 4)$?
3. Use differentials to estimate the change in temperature if the particle moves about .2 units in the direction of $\vec{u} = (3, 4)$.

We can define partial derivatives solely in terms of differential notation. We can define derivatives in any direction in terms of differential notation.

Problem 9.4 Suppose that $z = f(x, y)$ is a differentiable function (so the derivative is the matrix $\begin{bmatrix} f_x & f_y \end{bmatrix}$). Remember to use differential notation in this problem.

1. If $(dx, dy) = (1, 0)$, which means we've moved one unit in the x direction while holding y constant, what is dz?
2. If $(dx, dy) = (0, 1)$, which means we've moved one unit in the y direction while holding x constant, what is dz?
3. Consider the direction $\vec{u} = (2, 3)$. Find a unit vector in the direction of $\vec{u}$. If we move one unit in the direction of $\vec{u}$, what is dz? [It's all right to leave you answer as a dot product.]

Definition 9.1. The directional derivative of f in the direction of the unit vector $\vec{u}$ at a point P is defined to be

$$D_{\vec{u}}f(P) = Df(P)\vec{u} = \vec{\nabla} f \cdot \vec{u}.$$

We dot the gradient of f with the direction vector $\vec{u}$. The partial derivative of f with respect to x is precisely the directional derivative of f in the $(1, 0)$ direction. Similarly, the partial derivative of f with respect to y is precisely the directional derivative of f in the $(0, 1)$ direction. This definition extends to higher dimensions.

Note that in the definition above, we require the vector $\vec{u}$ to be a unit vector. If you are asked to find a directional derivative in some direction, make sure you start by finding a unit vector in that direction. We want to deal with unit vectors because when we say something has a slope of m units, we want to say "The function rises m units if we run 1 unit."

Problem 9.5 Consider the function $f(x, y) = 9 - x^2 - y^2$.

1. Draw several level curves of f.
2. At the point $P = (2, 1)$, place a dot on your graph. Then draw a unit vector based at P that points in the direction $\vec{u} = (3, 4)$ [not to the point $(3, 4)$, but in the direction $\vec{u} = (3, 4)$]. If you were to move in the direction $(3, 4)$, starting from the point $(2, 1)$, would the value of f increase or decrease?
3. Find the slope of f at $P = (2, 1)$ in the direction $\vec{u} = (3, 4)$ by finding the directional derivative. This should agree with your previous answer.
4. If you stand at $Q = (-2, 3)$ and move in the direction $\vec{v} = (1, -1)$, will f increase or decrease? Find the directional derivative of f in the direction $\vec{v} = (1, -1)$ at the point $Q = (-2, 3)$.

Problem 9.6 Recall that the directional derivative of $\vec{f}$ in the direction $\vec{u}$ is the dot product $\vec{\nabla} f \cdot \vec{u} = |\vec{\nabla} f||\vec{u}| \cos\theta$. In this problem, you'll explain why the gradient points in the direction of greatest increase.

1. Why is the directional derivative of $\vec{f}$ the largest when $\vec{u}$ points in the exact same direction as $\vec{\nabla} f$? [Hint: What angle maximizes the cosine function?]

2. When $\vec{u}$ points in the same direction as $\vec{\nabla} f$, show that $D_{\vec{u}} f = |\vec{\nabla} f|$. In other words, explain why the length of the gradient is precisely the slope of f in the direction of greatest increase (the slope in the steepest direction).

3. Which direction points in the direction of greatest decrease?

Problem 9.7 Suppose you are looking at a topographical map (see Wikipedia for an example). On this topographical map, each contour line represents 100 ft in elevation. You notice in one section of the map that the contour lines are really close together, and they start to form circles around a spot on the graph. You notice in another section of the map that the contour lines are spaced quite far apart. Let $f(x, y)$ be the elevation of the land, so that the topographical map is just a contour plot of f.

1. Where is the slope of the terrain larger, in the section with closely packed contour lines, or the section with contour lines that are spread out. In which section will the gradient be a longer vector?

2. At the very top of a mountain, or the very bottom of a valley, will the gradient be a long vector or a small vector? How do you locate a peak in a topographical map?

3. Create you own topographical map to illustrate the ideas above. Just make sure your map has a section with some contours that are closely packed together, and some that are far apart, as well as a contour that intersects itself. Then on your topographical map, please add a few gradient vectors, where you emphasize which ones are long, and which ones are short. Show us how to find a peak, as well as what the gradient vector would be at the peak.

