Multivariable Calculus

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Chapter 1

Review

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Give a summary of the ideas you learned in 112, including graphing, derivatives (product, quotient, power, chain, trig, exponential, and logarithm rules), and integration (u-sub and integration by parts).
2. Compute the differential dy of a function and use it to approximate the change in a function.
3. Explain how to perform matrix multiplication and compute determinants of square matrices.
4. Illustrate how to solve systems of linear equations, including how to express a solution parametrically (in terms of t) when there are infinitely solutions.
5. Extend the idea of differentials to approximate functions using parabolas, cubics, and polynomials of any degree.

You'll have a chance to teach your examples to your peers prior to the exam.

1.1 Review of First Semester Calculus

1.1.1 Graphing

We'll need to know how to graph by hand some basic functions. If you have not spent much time graphing functions by hand before this class, then you should spend some time graphing the following functions by hand. When we start drawing functions in 3D, we'll have to piece together infinitely many 2D graphs. Knowing the basic shape of graphs will help us do this.

Problem 1.1 Provide a rough sketch of the following functions, showing their basic shapes:

$$x^2, x^3, x^4, \frac{1}{x}, \sin x, \cos x, \tan x, \sec x, \arctan x, e^x, \ln x.$$

Then use a computer algebra system, such as Wolfram Alpha, to verify your work. I suggest Wolfram Alpha, because it is now built into Mathematica 8.0. If you can learn to use Wolfram Alpha, you will be able to use Mathematica.

1.1.2 Derivatives

In first semester calculus, one of the things you focused on was learning to compute derivatives. You'll need to know the derivatives of basic functions (found on the end cover of almost every calculus textbook). Computing derivatives accurately and rapidly will make learning calculus in high dimensions easier. The following rules are crucial.

- Power rule $(x^n)' = nx^{n-1}$
- Sum and difference rule $(f \pm g) = f' \pm g'$
- Product $(fg)' = f'g + fg'$ and quotient rule $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$
- Chain rule (arguably the most important) $(f \circ g)' = f'(g(x)) \cdot g'(x)$

Problem 1.2 Compute the derivative of $e^{\sec x} \cos(\tan(x) + \ln(x^2+4))$. Show each step in your computation, making sure to show what rules you used.

See sections 3.2-3.6 for more practice with derivatives. The later problems in 3.6 review of most of the entire differentiation chapter.

Problem 1.3 If $y(p) = \dfrac{e^{p^3} \cot(4p+7)}{\tan^{-1}(p^4)}$ find dy/dp. Again, show each step in your computation, making sure to show what rules you used.

The following problem will help you review some of your trigonometry, inverse functions, as well as implicit differentiation.

Problem 1.4 Use implicit differentiation to explain why the derivative of $y = \arcsin x$ is $y' = \dfrac{1}{\sqrt{1-x^2}}$. [Rewrite $y = \arcsin x$ as $x = \sin y$, differentiate both sides, solve for y', and then write the answer in terms of x].

See sections 3.7-3.9 for more examples involving inverse trig functions and implicit differentiation.

1.1.3 Integrals

Each derivative rule from the front cover of your calculus text is also an integration rule. In addition to these basic rules, we'll need to know three integration techniques. They are (1) u-substitution, (2) integration-by-parts, and (3) integration by using software. There are many other integration techniques, but we will not focus on them. If you are trying to compute an integral to get a number while on the job, then software will almost always be the tool you use. As we develop new ideas in this and future classes (in engineering, physics, statistics, math), you'll find that u-substitution and integrations-by-parts show up so frequently that knowing when and how to apply them becomes crucial.

Problem 1.5 Compute $\int x\sqrt{x^2+4}dx$.

For practice with u-substitution, see section 5.5 and 5.6.
For practice with integration by parts, see section 8.1.

Problem 1.6 Compute $\int x \sin 2x dx$.

Problem 1.7 Compute $\int \arctan x dx$.

Problem 1.8 Compute $\int x^2 e^{3x} dx$.

1.2 Differentials

The derivative of a function gives us the slope of a tangent line to that function. We can use this tangent line to estimate how much the output (y values) will change if we change the input (x-value). If we rewrite the notation $\frac{dy}{dx} = f'$ in the form $dy = f'dx$, then we can read this as "A small change in y (called dy) equals the derivative (f') times a small change in x (called dx)."

Definition 1.1. We call dx the differential of x. If f is a function of x, then the differential of f is $df = f'(x)dx$. Since we often write $y = f(x)$, we'll interchangeably use dy and df to represent the differential of f.

We will often refer to the differential notation $dy = f'dx$ as "a change in the output y equals the derivative times a change in the input x."

Problem 1.9 If $f(x) = x^2 \ln(3x+2)$ and $g(t) = e^{2t}\tan(t^2)$ then compute df and dg.

See 3.10:19-38.

Most of higher dimensional calculus can quickly be developed from differential notation. Once we have the language of vectors and matrices at our command, we will develop calculus in higher dimensions by writing $d\vec{y} = Df(\vec{x})d\vec{x}$. Variables will become vectors, and the derivative will become a matrix.

This problem will help you see how the notion of differentials is used to develop equations of tangent lines. We'll use this same idea to develop tangent planes to surfaces in 3D and more.

Problem 1.10 Consider the function $y = f(x) = x^2$. This problem has multiple steps, but each is fairly short.

See 3.11:39-44. Also see problems 3.11:1-18. The linearization of a function is just an equation of the tangent line where you solve for y.

1. Find the differential of y with respect to x.
2. Give an equation of the tangent line to $f(x)$ at $x = 3$.
3. Draw a graph of $f(x)$ and the tangent line on the same axes. Place a dot at the point $(3, 9)$ and label it on your graph. Place another dot on the tangent line up and to the right of (3,9). Label the point (x, y), as it will represent any point on the tangent line.
4. Using the two points $(3, 9)$ and (x, y), compute the slope of the line connecting these two points. Your answer should involve x and y. What is the rise (i.e, the change in y called dy)? What is the run (i.e, the change in x called dx)?
5. We already know the slope of the tangent line is the derivative $f'(3) = 6$. We also know the slope from the previous part. These two must be equal. Use this fact to give an equation of the tangent line to $f(x)$ at $x = 3$.

Problem 1.11 The manufacturer of a spherical storage tank needs to create a tank with a radius of 3 m. Recall that the volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$. No manufacturing process is perfect, so the resulting sphere will have a radius of 3 m, plus or minus some small amount dr. The actual radius will be $3 + dr$. Find the differential dV. Then use differentials to estimate the change in the volume of the sphere if the actual radius is 3.02 m instead of the planned 3 m.

See 3.11:45-62.

Problem 1.12 A forest ranger needs to estimate the height of a tree. The ranger stands 50 feet from the base of tree and measures the angle of elevation to the top of the tree to be about 60°. If this angle of 60° is correct, then what is the height of the tree? If the ranger's angle measurement could be off by as much as 5°, then how much could his estimate of the height be off? Use differentials to give an answer.

1.3 Matrices

We will soon discover that matrices represent derivatives in high dimensions. When you use matrices to represent derivatives, the chain rule is precisely matrix multiplication. For now, we just need to become comfortable with matrix multiplication.

We perform matrix multiplication "row by column". Wikipedia has an excellent visual illustration of how to do this. See Wikipedia for an explanation. See texample.net for a visualization of the idea.

The links will open your browser and take you to the web.

Problem 1.13 Compute the following matrix products.

For extra practice, make up two small matrices and multiply them. Use Sage or Wolfram Alpha to see if you are correct (click the links to see how to do matrix multiplication in each system).

- $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 6 & 1 \end{bmatrix}$

Problem 1.14 Compute the product $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

1.3.1 Determinants

Determinants measure area, volume, length, and higher dimensional versions of these ideas. Determinants will appear as we study cross products and when we get to the high dimensional version of u-substitution.

Associated with every square matrix is a number, called the determinant, which is related to length, area, and volume, and we use the determinant to generalize volume to higher dimensions. Determinants are only defined for square matrices.

Definition 1.2. The determinant of a 2×2 matrix is the number

We use vertical bars next to a matrix to state we want the determinant, so $\det A = |A|$.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The determinant of a 3×3 matrix is the number

Notice the negative sign on the middle term of the 3×3 determinant. Also, notice that we had to compute three determinants of 2 by 2 matrices in order to find the determinant of a 3 by 3.

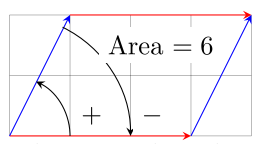
$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \det \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \det \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \det \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - hf) - b(di - gf) + c(dh - ge). \end{aligned}$$

Problem 1.15 Compute $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ and $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 2 & -3 & 1 \end{vmatrix}$.

For extra practice, create your own square matrix (2 by 2 or 3 by 3) and compute the determinant by hand. Then use Wolfram Alpha to check your work. Do this until you feel comfortable taking determinants.

What good is the determinant? The determinant was discovered as a result of trying to find the area of a parallelogram and the volume of the three dimensional version of a parallelogram (called a parallelepiped) in space. If we had a full semester to spend on linear algebra, we could eventually prove the following facts that I will just present here with a few examples.

Consider the 2 by 2 matrix $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ whose determinant is $3\cdot 2 - 0\cdot 1 = 6$. Draw the column vectors $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with their base at the origin (see figure 1.1). These two vectors give the edges of a parallelogram whose area is the determinant 6. If I swap the order of the two vectors in the matrix, then the determinant of $\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ is -6. The reason for the difference is that the determinant not only keeps track of area, but also order. Starting at the first vector, if you can turn counterclockwise through an angle smaller than 180° to obtain the second vector, then the determinant is positive. If you have to turn clockwise instead, then the determinant is negative. This is often termed "the right-hand rule," as rotating the fingers of your right hand from the first vector to the second vector will cause your thumb to point up precisely when the determinant is positive.



$$\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 6 \text{ and } \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -6$$

Figure 1.1: The determinant gives both area and direction. A counter clockwise rotation from column 1 to column 2 gives a positive determinant.

For a 3 by 3 matrix, the columns give the edges of a three dimensional parallelepiped and the determinant produces the volume of this object. The sign of the determinant is related to orientation. If you can use your right hand and place your index finger on the first vector, middle finger on the second vector, and thumb on the third vector, then the determinant is positive. For example, consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Starting from the origin, each column represents an edge of the rectangular box $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$ with volume (and determinant) $V = lwh = (1)(2)(3) = 6$. The sign of the determinant is positive because if you place your index finger pointing in the direction (1,0,0) and your middle finger in the direction (0,2,0), then your thumb points upwards in the direction (0,0,3). If you interchange two of the columns, for example $B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then the volume doesn't change since the shape is still the same. However, the sign of the determinant is negative because if you point your index finger in the direction (0,2,0) and your middle finger in the direction (1,0,0), then your thumb points down in the direction (0,0,-3). If you repeat this with your left hand instead of right hand, then your thumb points up.

Problem 1.16

- Use determinants to find the area of the triangle with vertices $(0,0)$, $(-2,5)$, and $(3,4)$.
- What would you change if you wanted to find the area of the triangle with vertices $(-3,1)$, $(-2,5)$, and $(3,4)$? Find this area.

1.4 Solving Systems of equations

Problem 1.17 Solve the following linear systems of equations.

For additional practice, make up your own systems of equations. Use Wolfram Alpha to check your work.

- $\begin{cases} x+y &= 3 \\ 2x-y &= 4 \end{cases}$
- $\begin{cases} -x+4y &= 8 \\ 3x-12y &= 2 \end{cases}$

Problem 1.18 Find all solutions to the linear system $\begin{cases} x+y+z &= 3 \\ 2x-y &= 4 \end{cases}$.

This link will show you how to specify which variable is t when using Wolfram Alpha.

Since there are more variables than equations, this suggests there is probably not just one solution, but perhaps infinitely many. One common way to deal with solving such a system is to let one variable equal t, and then solve for the other variables in terms of t. Do this three different ways.

- If you let $x=t$, what are y and z. Write your solution in the form (x,y,z) where you replace x, y, and z with what they are in terms of t.
- If you let $y=t$, what are x and z (in terms of t).
- If you let $z=t$, what are x and y.

1.5 Higher Order Approximations

When you ask a calculator to tell you what $e^{.1}$ means, your calculator uses an extension of differentials to give you an approximation. The calculator only uses polynomials (multiplication and addition) to give you an answer. This same process is used to evaluate any function that is not a polynomial (so trig functions, square roots, inverse trig functions, logarithms, etc.) The key idea needed to approximate functions is illustrated by the next problem.

Problem 1.19 Let $f(x)=e^x$. You should find that your work on each step can be reused to do the next step.

- Find a first degree polynomial $P_1(x)=a+bx$ so that $P_1(0)=f(0)$ and $P_1'(0)=f'(0)$. In other words, give me a line that passes through the same point and has the same slope as $f(x)=e^x$ does at $x=0$. Set up a system of equations and then find the unknowns a and b. The next two are very similar.

- Find a second degree polynomial $P_2(x) = a+bx+cx^2$ so that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$. In other words, give me a parabola that passes through the same point, has the same slope, and has the same concavity as $f(x) = e^x$ does at $x = 0$.
- Find a third degree polynomial $P_3(x) = a + bx + cx^2 + dx^3$ so that $P_3(0) = f(0)$, $P_3'(0) = f'(0)$, $P_3''(0) = f''(0)$, and $P_3'''(0) = f'''(0)$. In other words, give me a cubic that passes through the same point, has the same slope, the same concavity, and the same third derivative as $f(x) = e^x$ does at $x = 0$.
- Now compute $e^{.1}$ with a calculator. Then compute $P_1(.1)$, $P_2(.1)$, and $P_3(.1)$. How accurate are the line, parabola, and cubic in approximating $e^{.1}$?

Problem 1.20 Now let $f(x) = \sin x$. Find a 7th degree polynomial so that the function and the polynomial have the same value and same first seven derivatives when evaluated at $x = 0$. Evaluate the polynomial at $x = 0.3$. How close is this value to your calculator's estimate of $\sin(0.3)$? You may find it valuable to use the notation

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_7x^7.$$

The polynomial you are creating is often called a Taylor polynomial. (I'm giving you the name so that you can search online for more information if you are interested.)

The previous two problems involved finding polynomial approximations to the function at $x = 0$. The next problem shows how to move this to any other point, such as $x = 1$.

Problem 1.21 Let $f(x) = e^x$.

- Find a second degree polynomial
$$T(x) = a + bx + cx^2$$
so that $T(1) = f(1)$, $T'(1) = f'(1)$, and $T''(1) = f''(1)$. In other words, give me a parabola that passes through the same point, has the same slope, and the same concavity as $f(x) = e^x$ does at $x = 1$.
- Find a second degree polynomial written in the form
$$S(x) = a + b(x-1) + c(x-1)^2$$
so that $S(1) = f(1)$, $S'(1) = f'(1)$, and $S''(1) = f''(1)$. In other words, find a quadratic that passes through the same point, has the same slope, and the same concavity as $f(x) = e^x$ does at $x = 1$.

Notice that we just replaced x with $x - 1$. This centers, or shifts, the approximation to be at $x = 1$. The first part will be much simpler now when you let $x = 1$.

- Find a third degree polynomial written in the form
$$P(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3$$
so that $P(1) = f(1)$, $P'(1) = f'(1)$, $P''(1) = f''(1)$, and $P'''(1) = f'''(1)$. In other words, give me a cubic that passes through the same point, has the same slope, the same concavity, and the same third derivative as $f(x) = e^x$ does at $x = 1$.

Example 1.3. This example refers back to problem 1.11. We wanted a spherical tank of radius 3m, but due to manufacturing error the radius was slightly off. Let's now illustrate how we can use polynomials to give a first, second, and third order approximation of the volume if the radius is 3.02m instead of 3m.