If you're stuck, look at a contour plot of $f(x, y) = (x+1)^3 - 3(x+1)^2 - y^2 + 2$ in Sage. Then make your own example.

Theorem 9.2. *Let f be a continuously differentiable function, with $\vec{r}$ a level curve of the function.*

- *The gradient is always normal to level curves, meaning* $\vec{\nabla} f \cdot \dfrac{d\vec{r}}{dt} = 0$.
- *The gradient points in the direction of greatest increase.*
- *The directional derivative of f in the direction of the gradient is the length of the gradient. Symbolically, we write* $D_{\vec{\nabla} f} f = |\vec{\nabla} f|$.
- *At a maximum or minimum, the gradient is the zero vector.*

The next few problems have you practice using differentials, and then obtain tangent lines and planes to curves and surfaces using differentials.

Problem 9.8 The volume of a cylindrical can is $V(r, h) = \pi r^2 h$. Any manufacturing process has imperfections, and so building a cylindrical can with designed dimensions (r, h) will result in a can with dimensions $(r + dr, h + dh)$.

1. Compute both DV (the derivative of V) and dV (the differential of V).

2. If the can is tall and slender (h is big, r is small), which will cause a larger change in volume: an error in r or an error in h? Use dV to explain your answer.

3. If the can is short and wide (like a tuna can), which will cause a larger change in volume: an error in r or an error in h? Use dV to explain your answer.

Problem 9.9 Consider the function $f(x, y) = x^2 + y^2$. Consider the level curve C given by $f(x, y) = 25$. Our goal is to find an equation of the tangent line to C at $P = (3, -4)$.

1. Draw C. Compute $\vec{\nabla} f$ and add to your graph the vector $\vec{\nabla} f(P)$.

2. We know the point $P = (3, -4)$ is on the tangent line. Let $Q = (x, y)$ represent another point on the tangent line. Add to your graph the point Q and the vector $\vec{PQ} = (x - 3, y + 4)$.

3. Why are $\vec{\nabla} f(P)$ and $\vec{PQ}$ orthogonal? Use this fact to write an equation of the tangent line.

4. What is a normal vector to the line?

The previous problem had you give an equation of the tangent line to a level curve, by using differential notation. The next problems asks you to repeat this idea and give an equation of a tangent plane to a level surface.

Problem 9.10 Consider the function $f(x, y, z) = x^2 + y^2 + z^2$. Consider the level surface S given by $f(x, y, z) = 9$. Our goal is to find an equation of the tangent plane to S at $P = (1, 2, -2)$.

1. Draw S.

2. Compute $\vec{\nabla} f$. Add to your graph the vector $\vec{\nabla} f(P)$, with its base at P.

3. We know the point $P = (1, 2, -2)$ is on the tangent plane. Let $Q = (x, y, z)$ be any other point on the tangent plane. What is the component form of the vector $\vec{PQ}$?

4. Why are $\vec{\nabla} f(P)$ and $\vec{PQ}$ orthogonal? Use this fact to write an equation of the tangent plane.

5. What is a normal vector to the plane?

Problem 9.11 Find an equation of the tangent plane to the hyperboloid of one sheet $1 = x^2 - y^2 + z^2$ at the point $(-3, 3, 1)$.

Problem 9.12 The two surfaces $x^2 + y^2 + z^2 = 14$ and $3x + 4y - z = -1$ intersect in a curve C. Draw both surfaces, and show us the curve C. Then, at the point $(2, -1, 3)$, find an equation of the tangent line to this curve. [Hint: The line is in both tangent planes, so it is orthogonal to both normal vectors. The cross product gets you a vector that is orthogonal to two vectors.]

9.2 The Second Derivative Test

We start with a review problems from first-semester calculus.

Problem 9.13 Let $f(x) = x^3 - 3x^2$. Find the critical values of f by solving $f'(x) = 0$. Determine if each critical value leads to a local maximum or local minimum by computing the second derivative. State the local maxima/minima of f. Sketch the function using the information you discovered.

We now generalize the second derivative test to all dimensions. We've already seen that the second derivative of a function such as $z = f(x, y)$ is a square matrix. The second derivative test relied on understanding if a function was concave up or concave down. We need a way to examine the concavity of f as we approach a point (x, y) from any of the infinitely many directions. Such a method exists, and leads to an eigenvalue/eigenvector problem. I'm assuming that most of you have never heard the word "eigenvalue." We could spend an entire semester just studying eigenvectors. We'd need a few weeks to discover what they are from a problem-based approach. Instead, here is an example of how to find eigenvalues and eigenvectors.

Definition 9.3. Let A be a square matrix, so in 2D we have $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The identity matrix I is a square matrix with 1's on the diagonal and zeros everywhere else, so in 2D we have $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The eigenvalues of A are the solutions λ to the equation $|A - \lambda I| = 0$. Remember that $|A|$ means, "Compute the determinant of A." So in 2D, we need to find the value λ so that

$$\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0.$$

This definition extends to any square matrix. In 3D, the eigenvalues are the solutions to the equation

$$\left| \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} a - \lambda & b & c \\ d & e - \lambda & f \\ g & h & i - \lambda \end{vmatrix} = 0.$$

An eigenvector of A corresponding to λ is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda x$.

As you continue taking more upper level science courses (in physics, engineering, mathematics, chemistry, and more) you'll soon see that eigenvalues and eigenvectors play a huge role. You'll start to see them in most of your classes. For now, we'll use them without proof to apply the second derivative test. In class, make sure you ask me to show you pictures with each problem we do, so we can see how eigenvalues and eigenvectors appear in surfaces.

Theorem 9.4 (The Second Derivative Test). *Let $f(x, y)$ be a function so that all the second partial derivatives exist and are continuous. The second derivative of f, written $D^2 f$ and sometimes called the Hessian of f, is a square matrix. Let λ_1 be the largest eigenvalue of $D^2 f$, and λ_2 be the smallest eigenvalue. Then λ_1 is the largest possible second derivative obtained in any direction. Similarly, the smallest possible second derivative obtained in any direction is λ_2. The eigenvectors give the directions in which these extreme second derivatives are obtained. The second derivative test states the following.*

Suppose (a, b) is a critical point of f, meaning $Df(a, b) = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

- *If all the eigenvalues of $D^2f(a,b)$ are positive, then in every direction the function is concave upwards at (a,b) which means the function has a local minimum at (a,b).*
- *If all the eigenvalues of $D^2f(a,b)$ are negative, then in every direction the function is concave downwards at (a,b). This means the function has a local maximum at (a,b).*
- *If the smallest eigenvalue of $D^2f(a,b)$ is negative, and the largest eigenvalue of $D^2f(a,b)$ is positive, then in one direction the function is concave upwards, and in another the function is concave downwards. The point (a,b) is called a saddle point.*
- *If the largest or smallest eigenvalue of f equals 0, then the second derivative tests yields no information.*

Example 9.5. Consider the function $f(x,y) = x^2 - 2x + xy + y^2$. The first and second derivatives are

$$Df(x,y) = \begin{bmatrix}2x-2+y, x+2y\end{bmatrix} \quad \text{and} \quad D^2f = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}.$$

The first derivative is zero (the zero matrix) when both $2x-2+y=0$ and $x+2y=0$. We need to solve the system of equations $2x+y=2$ and $x+2y=0$. Double the second equation, and then subtract it from the first to obtain $0x-3y=2$, or $y=-2/3$. The second equation says that $x=-2y$, or that $x=4/3$. So the only critical point is $(4/3,-2/3)$.

We find the eigenvalues of $D^2f(4/3,-2/3)$ by solving the equation

$$\begin{vmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{vmatrix} = (2-\lambda)(2-\lambda)-1=0.$$

In this example, the second derivative is constant, so the point $(4/3,-2/3)$ did not change the matrix. In general, the point will affect your matrix. See Sage to see a graph which shows the eigenvectors in which the largest and smallest second derivatives occur.

Expanding the left hand side gives us $4-4\lambda+\lambda^2-1=0$. Simplifying and factoring gives us $\lambda^2-4\lambda+3=(\lambda-3)(\lambda-1)=0$. This means the eigenvalues are $\lambda=1$ and $\lambda=3$. Since both numbers are positive, the function is concave upwards in every direction. The critical point $(4/3,-2/3)$ corresponds to a local minimum of the function. The local minimum is the output $f(4/3,-2/3)=(4/3)^2-2(4/3)+(4/3)(-2/3)+(-2/3)^2$.

Problem 9.14 Consider the function $f(x,y) = x^2+4xy+y^2$.

See 14.7 for more practice.

1. Find the critical points of f by finding when $Df(x,y)$ is the zero matrix.
2. Find the eigenvalues of D^2f at any critical points.
3. Label each critical point as a local maximum, local minimum, or saddle point, and state the value of f at the critical point.

Problem 9.15 Consider the function $f(x,y) = x^3-3x+y^2-4y$.