We start with $V = \frac{4}{3}\pi r^3$ and then compute the derivatives

$$V' = 4\pi r^2, V'' = 8\pi r, \text{ and } V''' = 8\pi.$$

Because we are approximating the increase in volume from $r = 3$ to something new, we'll create our polynomial approximations centered at $r = 3$. We'll consider the polynomial

$$P(r) = a_0 + a_1(r-3) + a_2(r-3)^2 + a_3(r-3)^3,$$

whose derivatives are

$$P' = a_1 + 2a_2(r-3) + 3a_3(r-3)^2, P'' = 2a_2 + 6a_3(r-3), P''' = 6a_3.$$

So that the derivatives of the volume function match the derivatives of the polynomial (at $r = 3$), we need to satisfy the equations in the table below.

k	Value of V at the kth derivative	Value of P at the the kth derivative	Equation
0	$V(3) = \frac{4}{3}\pi(3)^3 = 36\pi$	$P(3) = a_0$	$a_0 = 36\pi$
1	$V'(3) = 4\pi(3)^2 = 36\pi$	$P'(3) = a_1$	$a_1 = 36\pi$
2	$V''(3) = 8\pi(3) = 24\pi$	$P''(3) = 2a_2$	$2a_2 = 24\pi$
3	$V'''(3) = 8\pi$	$P'''(3) = 6a_3$	$6a_3 = 8\pi$

This tells us that the third order polynomial is

$$P(r) = a_0+a_1(r-3)+a_2(r-3)^2+a_3(r-3)^3 = 36\pi+36\pi(r-3)+12\pi(r-3)^2+\frac{4}{3}\pi(r-3)^3.$$

We wanted to approximate the volume if $r = 3.2$, so our change in r is $dr = 3.2 - 3 = 0.2$. We can rewrite our polynomial as

$$P(r) = 36\pi + 36\pi(dr) + 12\pi(dr)^2 + \frac{4}{3}\pi(dr)^3.$$

We are now prepared to approximate the volume using a first, second, and third order approximation.

1. A first order approximation yields $P = 36\pi + 36\pi \cdot 0.02 = 36.72\pi$. The volume increased by 0.72π m^3.
2. A second order approximation yields
$$P = 36\pi + 36\pi \cdot 0.02 + 12\pi(0.02)^2 = 36.7248\pi.$$
3. A third order approximation yields
$$P = 36\pi + 36\pi \cdot 0.02 + 12\pi(0.02)^2 + \frac{4}{3}\pi(0.02)^3 = 36.724810\bar{6}\pi.$$

With each approximation, we add on a little more volume to get closer to the actual volume of a sphere with radius $r = 3.02$. The actual volume of a sphere involves a cubic function, so when we approximate the volume with a cubic, we should get an exact approximation (and $V(3.02) = \frac{4}{3}\pi(3.02)^3 = (36.724810\bar{6})\pi$.)

We'll end this section with a problem to practice the example above.

Problem 1.22 Suppose you are constructing a cube whose side length should be $s = 2$ units. The manufacturing process is not exact, but instead creates a cube with side lengths $s = 2 + ds$ units. (You should assume that all sides are still the same, so any error on one side is replicated on all. We have to assume this for now, but before the semester ends we'll be able to do this with high dimensional calculus.)

Suppose that the machine creates a cube with side length 2.3 units instead of 2 units. Note that the volume of the cube is $V = s^3$. Use a first, second, and third order approximation to estimate the increase in volume caused by the .3 increase in side length. Then compute the actual increase in volume $V(2.3) - V(2)$.

Ask me in class to draw a 3D graph which illustrates the volume added on by each successive approximation. As a challenge, try to construct this graph yourself first. If you have it before I put it up in class, let me know and I'll let you share what you have discovered with the class.

1.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to Brainhoney and download the quiz. Once you have taken the quiz, you can upload your work back to brainhoney and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 2

Vectors

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Define, draw, and explain what a vector is in 2 and 3 dimensions.
2. Add, subtract, multiply (scalar, dot product, cross product) vectors. Be able to illustrate each operation geometrically.
3. Use vector products to find angles, length, area, projections, and work.
4. Use vectors to give equations of lines and planes, and be able to draw lines and planes in 3D.

You'll have a chance to teach your examples to your peers prior to the exam.

2.1 Vectors and Lines

Learning to work with vectors will be key tool we need for our work in high dimensions. Let's start with some problems related to finding distance in 3D, drawing in 3D, and then we'll be ready to work with vectors.

Problem 2.1 To find the distance between two points (x_1, y_1) and (x_2, y_2) in the plane, we create a triangle connecting the two points. The base of the triangle has length $\Delta x = (x_2 - x_1)$ and the vertical side has length $\Delta y = (y_2 - y_1)$. The Pythagorean theorem gives us the distance between the two points as $\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Show that the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in 3-dimensions is $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Problem 2.2 Find the distance between the two points $P = (2, 3, -4)$ and $Q = (0, -1, 1)$. Then find an equation of the sphere passing though point Q whose center is at P.

See 12.1:41-58.

Problem 2.3 For each of the following, construct a rough sketch of the set of points in space (3D) satisfying:

See 12.1:1-40.

1. $2 \leq z \leq 5$

2. $x = 2, y = 3$
3. $x^2 + y^2 + z^2 = 25$

Definition 2.1. A vector is a magnitude in a certain direction. If P and Q are points, then the vector $\vec{PQ}$ is the directed line segment from P to Q. This definition holds in 1D, 2D, 3D, and beyond. If $V = (v_1, v_2, v_3)$ is a point in space, then to talk about the vector $\vec{v}$ from the origin O to V we'll use any of the following notations:

$$\vec{v} = \vec{OV} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} = (v_1, v_2, v_3) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

The entries of the vector are called the x, y, and z components of the vector.

Note that (v_1, v_2, v_3) could refer to either the point V or the vector $\vec{v}$. The context of the problem we are working on will help us know if we are dealing with a point or a vector.

Definition 2.2. Let $\mathbb{R}$ represent the set real numbers. Real numbers are actually 1D vectors. Let $\mathbb{R}^2$ represent the set of vectors (x_1, x_2) in the plane. Let $\mathbb{R}^3$ represent the set of vectors (x_1, x_2, x_3) in space. There's no reason to stop at 3, so let $\mathbb{R}^n$ represent the set of vectors $(x_1, x_2, \ldots, x_n)$ in n dimensions.

In first semester calculus and before, most of our work dealt with problem in $\mathbb{R}$ and $\mathbb{R}^2$. Most of our work now will involve problems in $\mathbb{R}^2$ and $\mathbb{R}^3$. We've got to learn to visualize in $\mathbb{R}^3$.

Definition 2.3. The magnitude, or length, or norm of a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. It is just the distance from the point (v_1, v_2, v_3) to the origin. A unit vector is a vector whose length is one unit.

The standard unit vectors are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.

Note that in 1D, the length of the vector $\langle -2 \rangle$ is simply $|-2| = \sqrt{(-2)^2} = 2$, the distance to 0. Our use of the absolute value symbols is appropriate, as it generalizes the concept of absolute value (distance to zero) to all dimensions.

Definition 2.4. Suppose $\vec{x} = \langle x_1, x_2, x_3 \rangle$ and $\vec{y} = \langle y_1, y_2, y_3 \rangle$ are two vectors in 3D, and c is a real number. We define vector addition and scalar multiplication as follows:

- Vector addition: $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ (add component-wise).
- Scalar multiplication: $c\vec{x} = (cx_1, cx_2, cx_3)$.

Problem 2.4 Consider the vectors $\vec{u} = (1, 2)$ and $\vec{v} = \langle 3, 1 \rangle$. Draw $\vec{u}$, $\vec{v}$, $\vec{u} + \vec{v}$, and $\vec{u} - \vec{v}$ with their tail placed at the origin. Then draw $\vec{v}$ with its tail at the head of $\vec{u}$. See 12.2:23-24.

Problem 2.5 Consider the vector $\vec{v} = (3, -1)$. Draw $\vec{v}$, $-\vec{v}$, and $3\vec{v}$. Suppose a donkey travels along the path given by $(x, y) = \vec{v}t = (3t, -t)$, where t represents time. Draw the path followed by the donkey. Where is the donkey at time $t = 0, 1, 2$? Put markers on your graph to show the donkey's location. Then determine how fast the donkey is traveling. See 11.1: 3,4.