1. Find the critical points of f by finding when $Df(x,y)$ is the zero matrix.
2. Find the eigenvalues of D^2f at any critical points. [Hint: First compute D^2f. Since there are two critical points, evaluate the second derivative at each point to obtain 2 different matrices. Then find the eigenvalues of each matrix.]
3. Label each critical point as a local maximum, local minimum, or saddle point, and state the value of f at the critical point.

Problem 9.16 Consider the function $f(x, y) = x^3 + 3xy + y^3$.

1. Find the critical points of f by finding when $Df(x, y)$ is the zero matrix.
2. Find the eigenvalues of $D^2 f$ at any critical points.
3. Label each critical point as a local maximum, local minimum, or saddle point, and state the value of f at the critical point.

You now have the tools needed to find optimal solutions to problems in any dimension. Here's a silly problem that demonstrates how we can use what we've just learned.

Problem 9.17: Optional For my daughter's birthday, she has asked for a Barbie princess cake. I purchased a metal pan that's roughly in the shape of a paraboloid $z = f(x, y) = 9 - x^2 - y^2$ for $z \geq 0$. To surprise her, I want to hide a present inside the cake. The present is a bunch of small candy that can pretty much fill a box of any size. I'd like to know how large (biggest volume) of a rectangular box I can fit under the cake, so that when we start cutting the cake, she'll find her surprise present. The box will start at $z = 0$ and the corners of the box (located at $(x, \pm y)$ and $(-x, \pm y)$) will touch the surface of the cake $z = 9 - x^2 - y^2$.

1. What is the function $V(x, y)$ that we are trying to maximize?
2. If you find all the critical points of V, you'll discover there are 9. However, only one of these critical points makes sense in the context of this problem. Find that critical point.
3. Use the second derivative test to prove that the critical point yields a maximum volume.
4. What are the dimensions of the box? What's the volume of the box?

The only thing left for me is to now determine how much candy I should buy to fill the box. I'll take care of that.

In this problem, we'll derive the version of the second derivative test that is found in most multivariate calculus texts. The test given below only works for functions of the form $f : \mathbb{R}^2 \to \mathbb{R}$. The eigenvalue test you have been practicing will work with a function of the form $f : \mathbb{R}^n \to \mathbb{R}$, for any natural number n.

Problem 9.18: Optional Suppose that $f(x, y)$ has a critical point at (a, b).

1. Find a general formula for the eigenvalues of $D^2 f(a, b)$. Your answer will be in terms of the second partials of f.
2. Let $D = f_{xx} f_{yy} - f_{xy}^2$.
   - If $D < 0$, explain why f has a saddle point at (a, b).
   - If $D = 0$, explain why the second derivative test fails.
   - If $D > 0$, explain why f has either a maximum or minimum at (a, b).
   - If $D > 0$, and $f_x(a, b) > 0$, does f have a local max or local min at (a, b). Explain.

3. The only critical point of $f(x,y) = x^2 + 3xy + 2y^2$ is at $(0,0)$. Does this point correspond to a local maximum, local minimum, or saddle point? Give the eigenvalues (which should come instantly out of part 1). Find D, from part 2, to answer the question.

9.3 Lagrange Multipliers

The last problem was an example of an optimization problem where we wish to optimize a function (the volume of a box) subject to a constraint (the box has to fit inside a cake). If you are economics student, this section may be the key reason why you were asked to take multivariate calculus. In the business world, we often want to optimize something (profit, revenue, cost, utility, etc.) subject to some constraint (a limited budget, a demand curve, warehouse space, employee hours, etc.). An aerospace engineer will build the best wing that can withstand given forces. Everywhere in the engineering world, we often seek to create the "best" thing possible, subject to some outside constraints. Lagrange discovered an extremely useful method for answering this question, and today we call it "Lagrange Multipliers."

Rather than introduce Cobb-Douglass production functions (from economics) or sheer-stress calculations (from engineering), we'll work with simple examples that illustrate the key points. Sometimes silly examples carry the message across just as well.

Problem 9.19 Suppose an ant walks around the circle $g(x,y) = x^2 + y^2 = 1$. As the ant walks around the circle, the temperature is $f(x,y) = x^2 + y + 4$. Our goal is to find the maximum and minimum temperatures reached by the ant as it walks around the circle. We want to optimize $f(x,y)$ subject to the constraint $g(x,y) = 1$.