In the previous problem you encountered $(x, y) = (3t, -t)$. This is an example of a function where the input is t and the output is a vector (x, y). For each input t, you get a single vector output (x, y). Such a function is called a parametrization of the donkey's path. Because the output is a vector, we call the function a vector-valued function. Often, we'll use the variable $\vec{r}$ to represent the radial vector (x, y), or (x, y, z) in 3D. So we could rewrite the position of the donkey as $\vec{r}(t) = (3, -1)t$. We use $\vec{r}$ instead of r to remind us that the output is a vector.

Problem 2.6 Suppose a horse races down a path given by the vector-valued function $\vec{r}(t) = (1, 2)t + (3, 4)$. (Remember this is the same as writing $(x, y) = (1, 2)t + (3, 4)$ or similarly $(x, y) = (1t + 3, 2t + 4)$.) Where is the horse at time $t = 0, 1, 2$? Put markers on your graph to show the horse's location. Draw the path followed by the horse. Give a unit vector that tells the horse's direction. Then determine how fast the horse is traveling. See 12.2: 1.

Problem 2.7 Consider the two points $P = (1, 2, 3)$ and $Q = (2, -1, 0)$. Write the vector $\vec{PQ}$ in component form (a, b, c). Find the length of vector $\vec{PQ}$. Then find a unit vector in the same direction as $\vec{PQ}$. Finally, find a vector of length 7 units that points in the same direction as $\vec{PQ}$. See 12.2: 9,17,25,33 and surrounding.

Problem 2.8 A raccoon is sitting at point $P = (0, 2, 3)$. It starts to climb in the direction $\vec{v} = \langle 1, -1, 2 \rangle$. Write a vector equation $(x, y, z) = (?, ?, ?)$ for the line that passes through the point P and is parallel to $\vec{v}$. [Hint, study problem 2.6, and base your work off of what you saw there. It's almost identical.] See 12.5: 1-12.

Then generalize your work to give an equation of the line that passes through the point $P = (x_1, y_1, z_1)$ and is parallel to the vector $\vec{v} = (v_1, v_2, v_3)$.

Make sure you ask me in class to show you how to connect the equation developed above to what you have been doing since middle school. If you can remember $y = mx + b$, then you can quickly remember the equation of a line. If I don't show you in class, make sure you ask me (or feel free to come by early and ask before class).

Problem 2.9 Let $P = (3, 1)$ and $Q = (-1, 4)$. See 12.5: 13-20.

- Write a vector equation $\vec{r}(t) = (x, y)$ for (i.e, give a parametrization of) the line that passes through P and Q, with $\vec{r}(0) = P$ and $\vec{r}(1) = Q$.
- Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is twice the speed of the first line.
- Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is one unit per second.

2.2 The Dot Product

Now that we've learned how to add and subtract vectors, stretch them by scalars, and use them to find lines, it's time to introduce a way of multiplying vectors called the dot product. We'll use the dot product to help us find find angles. First, we need to recall the law of cosines.

Theorem (The Law of Cosines)**.** *Consider a triangle with side lengths a, b, and c. Let θ be the angle between the sides of length a and b. Then the law of cosines states that*

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

If $\theta = 90^\circ$, then $\cos\theta = 0$ and this reduces to the Pythagorean theorem.

Definition 2.5: The Dot Product. If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$, then we define the dot product of these two vectors to be

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

A similar definition holds for vectors in $\mathbb{R}^n$, where $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$. You just multiply corresponding components together and then add. It is the same process used in matrix multiplication.

Problem 2.10 If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$ (which is often written $\vec{u}, \vec{v} \in \mathbb{R}^3$), then show that

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2.$$

Page 693 has the solution if you are struggling.

Problem 2.11 Sketch in $\mathbb{R}^2$ the vectors $\langle 1, 2\rangle$ and $\langle 3, 5\rangle$. Use the law of cosines to find the angle between the vectors.

See 12.3: 9-12.

Problem 2.12 Let $\vec{u}, \vec{v} \in \mathbb{R}^3$. Let θ be the angle between $\vec{u}$ and $\vec{v}$.

See page 693.

1. Use the law of cosines to explain why $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$.
2. Use the above together with problem 2.10 to explain why

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta.$$

Problem 2.13 Sketch in $\mathbb{R}^3$ the vectors $\langle 1, 2, 3\rangle$ and $\langle -2, 1, 0\rangle$. Use the law of cosines to find the angle between the vectors. Then use the formula $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$ to find the angle between them. Which was easier?

See 12.3: 9-12.

Definition 2.6. We say that the vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$.

Problem 2.14 Find two vectors orthogonal to $(1, 2)$. Then find 4 vectors orthogonal to $(3, 2, 1)$.

Problem 2.15 Mark each statement true or false. Explain. You can assume that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and that $c \in \mathbb{R}$.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
2. $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$.
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$.
4. $\vec{u} + (\vec{v} \cdot \vec{w}) = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{w})$.
5. $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$.
6. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

Problem 2.16 Show that if two nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal, then the angle between them is $90°$. Then show that if the angle between them is $90°$, then the vectors are orthogonal. See page 694.

The dot product provides a really easy way to find when two vectors meet at a right angle. The dot product is precisely zero when this happens.

2.2.1 Projections and Work

Suppose a heavy box needs to be lowered down a ramp. The box exerts a downward force of 200 Newtons, which we will write in vector notation as $\vec{F} = \langle 0, -200\rangle$. The ramp was placed so that the box needs to be moved right 6 m, and down 3 m, so we need to get from the origin $(0, 0)$ to the point $(6, -3)$. This displacement can be written as $\vec{d} = \langle 6, -3\rangle$. The force F acts straight down, which means the ramp takes some of the force. Our goal is to find out how much of the 200N the ramp takes, and how much force must be applied to prevent the box from sliding down the ramp (neglecting friction). We are going to break the force $\vec{F}$ into two components, one component in the direction of $\vec{d}$, and another component orthogonal to $\vec{d}$.

Problem 2.17 Read the preceding paragraph. We want to write $\vec{F}$ as the sum of two vectors $\vec{F} = \vec{w} + \vec{n}$, where $\vec{w}$ is parallel to $\vec{d}$ and $\vec{n}$ is orthogonal to $\vec{d}$. Since $\vec{w}$ is parallel to $\vec{d}$, we can write $\vec{w} = c\vec{d}$ for some unknown scalar c. This means that $\vec{F} = c\vec{d} + \vec{n}$. Use the fact that $\vec{n}$ is orthogonal to $\vec{d}$ to solve for the unknown scalar c. [Hint: dot each side of $\vec{F} = c\vec{d} + \vec{n}$ with $\vec{d}$. This should turn the vectors into numbers, so you can use division.]

The solution to the previous problem gives us the definition of a projection.

Definition 2.7. The projection of $\vec{F}$ onto $\vec{d}$, written $\text{proj}_{\vec{d}}\vec{F}$, is defined as

$$\text{proj}_{\vec{d}}\vec{F} = \left(\frac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\right)\vec{d}.$$

Problem 2.18 Let $\vec{u} = (-1, 2)$ and $\vec{v} = (3, 4)$. Draw $\vec{u}$, $\vec{v}$, and $\text{proj}_{\vec{v}}\vec{u}$. Then draw a line segment from the head of $\vec{u}$ to the head of the projection. See 12.3:1-8 (part d).

Now let $\vec{u} = (-2, 0)$ and keep $\vec{v} = (3, 4)$. Draw $\vec{u}$, $\vec{v}$, and $\text{proj}_{\vec{v}}\vec{u}$. Then draw a line segment from the head of $\vec{u}$ to the head of the projection.

One final application of projections pertains to the concept of work. Work is the transfer of energy. If a force F acts through a displacement d, then the most basic definition of work is $W = Fd$, the product of the force and the displacement. This basic definition has a few assumptions.

- The force F must act in the same direction as the displacement.
- The force F must be constant throughout the entire displacement.
- The displacement must be in a straight line.

Before the semester ends, we will be able to remove all 3 of these assumptions. The next problem will show you how dot products help us remove the first assumption.