1. Draw the circle $g(x,y) = 1$. Then, on the same set of axes, draw several level curves of f. The level curves $f = 3, 4, 5, 6$ are a good start. Then add more (maybe at each 1/4th). If you make a careful, accurate graph, it will help a lot below.

2. Based solely on your graph, where does the minimum temperature occur? What is the minimum temperature?

3. If the ant is at the point $(0,1)$, and it moves left, will the temperature rise or fall? What if the ant moves right?

4. On your graph, place a dot(s) where you believe the ant reaches a maximum temperature (it may occur at more than one spot). Explain why you believe this is the spot where the maximum temperature occurs. What about the level curves tells you that these spots should be a maximum.

5. Draw the gradient of f at the places where the minimum and maximum temperatures occur. Also draw the gradient of g at these spots. How are the gradients of f and g related at these spots?

Theorem 9.6 (Lagrange Multipliers)**.** *Suppose f and g are continuously differentiable functions. Suppose that we want to find the maximum and minimum values of f subject to the constraint $g(x,y) = c$ (where c is some constant). Then if a maximum or minimum occurs, it must occur at a spot where the gradient* *of f and the gradient of g point in the same, or opposite, directions. So the gradient of g must be a multiple of the gradient of f. To find the maximum and minimum values (if they exist), we just solve the system of equations that result from*

$$\vec{\nabla} f = \lambda \vec{\nabla} g, \quad \textit{and} \quad g(x,y) = c$$

where λ is the proportionality constant. The maximum and minimum values will be among the solutions of this system of equations.

Problem 9.20 Suppose an ant walks around the circle $x^2 + y^2 = 1$. As the ant walks around the circle, the temperature is $T(x,y) = x^2 + y + 4$. Our goal is to find the maximum and minimum temperatures T reached by the ant as it walks around the circle.

1. What function $f(x,y)$ do we wish to optimize? What is the constraint $g(x,y) = c$?
2. Find the gradient of f and the gradient of g. Then solve the system of equations that you get from the equations

   $$\vec{\nabla} f = \lambda \vec{\nabla} g, \qquad x^2 + y^2 = 1.$$

   You should obtain 4 ordered pairs (x,y).
3. At each ordered pair, find the temperature. What is the maximum temperature obtained? What is the minimum temperature obtained.

The most common error on this problem is to divide both sides of an equation by x, which could be zero. If you do this, you'll only get 2 ordered pairs.

Problem 9.21 Consider the curve $xy^2 = 54$ (draw it). The distance from each point on this curve to the origin is a function that must have a minimum value. Find a point (a,b) on the curve that is closest to the origin.

[The distance to the origin is $d(x,y) = \sqrt{x^2+y^2}$. This distance is minimized when $f(x,y) = x^2 + y^2$ is minimized. So use $f(x,y) = x^2 + y^2$ as the function you wish to minimize. What's the constraint $g(x,y) = c$?]

See 14.8 for more practice.

Problem 9.22 Find the dimensions of the rectangular box with maximum volume that can be inscribed inside the ellipsoid

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{5^2} = 1.$$

[What is the function f you wish to optimize? What is the constraint $g = c$? Try solving each equation for λ so you can eliminate it from the problem.]

Problem 9.23 Repeat problem 9.17, but this time use Lagrange multipliers. Find the dimensions of the rectangular box of maximum volume that fits underneath the surface $z = f(x,y) = 9 - x^2 - y^2$ for $z \geq 0$.

[Hint: Let $f(x,y,z) = (2x)(2x)(z)$ and $g(x,y,z) = z + x^2 + y^2 = 9$. You'll get a system of 4 equations with 4 unknowns (x,y,z,λ). Try solving each equation for lambda. You know x,y,z can't be zero or negative, so ignore those possible cases.]

9.4 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 10

Integration

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Explain how to setup and compute a double integral. Show how to interchange the bounds of integration.

2. For planar regions, find area, mass, centroids, center of mass, moments of inertia, and radii of gyration.

3. Explain how to change coordinate systems in integration, in particular to polar coordinates. Explain what the Jacobian is, and show how to use it.

4. Explain how to use Green's theorem to compute flow along and flux across a curve.

You'll have a chance to teach your examples to your peers prior to the exam.

The following homework problems line up with the topics in class.