Recall the set up to problem 2.17. We want to lower a box down a ramp (which we will assume is frictionless). Gravity exerts a force of $\vec{F} = \langle 0, -200 \rangle$ N. If we apply no other forces to this system, then gravity will do work on the box through a displacement of $\langle 6, -3 \rangle$ m. The work done by gravity will transfer the potential energy of the box into kinetic energy (remember that work is a transfer of energy). How much energy is transferred?

Problem 2.19 Find the amount of work done by the force $\vec{F} = \langle 0, -200 \rangle$ through the displacement $\vec{d} = \langle 6, -3 \rangle$. Find this by doing the following:

See 12.3: 24, 41-44.

1. Find the projection of $\vec{F}$ onto $\vec{d}$. This tells you how much force acts in the direction of the displacement. Find the magnitude of this projection.
2. Since work equals $W = Fd$, multiply your answer above by $|\vec{d}|$.
3. Now compute $\vec{F} \cdot \vec{d}$. You have just shown that $W = \vec{F} \cdot \vec{d}$ when $\vec{F}$ and $\vec{d}$ are not in the same direction.

2.3 The Cross Product and Planes

The dot product gave us a way of multiplying two vectors together, but the result was a number, not a vectors. We now define the cross product, which will allow us to multiply two vectors together to give us another vector. We were able to define the dot product in all dimensions. The cross product is only defined in $\mathbb{R}^3$.

Definition 2.8: The Cross Product. The cross product of two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is a new vector $\vec{u} \times \vec{v}$. This new vector is (1) orthogonal to both $\vec{u}$ and $\vec{v}$, (2) has a length equal to the area of the parallelogram whose sides are these two vectors, and (3) points in the direction your thumb points as you curl the base of your right hand from $\vec{u}$ to $\vec{v}$. The formula for the cross product is

$$\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

This definition is not really a definition. It is actually a theorem. If you use the formula given as the definition, then you would need to prove the three facts. We have the tools to give a complete proof of (1) and (3), but we would need a course in linear algebra to prove (2). It shouldn't be too much of a surprise that the cross product is related to area, since it is defined in terms of determinants

Problem 2.20 Let $\vec{u} = (1, -2, 3)$ and $\vec{v} = (2, 0, -1)$.

See 12.4: 1-8.

- Compute $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$. How are they related?
- Compute $\vec{u} \cdot (\vec{u} \times \vec{v})$ and $\vec{v} \cdot (\vec{u} \times \vec{v})$. Why did you get the answer you got?
- Compute $\vec{u} \times (2\vec{u})$. Why did you get the answer you got?
- Compute $|\vec{u} \times \vec{v}|$. Compute the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$ using trigonometry and $|\vec{u}|$, $|\vec{v}|$, and the angle θ between the two vectors. Compare your answer with $|\vec{u} \times \vec{v}|$.

Problem 2.21 Let $P = (2, 0, 0)$, $Q = (0, 3, 0)$, and $R = (0, 0, 4)$. Find a vector that orthogonal to both $\vec{PQ}$ and $\vec{PR}$. Then find the area of the triangle PQR. Construct a 3D graph of this triangle. See 12.4: 15-18.

Problem 2.22 Consider the vectors $\vec{\imath} = (1, 0, 0)$, $2\vec{\jmath} = (0, 2, 0)$, and $3\vec{k} = (0, 0, 3)$. See 12.3: 9-14.

- Compute $\vec{\imath} \times 2\vec{\jmath}$ and $2\vec{\jmath} \times \vec{\imath}$.
- Compute $\vec{\imath} \times 3\vec{k}$ and $3\vec{k} \times \vec{\imath}$.
- Compute $2\vec{\jmath} \times 3\vec{k}$ and $3\vec{k} \times 2\vec{\jmath}$.

Give a geometric reason as to why some vectors above have a plus sign, and some have a minus sign.

We will now combine the dot product with the cross product to develop an equation of a plane in 3D. Before doing so, let's look at what information we need to obtain a line in 2D, and a plane in 3D. To obtain a line in 2D, one way is to have 2 points. The next problem introduces the new idea by showing you how to find an equation of a line in 2D.

Problem 2.23 Suppose the point $P = (1, 2)$ lies on line L. Suppose that the angle between the line and the vector $\vec{n} = \langle 3, 4\rangle$ is 90° (whenever this happens we say the vector $\vec{n}$ is normal to the line). Let $Q = (x, y)$ be another point on the line L. Use the fact that $\vec{n}$ is orthogonal to $\vec{PQ}$ to obtain an equation of the line L.

Problem 2.24 Let $P = (a, b, c)$ be a point on a plane in 3D. Let $\vec{n} = (A, B, C)$ be a normal vector to the plane (so the angle between the plane and and $\vec{n}$ is 90°). Let $Q = (x, y, z)$ be another point on the plane. Show that an equation of the plane through point P with normal vector $\vec{n}$ is See page 709.

$$A(x - a) + B(y - b) + C(z - c) = 0.$$

Problem 2.25 Consider the three points $P = (1, 0, 0), Q = (2, 0, -1), R = (0, 1, 3)$. Find an equation of the plane which passes through these three points. [Hint: first find a normal vector to the plane.] See 12.5: 21-28.

Problem 2.26 Consider the two planes $x + 2y + 3z = 4$ and $2x - y + z = 0$. These planes meet in a line. Find a vector that is parallel to this line. Then find a vector equation of the line. See 12.5: 57-60.

Problem 2.27 Find an equation of the plane containing the lines $\vec{r}_1(t) = (1, 3, 0)t + (1, 0, 2)$ and $\vec{r}_2(t) = (2, 0, -1)t + (2, 3, 2)$.

2.4 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to Brainhoney and download the quiz. Once you have taken the quiz, you can upload your work back to brainhoney and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 3

Curves

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Be able to describe, graph, give equations of, and find foci for conic sections (parabolas, ellipses, hyperbolas).

2. Model motion in the plane using parametric equations. In particular, describe conic sections using parametric equations.

3. Find derivatives and tangent lines for parametric equations. Explain how to find velocity, speed, and acceleration from parametric equations.

4. Use integrals to find the lengths of parametric curves.

You'll have a chance to teach your examples to your peers prior to the exam.

3.1 Conic Sections

Before we jump fully into $\mathbb{R}^3$, we need some good examples of planar curves (curves in $\mathbb{R}^2$) that we'll extend to object in 3D. These examples are conic sections. We call them conic sections because you can obtain each one by intersecting a cone and a plane (I'll show you in class how to do this). Here's a definition.

Definition 3.1. Consider two identical, infinitely tall, right circular cones placed vertex to vertex so that they share the same axis of symmetry. A conic section is the intersection of this three dimensional surface with any plane that does not pass through the vertex where the two cones meet.

These intersections are called circles (when the plane is perpendicular to the axis of symmetry), parabolas (when the plane is parallel to one side of one cone), hyperbolas (when the plane is parallel to the axis of symmetry), and ellipses (when the plane does not meet any of the three previous criteria).

The definition above provides a geometric description of how to obtain a conic section from cone. We'll not introduce an alternate definition based on distances between points and lines, or between points and points. Let's start with one you are familiar with.

Definition 3.2. Consider the point $P = (a, b)$ and a positive number r. A circle circle with center (a, b) and radius r is the set of all points $Q = (x, y)$ in the plane so that the segment PQ has length r.

Using the distance formula, this means that every circle can be written in the form $(x-a)^2+(y-b)^2=r^2$.

Problem 3.1 The equation $4x^2+4y^2+6x-8y-1=0$ represents a circle (though initially it does not look like it). Use the method of completing the square to rewrite the equation in the form $(x-a)^2+(y-b)^2=r^2$ (hence telling you the center and radius). Then generalize your work to find the center and radius of any circle written in the form $x^2+y^2+Dx+Ey+F=0$.

3.1.1 Parabolas

Before proceeding to parabolas, we need to define the distance between a point and a line.

Definition 3.3. Let P be a point and L be a line. Define the distance between P and L (written $d(P,L)$) to be the length of the shortest line segment that has one end on L and the other end on P. Note: This segment will always be perpendicular to L.

Definition 3.4. Given a point P (called the focus) and a line L (called the directrix) which does not pass through P, we define a parabola as the set of all points Q in the plane so that the distance from P to Q equals the distance from Q to L. The vertex is the point on the parabola that is closest to the directrix.