Topic (11th ed)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Double Integrals	15.1	1-16, 21-50	17-20, 51-54, 57-66	55-56	67-76
Double Integral Applications	15.2	1-12, 15-18, 19-40	13, 14, 41-48, 53-56	49-52	
Polar Coordinates	15.3	1-22,23-32	33-42 (do 37 and 40 for sure)		43-46

Topic 12th ed	Sec	Basic Practice	Good Problems	Thy/App	Comp
Double Integrals (rect.)	15.1	1-28			
Double Integrals	15.2	1-24,33-46	19-32, 47-56,57-68	69-84	85-94
Area, Average Value	15.3	1-22	23-25	26	
Polar Integrals	15.4	1-16	17-26, 27-36,41	37-40, 42-46	47-50
Double Integral Applications	15.6	1-20			

10.1 Double Integrals and Applications

Before we introduce integration, let's practice using inequalities to describe regions in the plane. In first semester calculus, we often use the inequalities $a \leq x \leq b$ and $g(x) \leq y \leq f(x)$ to describe the region above g below f for x between a and b. We trapped x between two constants, and y between two functions. Sometimes we wrote $c \leq y \leq d$ where $g(y) \leq x \leq f(y)$ to describe the region to the right of g and left of f for y between c and d. We need to practice writing inequalities in this form, as these inequalities provide us the bounds of integration for double integrals.

Problem 10.1 Consider the region R in the xy-plane that is below the line $y = x + 2$, above the line $y = 2$, and left of the line $x = 5$. We can describe this region by saying for each x with $0 \leq x \leq 5$, we want y to satisfy $2 \leq y \leq x + 2$. In set builder notation, we would write

$$R = \{(x, y) \mid 0 \leq x \leq 5, 2 \leq y \leq x + 2\}.$$

The symbols { and } are used to enclose sets, and the symbol | stands for "such that". We read the above line as "R equals the set of (x, y) such that zero is less than x which is less than 5, and 2 is less than y which is less than $x + 2$."

1. Describe the region R by saying for each y with $c \leq y \leq d$, we want x to satisfy $a(y) \leq x \leq b(y)$. In other words, find constants c and d, and functions $a(y)$ and $b(y)$, so that for each y between c and d, the x values must be between the functions $a(y)$ and $b(y)$.

2. Write your last answer in the set builder notation

$$R = \{(x, y) \mid c \leq y \leq d, a(y) \leq x \leq b(y)\}.$$

[Hint: If your struggling, then draw the 4 curves given by $0 = x$, $x = 5$, $2 = y$ and $y = x + 2$. Then shade either above, below, left, or right of the line (as appropriate).]

Problem 10.2 For each region R below, draw the region and give a set of inequalities of the form $a \leq x \leq b, c(x) \leq y \leq d(x)$ or in the form $c < y < d, a(y) \leq x \leq b(y)$. In class, we'll give whichever one you did not.

1. The region R is above the line $x + y = 1$ and inside the circle $x^2 + y^2 = 1$.

2. The region R is below the line $y = 8$, above the curve $y = x^2$, and to the right of the y-axis.

3. The region R is bounded by $2x + y = 3$, $y = x$, and $x = 0$.

We can use iterated double integrals to find the areas of any of the regions above. We can set up the integrals with dx on the inside, or with dy on the inside. Whichever way we set up the iterated integral, we'll get the same answer. We can also integrate any function $f(x, y)$ over the region described by the bounds (such as a density, or a height, or some other quantity). When we set up the integral with bounds, we call it an iterated integral, and we write

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y) dy dx \quad \text{or} \quad \int_c^d \int_{a(y)}^{b(y)} f(x, y) dx dx y.$$

Definition 10.1: Double and Iterated Integrals. A double integral is written $\iint_R f(x, y) dA$. We just have to state what the region R is to talk about a double integral. The formal definition of a double integrals involves slicing the region R up into tiny rectangles of area $dxdy$, multiplying each rectangle by a function f, and then summing over all rectangles. This process is repeated as the length and width of the rectangles shrink to zero at similar rates, with the double integral being the limit of this process.

An iterated integral is a double integral where we have actually set up the bounds as either

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)dydx \quad \text{or} \quad \int_c^d \int_{a(y)}^{b(y)} f(x,y)dxdxy.$$

We'll focus mostly on setting up iterated integrals in this course.

Problem 10.3 Consider the region R in the plane that is bounded by the line $y = x + 2$ and the parabola $y = x^2 - 4$. Distances are measured in cm.

1. Draw the region R, and give bounds of the form $a \leq x \leq b$, $c(x) \leq y \leq d(x)$ to describe the region.

2. A metal plate occupies the region R. The metal plate was constructed to have a density of $\delta(x, y) = (y + 4)$ g/cm^2. Explain why the mass of the plate is the double integral $\iint_R \delta dA$.

3. Compute the double integral $\iint_R (y + 4)dA$ by setting up an iterated integral (use the bounds from part 1) and then performing each integral. Check your work with the link in the margin.

Check your work with this Double Integral Checker written in Sage.

You can use this Sage link to check any double integral. If you think you have the bounds right, use this Sage link to draw the region your bounds describe. If it doesn't draw the region you thought, then your bounds are off. Trial and error is a powerful tool here. You've got to try, and fail, and then make adjustments. This is the key to mastering high dimensional integration.

Problem 10.4 Consider the iterated integral $\int_0^3 \int_x^3 e^{y^2} dydx$.

1. Write the bounds as two inequalities ($0 \leq x \leq 3$ and $? \leq y \leq ?$). Then draw and shade the region R described by these two inequalities.

2. Swap the order of integration from $dydx$ to $dxdy$. This forces you to describe the region using two inequalities of the form $c \leq y \leq d$ and $a(y) \leq x \leq b(y)$. This is the key.

3. Use your new bounds to compute the integral by hand (you'll need a u-substitution $u = y^2$ on the outer integral).

4. Now use Sage to check your work. Then also use Sage to compute the original integral $\int_0^3 \int_x^3 e^{y^2} dydx$, and tell us what the inner integral equals (if you see i, $\sqrt{\pi}$, and erf, then you did this correctly).

Problem 10.5 Consider the region R in the plane that is trapped between the curves $x = 2y$ and $x = y^2$. We would like to compute $\iint_R (-y)dA$ over this region R. Set up both iterated integrals. Then compute one of them.

In the line integral chapter, we introduce the ideas of average value, centroid, and center of mass. We now extend those ideas to regions in the plane, in exactly the same way. For example, the average value formula in the line integral section was $\bar{f} = \dfrac{\int_C f dx}{\int_C ds}$. For double integrals, we just change ds to dA, and add an integral. This gives the formula $\bar{f} = \dfrac{\iint_R f dA}{\iint_R dA}$. The same substitution works on all the integrals from before. We now have $dm = \delta dA$ instead of $dm = \delta ds$, as now density is a mass per area, instead of a mass per length. We obtained the arc length of a curve C by computing $s = \int_C ds$, as we just add up little bits of arc length. We can obtain the area of a region R by computing $A = \iint_R dA$, as we just add up little bits of area. The centroid of a region R in the plane is

Average value formula

Centroid Formula

$$\left(\bar{x} = \frac{\iint_R x dA}{\iint_R dA}, \bar{y} = \frac{\iint_R y dA}{\iint_R dA}\right)$$

and the center of mass is

Center of Mass Formula

$$\left(\bar{x} = \frac{\iint_R x dm}{\iint_R dm}, \bar{y} = \frac{\iint_R y dm}{\iint_R dm}\right), \text{ where } dm = \delta dA.$$

Problem 10.6 Consider the rectangular region R in the xy-plane described by $\{(x, y) \mid 2 \leq x \leq 11, 3 \leq y \leq 7\}$.

1. Set up an integral formula which would give $\bar{y}$ for the centroid of R. Then evaluate the integral.

2. State $\bar{x}$ from geometric reasoning.

Problem 10.7 Consider the region in the plane that is bounded by the curves $x = y^2 - 3$ and $x = y - 1$. A metal plate occupies this region in space, and its temperature function on the plate is give by the function $T(x, y) = 2x + y$. Find the average temperature of the metal plate.

Problem 10.8 Consider the region R that is the circular disc which is inside the circle $(x - 2)^2 + (y + 1)^2 = 9$. The centroid is clearly $(2, -1)$, and the area is $A = \pi(3)^2 = 9\pi$. We can use these fact to simplify many integrals that require integrating over the region R.

1. Compute $\iint_R 3dA = 3 \iint_R dA$. [How can area help you?]

2. Explain why $\iint_R x dA = \bar{x}A$ for any region R, and then compute $\iint_R x dA$ for the circular disc. [You don't need to set up any integrals at all.]

3. Compute the integral $\iint_R 5x + 2y dA$ by using centroid and area facts.

Problem 10.9: Centroid of a Triangular Region Consider the triangular region R in the first quadrant, bounded by the line $\dfrac{x}{5} + \dfrac{y}{7} = 1$. Assume that the density of the object is a constant $\delta = c$.

1. Draw the region R, and give bounds for performing double integrals over this region. Check your answer with Sage (use any f you want for the integrand, it doesn't matter as you just want to make sure you got the bounds right).