Problem 3.2 Consider the line $L: y=-p$, the point $P=(0,p)$, and another point $Q=(x,y)$. Use the distance formula to show that an equation of a parabola with directrix L and focus P is $x^2=4py$. Then use your work to explain why an equation of a parabola with directrix $x=-p$ and focus $(p,0)$ is $y^2=4px$. See page 658.

Ask me about the reflective properties of parabolas in class, if I have not told you already. They are used in satellite dishes, long range telescopes, solar ovens, and more. The following problem provides the basis to these reflective properties and is optional. If you wish to present it, let me know. I'll have you type it up prior to presenting in class.

Problem: Optional Consider the parabola $x^2=4py$ with directrix $y=-p$ and focus $(0,p)$. Let $Q=(a,b)$ be some point on the parabola. Let T be the tangent line to L at point Q. Show that the angle between PQ and T is the same as the angle between the line $x=a$ and T. This shows that a vertical ray coming down towards the parabola will reflect of the wall of a parabola and head straight towards the vertex.

The next two problems will help you use the basic equations of a parabola, together with shifting and reflecting, to study all parabolas whose axis of symmetry is parallel to either the x or y axis.

Problem 3.3 Once the directrix and focus are known, we can give an equation of a parabola. For each of the following, give an equation of the parabola with the stated directrix and focus. Provide a sketch of each parabola. See 11.6: 9-14

1. The focus is $(0,3)$ and the directrix is $y=-3$.

2. The focus is $(0,3)$ and the directrix is $y=1$.

Problem 3.4 Give an equation of each parabola with the stated directrix and focus. Provide a sketch of each parabola.

1. The focus is $(2, -5)$ and the directrix is $y = 3$.
2. The focus is $(1, 2)$ and the directrix is $x = 3$.

Problem 3.5 Each equation below represents a parabola. Find the focus, directrix, and vertex of each parabola, and then provide a rough sketch. See 11.6: 9-14

1. $y = x^2$
2. $(y - 2)^2 = 4(x - 1)$

Problem 3.6 Each equation below represents a parabola. Find the focus, directrix, and vertex of each parabola, and then provide a rough sketch.

1. $y = -8x^2 + 3$
2. $y = x^2 - 4x + 5$

3.1.2 Ellipses

Definition 3.5. Given two points F_1 and F_2 (called foci) and a fixed distance d, we define an ellipse as the set of all points Q in the plane so that the sum of the distances F_1Q and F_2Q equals the fixed distance d. The center of the ellipse is the midpoint of the segment F_1F_2. The two foci define a line. Each of the two points on the ellipse that intersect this line is called a vertex. The major axis is the segment between the two vertexes. The minor axis is the largest segment perpendicular to the major axis that fits inside the ellipse.

We can derive an equation of an ellipse in a manner very similar to how we obtained an equation of a parabola. The following problem will walk you through this. We will not have time to present this problem in class. However, if you would like to complete the problem and write up your solution on the wiki, you can obtain presentation points for doing so. Let me know if you are interested.

Problem: Optional Consider the ellipse produced by the fixed distance d and the foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $(a, 0)$ and $(-a, 0)$ be the vertexes of the ellipse.

1. Show that $d = 2a$ by considering the distances from F_1 and F_2 to the point $Q = (a, 0)$.
2. Let $Q = (0, b)$ be a point on the ellipse. Show that $b^2 + c^2 = a^2$ by considering the distance between Q and each focus.
3. Let $Q = (x, y)$. By considering the distances between Q and the foci, show that an equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

4. Suppose the foci are along the y-axis (at $(0, \pm c)$) and the fixed distance d is now $d = 2b$, with vertexes $(0, \pm b)$. Let $(a, 0)$ be a point on the x axis that intersect the ellipse. Show that we still have
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$
but now we instead have $a^2 + c^2 = b^2$.

You'll want to use the results of the previous problem to complete the problems below. The key equation above is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The foci will be on the x-axis if $a > b$, and will be on the y-axis if $b > a$. The second part of the problem above shows that the distance from the center of the ellipse to the vertex is equal to the hypotenuse of a right triangle whose legs go from the center to a focus, and from the center to an end point of the minor axis.

The next three problems will help you use the basic equations of an ellipse, together with shifting and reflecting, to study all ellipses whose major axis is parallel to either the x- or y-axis.

Problem 3.7 For each ellipse below, graph the ellipse and give the coordinates of the foci and vertexes. See 11.6: 17-24

1. $16x^2 + 25y^2 = 400$ [Hint: divide by 400.]
2. $\dfrac{(x-1)^2}{5} + \dfrac{(y-2)^2}{9} = 1$

Problem 3.8 For the ellipse $x^2 + 2x + 2y^2 - 8y = 9$, sketch a graph and give the coordinates of the foci and vertexes.

Problem 3.9 Given an equation of each ellipse described below, and provide a rough sketch. See 11.6: 25-26

1. The foci are at $(2 \pm 3, 1)$ and vertices at $(2 \pm 5, 1)$.
2. The foci are at $(-1, 3 \pm 2)$ and vertices at $(-1, 3 \pm 5)$.

Ask me about the reflective properties of an ellipse in class, if I have not told you already. The following problem provides the basis to these reflective properties and is optional. If you wish to present it, let me know. I'll have you type it up prior to presenting in class.

Problem: Optional Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $Q = (x, y)$ be some point on the ellipse. Let T be the tangent line to the ellipse at point Q. Show that the angle between F_1Q and T is the same as the angle between F_2Q and T. This shows that a ray from F_1 to Q will reflect off the wall of the ellipse at Q and head straight towards the other focus F_2.

3.1.3 Hyperbolas

Definition 3.6. Given two points F_1 and F_2 (called foci) and a fixed number d, we define a hyperbola as the set of all points Q in the plane so that the difference of the distances F_1Q and F_2Q equals the fixed number d or $-d$. The center of the hyperbola is the midpoint of the segment F_1F_2. The two foci define a line. Each of the two points on the hyperbola that intersect this line is called a vertex.

We can derive an equation of a hyperbola in a manner very similar to how we obtained an equation of an ellipse. The following problem will walk you through this. We will not have time to present this problem in class.

Problem: Optional Consider the hyperbola produced by the fixed number d and the foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $(a, 0)$ and $(-a, 0)$ be the vertexes of the hyperbola.

1. Show that $d = 2a$ by considering the difference of the distances from F_1 and F_2 to the vertex $(a, 0)$.
2. Let $Q = (x, y)$ be a point on the hyperbola. By considering the difference of the distances between Q and the foci, show that an equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1$, or if we let $c^2 - a^2 = b^2$, then the equation is
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$
3. Suppose the foci are along the y-axis (at $(0, \pm c)$) and the number d is now $d = 2b$, with vertexes $(0, \pm b)$. Let $a^2 = c^2 - b^2$. Show that an equation of the hyperbola is
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

You'll want to use the results of the previous problem to complete the problems below.

Problem 3.10 Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Construct a box centered at the origin with corners at $(a, \pm b)$ and $(-a, \pm b)$. Draw lines through the diagonals of this box. Rewrite the equation of the hyperbola by solving for y and then factoring to show that as x gets large, the hyperbola gets really close to the lines $y = \pm\frac{b}{a}x$. [Hint: rewrite so that you obtain $y = \pm\frac{b}{a}x\sqrt{\text{something}}$]. These two lines are often called oblique asymptotes. See 11.6: 27-34

Now apply what you have just done to sketch the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$ and give the location of the foci.

The next three problems will help you use the basic equations of a hyperbola, together with shifting and reflecting, to study all ellipses whose major axis is parallel to either the x- or y-axis.

Problem 3.11 For each hyperbola below, graph the hyperbola (include the box and asymptotes) and give the coordinates of the foci and vertexes. See 11.6: 27-34

1. $16x^2 - 25y^2 = 400$ [Hint: divide by 400.]
2. $\dfrac{(x-1)^2}{5} - \dfrac{(y-2)^2}{9} = 1$

Problem 3.12 For the hyperbola $x^2 + 2x - 2y^2 + 8y = 9$, sketch a graph (include the box and asymptotes) and give the coordinates of the foci and vertexes.