2. Set up an integral formula to compute the center of mass $\bar{x}$ of the region R. Compute any integrals by hand to show that $\bar{x} = \frac{5}{3}$. Then state a guess for $\bar{y}$.

Remember you can check your work with Sage.

Problem 10.10 Let R be the region in the plane with $a \leq x \leq b$ and $g(x) \leq y \leq f(x)$. Let A be the area of R.

1. Set up an iterated integral to compute the area of R. Then compute just the inside integral. You should obtain a familiar formula.
2. Set up an iterated integral formula to compute $\bar{x}$ for the centroid. Compute just the inside integral. You should obtain $\bar{x} = \dfrac{1}{A}\displaystyle\int_a^b x(f-g)dx$.
3. Set up an iterated integral formula to compute $\bar{y}$ for the center of mass. Compute just the inside integral.

When you use double integrals to find centroids, the formulas for the centroid are the same for both $\bar{x}$ and $\bar{y}$. In other courses, you may see the formulas on the left, because the ideas will be presented without requiring knowledge of double integrals. Integrating the inside integral from the double integral formula gives the single variable formulas.

Earlier in the semester we explored what happens if we change from Cartesian coordinates to another coordinate system. We found that if we want to change from an integral containing $dxdy$, then we had to replace $dxdy$ with $rdrd\theta$. The next problem has you review why.

Problem 10.11 Consider the polar change of coordinates $x = r\cos\theta$ and $y = r\sin\theta$, which we could just write as

$$\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta).$$

1. Compute the derivative $D\vec{T}(r,\theta)$. You should have a 2 by 2 matrix.
2. We need a single number from this matrix that tells us something about area. Determinants are connected to area. Compute the determinant of $D\vec{T}(r,\theta)$ and simplify.

The determinant you found above we called the Jacobian of the polar coordinate transformation. Let's summarize these results in a theorem.

Theorem 10.2. *If we use the polar coordinate transformation $x = r\cos\theta, y = r\sin\theta$, then we can switch from (x,y) coordinates to (r,θ) coordinates if we use*

$$dxdy = |r|drd\theta.$$

The number $|r|$ we call the Jacobian of x and y with respect to r and θ. If we require all bounds for r to be nonnegative, we can ignore the absolute value. If R_{xy} is a region in the xy plane that corresponds to the region $R_{r\theta}$ in the $r\theta$ plane (where $r \geq 0$), then we can write

$$\iint_{R_{xy}} f(x,y)dxdy = \iint_{R_{r\theta}} f(r\cos\theta, r\sin\theta)r\ drd\theta.$$

Ask me in class to give you an informal picture approach that explains why $dxdy = rdrd\theta$.

We need some practice using this idea. We'll start by describing regions using inequalities on r and θ.

Problem 10.12 For each region R below, draw the region in the xy-plane. Then give a set of inequalities of the form $a \leq r \leq b, \alpha(r) \leq \theta \leq \beta(r)$ or $\alpha < \theta < \beta, a(\theta) \leq r \leq b(\theta)$. For example, if the region is the inside of the circle $x^2 + y^2 = 9$, then we could write $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 3$.

1. The region R is the quarter circle in the first quadrant inside the circle $x^2 + y^2 = 25$.
2. The region R is below $y = \sqrt{9 - x^2}$, above $y = x$, and to the right of $x = 0$.
3. The region R is the triangular region below $y = \sqrt{3}x$, above the x-axis, and to the left of $x = 1$.

Problem 10.13 Consider the region R in the xy plane described by the bounds $-3 \leq y \leq 3$, and $0 \leq x \leq \sqrt{9 - y^2}$. Let's compute the iterated integral

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-y^2}} (9 - x^2 - y^2) dx dy.$$

1. Draw the region R in the xy plane and then write polar bounds for the region R by giving bounds for r and θ.
2. Rewrite the double integral as an iterated integral with bounds for r and θ. Don't forget the Jacobian (as $dxdy = r dr d\theta$).
3. Compute the integral in the previous part by hand. [Suggestion: you'll want to simplify $9 - x^2 - y^2$ to $9 - r^2$ before integrating.]

Problem 10.14 Find the centroid of a semicircular disc of radius a ($y \geq 0$). Actually compute any integrals.

Problem 10.15 Compute the integral $\displaystyle\int_{0}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1 + x^2 + y^2)^2} dy dx$. [Hint: try switching coordinate systems to polar coordinates. This will require you to first draw the region of integration, and then then obtain bounds for the region in polar coordinates.]