Problem 3.13 Given an equation of each hyperbola described below, and provide a rough sketch. See 11.6: 35-38

1. The vertexes are at $(2 \pm 3, 1)$ and foci at $(2 \pm 5, 1)$.
2. The vertexes are at $(-1, 3 \pm 2)$ and foci at $(-1, 3 \pm 5)$.

Ask me about the reflective properties of a hyperbola in class, if I have not told you already. In particular, we can discuss lasers and long range telescopes. The following problem provides the basis to these reflective properties and is optional. If you wish to present it, let me know. I'll have you type it up prior to presenting in class.

Problem: Optional Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $Q = (x, y)$ be a point on the hyperbola. Let T be the tangent line to the hyperbola at point Q. Show that the angle between F_1Q and T is the same as the angle between F_2Q and T. This shows that if you begin a ray from a point in the plane and head towards F_1 (where the wall of the hyperbola lies between the start point and F_1), then when the ray hits the wall at Q, it reflects off the wall and heads straight towards the other focus F_2.

3.2 Parametric Equations

In middle school, you learned to write an equation of a line as $y = mx + b$. In the vector unit, we learned to write this in vector form as $(x, y) = (1, m)t + (0, b)$. The equation to the left is called a vector equation. It is equivalent to writing the two equations

$$x = 1t + 0, y = mt + b,$$

which we will call parametric equations of the line. We were able to quickly develop equations of lines in space, by just adding a third equation for z.

Parametric equations provide us with a way of specifying the location (x, y, z) of an object by giving an equation for each coordinate. We will use these equations to model motion in the plane and in space. In this section we'll focus mostly on planar curves.

Definition 3.7. If each of f and g are continuous functions, then the curve in the plane defined by $x = f(t), y = g(t)$ is called a parametric curve, and the equations $x = f(t), y = g(t)$ are called parametric equations for the curve. You can generalize this definition to 3D and beyond by just adding more variables.

Problem 3.14 By plotting points, construct graphs of the three parametric curves given below (just make a t, x, y table, and then plot the (x, y) coordinates). Place an arrow on your graph to show the direction of motion. See 11.1: 1-18. This is the same for all the problems below.

1. $x = \cos t, y = \sin t$, for $0 \leq t \leq 2\pi$.

2. $x = \sin t, y = \cos t$, for $0 \le t \le 2\pi$.
3. $x = \cos t, y = \sin t, z = t$, for $0 \le t \le 4\pi$.

Problem 3.15 Plot the path traced out by the parametric curve $x = 1 + 2\cos t, y = 3 + 5\sin t$. Then use the trig identity $\cos^2 t + \sin^2 t = 1$ to give a Cartesian equation of the curve (an equation that only involves x and y). What are the foci of the resulting object (it's a conic section).

Problem 3.16 Find parametric equations for a line that passes through the points $(0, 1, 2)$ and $(3, -2, 4)$.

What we did in the previous chapter should help here.

Problem 3.17 Plot the path traced out by the parametric curve $\vec{r}(t) = (t^2 + 1, 2t - 3)$. Give a Cartesian equation of the curve (eliminate the parameter t), and then find the focus of the resulting curve.

Problem 3.18 Consider the parametric curve given by $x = \tan t, y = \sec t$. Plot the curve for $-\pi/2 < t < \pi/2$. Give a Cartesian equation of the curve (a trig identity will help). Then find the foci of the resulting conic section. [Hint: this problem will probably be easier to draw if you first find the Cartesian equation, and then plot the curve.]

3.2.1 Derivatives and Tangent lines

We're now ready to discuss calculus on parametric curves. The derivative of a vector valued function is defined using the same definition as first semester calculus.

Definition 3.8. If $\vec{r}(t)$ is a vector equation of a curve (or in parametric form just $x = f(t), y = g(t)$), then we define the derivative to be

$$\frac{d\vec{r}}{dt} = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

The subtraction above requires vector subtraction. The following problem will provide a simple way to take derivatives which we will use all semester long.

Problem 3.19 Show that if $\vec{r}(t) = (f(t), g(t))$, then the derivative is just $\frac{d\vec{r}}{dt} = (f'(t), g'(t))$. In other words, you can take the derivative by just differentiating each component separately.

See page 728.

Problem 3.20 Consider the parametric curve given by $\vec{r}(t) = (3\cos t, 3\sin t)$.

See 13.1:5-8 and 13.1:19-20

1. Compute $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$.
2. Construct a graph of the curve given by $\vec{r}$.
3. On your graph, draw the vectors $\frac{d\vec{r}}{dt}\left(\frac{\pi}{4}\right)$ and $\frac{d^2\vec{r}}{dt^2}\left(\frac{\pi}{4}\right)$ with their tail placed on the curve at $\vec{r}\left(\frac{\pi}{4}\right)$. These vectors represent the velocity and acceleration vectors.

4. Give a vector equation of the tangent line to this curve at $t = \frac{\pi}{4}$.

Definition 3.9. If an object moves along a path $\vec{r}(t)$, we can find the velocity and acceleration by just computing the first and second derivatives. The velocity is $\frac{d\vec{r}}{dt}$, and the acceleration is $\frac{d^2\vec{r}}{dt^2}$. Speed is a scalar, not a vector. The speed of an object is just the length of the velocity vector.

Problem 3.21 Consider the curve $\vec{r}(t) = (2t+3, 4(2t-1)^2)$.

1. Construct a graph of $\vec{r}$ for $0 \leq t \leq 2$.
2. If this curve represented the path of a horse running through a pasture, find the velocity of the horse at any time t, and then specifically at $t = 1$. What is the horse's speed at $t = 1$?
3. Find a vector equation of the tangent line to $\vec{r}$ at $t = 1$. Include this on your graph.
4. Show that the slope of the line is $\left.\dfrac{dy}{dx}\right|_{x=5} = \dfrac{\left.\frac{dy}{dt}\right|_{t=1}}{\left.\frac{dx}{dt}\right|_{t=1}}$.

3.2.2 Arc Length

If an object moves at a constant speed, then the distance travelled is

$$\text{distance} = \text{speed} \times \text{time}.$$

This requires that the speed be constant. What if the speed is not constant? Over a really small time interval dt, the speed is almost constant, so we can still use the idea above. The following problem will help you develop the key formula for arc length.

Problem 3.22: Derivation of the arc length formula Suppose an object moves along the path given by $\vec{r}(t) = (x(t), y(t))$ for $a \leq t \leq b$.

1. Show that the object's speed at any time t is $\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$.
2. If you move over a really small time interval, say of length dt, then the speed is almost constant. Give a formula for the small distance ds you have travelled through a small time dt, provided you are moving at the speed given above.
3. Explain why the length of the path given by $\vec{r}$ is

   This is the arc length formula.

$$s = \int ds = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Problem 3.23 Find the length of the curve $\vec{r}(t) = \left(t^3, \dfrac{3t^2}{2}\right)$ for $t \in [1, 3]$. See 11.2: 25-30
The notation $t \in [1, 3]$ means $1 \leq t \leq 3$.

Problem 3.24 Set up an integral formula which would give the length of the following curves. Sketch the curve. Do not worry about integrating them.

1. The parabola $\vec{p}(t) = (t, t^2)$ for $t \in [0, 3]$.
2. The ellipse $\vec{e}(t) = (4\cos t, 5\sin t)$ for $t \in [0, 2\pi]$.
3. The hyperbola $\vec{h}(t) = (\tan t, \sec t)$ for $t \in [-\pi/4, \pi/4]$.

The reason I don't want you to actually compute the integrals is that they will get ugly really fast. Try doing one in Wolfram Alpha and see what the computer gives.

To actually compute the integrals above and find the lengths, we would use a numerical technique to approximate the integral (something akin to adding up the areas of lots and lots of rectangles as you did in first semester calculus).

3.3 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to Brainhoney and download the quiz. Once you have taken the quiz, you can upload your work back to brainhoney and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 4

New Coordinates

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Be able to convert between rectangular and polar coordinates in 2D. Convert between rectangular and cylindrical or spherical in 3D.
2. Graph polar functions in the plane. Find intersections of polar equations, and illustrate that not every intersection can be obtained algebraically (you may have to graph the curves).
3. Find derivatives and tangent lines in polar coordinates.
4. Find area and arc length using polar equations.

You'll have a chance to teach your examples to your peers prior to the exam.

4.1 Polar Coordinates

Up to now, we most often give the location of a point (or coordiantes of a vector) by stating the (x, y) coordinates. These are called the Cartesian (or rectangular) coordinates. Some problems are much easier to work with if we know how far a point is from the origin, together with the angle between the x-axis and a ray from the origin to the point.

Problem 4.1 There are two parts to this problem. See 11.3:5-10.

1. Consider the point P with Cartesian (rectangular) coordinates $(2, 1)$. Find the distance r from P to the origin. Consider the ray $\vec{OP}$ from the origin through P. Find an angle between $\vec{OP}$ and the x-axis.
2. Suppose that a point $Q = (a, b)$ is 6 units from the origin, and the angle the ray $\vec{OP}$ makes with the x-axis is $\pi/4$ radians. Find the Cartesian (rectangular) coordinates (a, b) of Q.

Definition 4.1. Let Q be be a point in the plane with Cartesian coordinates (x, y). Let $O = (0, 0)$ be the origin. We define the polar coordinates of Q to be the ordered pair (r, θ) where r is the displacement from the origin to Q, and θ is an angle of rotation (counter-clockwise) from the x-axis to the ray $\vec{OP}$.

Problem 4.2 The following points are given using their polar coordinates. Plot the points in the Cartesian plane, and give the Cartesian (rectangular) coordinates of each point. The points are

$$(1,\pi), \left(3, \frac{5\pi}{4}\right), \left(-3, \frac{\pi}{4}\right), \text{ and } \left(-2, -\frac{\pi}{6}\right).$$

See 11.3:5-10.

The next problem provides general formulas for converting between the Cartesian (rectangular) and polar coordinate systems.

Problem 4.3 Suppose that Q is a point in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ). Write formulas for x and y in terms of r and θ. Then write a formula to find the distance r from Q to the origin (in terms of x and y) as well as a formula to find the angle θ between the x-axis and a line connecting Q to the origin. [Hint: A picture of a triangle will help here.]

See page 647.

In problem 4.3, you should have obtained the equations

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We can write this in vector notation as $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$. This is a vector equation in which you input polar coordinates (r, θ) and get out Cartesian coordinates (x, y). So you input one thing to get out one thing, which means that we have a function. We could write $\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta)$, where we've used the letter T as the name of the function because it is a transformation between coordinate systems. To emphasize that the domain and range are both two dimensional systems, we could also write $T : \mathbb{R}^2 \to \mathbb{R}^2$. In the next chapter, we'll spend more time with this notation. The following problem will show you how to graph a coordinate transformation. When you're done, you should essentially have polar graph paper.

Problem 4.4 Consider the coordinate transformation

$$\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta).$$

For this problem, you are just drawing many parametric curves. This is what we did in the previous chapter.

1. Let $r = 3$ and then graph $\vec{T}(3,\theta) = (3\cos\theta, 3\sin\theta)$ for $\theta \in [0, 2\pi]$.
2. Let $\theta = \frac{\pi}{4}$ and then, on the same axes as above, add the graph of $\vec{T}\left(r, \frac{\pi}{4}\right) = \left(r\frac{\sqrt{2}}{2}, r\frac{\sqrt{2}}{2}\right)$ for $r \in [0, 5]$.
3. To the same axes as above, add the graphs of $\vec{T}(1,\theta), \vec{T}(2,\theta), \vec{T}(4,\theta)$ for $\theta \in [0, 2\pi]$ and $\vec{T}(r,0), \vec{T}(r,\pi/2), \vec{T}(r,3\pi/4), \vec{T}(r,\pi)$ for $r \in [0, 5]$.

Problem 4.5 In the plane, graph the curve $y = \sin x$ for $x \in [0, 2\pi]$ and the curve $r = \sin\theta$ for $\theta \in [0, 2\pi]$ (just make an r, θ table).

Problem 4.6 Each of the following equations is written in the Cartesian (rectangular) coordinate system. Convert each to an equation in polar coordinates, and then solve for r so that the equation is in the form $r = f(\theta)$.

See 11.3: 53-66.

1. $x^2 + y^2 = 7$

2. $2x + 3y = 5$
3. $x^2 = y$

Problem 4.7 Each of the following equations is written in the polar coordinate system. Convert each to an equation in the Cartesian coordinates.

See 11.3: 27-52. I strongly suggest that you do many of these as practice.

1. $r = 9\cos\theta$
2. $r = \dfrac{4}{2\cos\theta + 3\sin\theta}$
3. $\theta = 3\pi/4$

4.1.1 Graphing and Intersections

To construct a graph of a polar curve, just create an r, θ table. Choose values for θ that will make it easy to compute any trig functions involved. Then connect the points in a smooth manner, making sure that your radius grows or shrinks appropriately as your angle increases.

Problem 4.8 Graph the polar curve $r = 2 + 2\cos\theta$.

See 11.4: 1-20.

Problem 4.9 Graph the polar curve $r = 2\sin 3\theta$.

Problem 4.10 Graph the polar curve $r = 3\cos 2\theta$.

Problem 4.11 Find the points of intersection of $r = 3 - 3\cos\theta$ and $r = 3\cos\theta$. (If you don't graph the curves, you'll probably miss a few points of intersection.)

Problem 4.12 Find the points of intersection of $r = 2\cos 2\theta$ and $r = \sqrt{3}$. (If you don't graph the curves, you'll probably miss a few points of intersection.)

4.1.2 Calculus with Polar Coordinates

Recall that for parametric curves $\vec{r}(t) = (x(t), y(t))$, to find the slope of the curve we just compute

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

A polar curve of the form $r = f(\theta)$ can be thought of as just the parametric curve $(x, y) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. So you can find the slope by computing

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Problem 4.13 Consider the polar curve $r = 1 + 2\cos\theta$. (It wouldn't hurt to provide a quick sketch of the curve.) See 11.2: 1-14.

1. Compute both $dx/d\theta$ and $dy/d\theta$.
2. Find the slope dy/dx of the curve at $\theta = \pi/2$.
3. Give both a vector equation of the tangent line, and a Cartesian equation of the tangent line at $\theta = \pi/2$.

We showed in the curves section that you can find arc length for parametric curves using the formula

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

If we replace t with θ, this becomes a formula for arc length in polar coordinates. However, the formula can be simplified.

Problem 4.14 Recall that $x = r\cos\theta$ and $y = r\sin\theta$. Suppose that $r = f(\theta)$ for $\theta \in [\alpha, \beta]$ is a continuous function, and that f' is continuous. Show that the arc length formula can be simplified to See 11.5: 29.

$$s = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

[Hint: the product rule and Pythagorean identity will help.]

Problem 4.15 Set up (do not evaluate) an integral formula to compute the length of See 11.5: 21-28.

1. the rose $r = 2\cos 3\theta$, and
2. the rose $r = 3\sin 2\theta$.

Problem 4.16 In this problem, you will develop a formula for finding area inside a polar curve. See page 653.

1. Consider a circle of radius r. The area inside the circle is πr^2. This is the area inside when you traverse around the circle for a full 2π radians. Fill in the following table by finding the pattern that connects angle traversed to area inside.

Angle traversed	Area inside
2π	$A = \pi r^2$
π	
$\pi/2$	
$\pi/4$	
$d\theta$	$dA =$

2. Explain why the area inside a polar curve $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$ is

$$A = \int dA = \int_\alpha^\beta \frac{1}{2} r^2 d\theta.$$

What must be true about the curve $r = f(\theta)$ for this formula to be valid?

Problem 4.17 Find the area inside of the polar curve $r = \sin\theta$. [Hint: Construct a graph to determine the appropriate bounds for the integral. When you integrate, you'll need to use the half angle identity.]

See 11.5: 1-20.

Problem 4.18 Set up (do not evaluate) an integral to compute the area

1. inside the cardioid $r = 2 + 2\sin\theta$, and
2. inside the circle $r = 3\cos\theta$.

Problem 4.19 Set up (do not evaluate) an integral formula to compute the area that lies inside both $r = 2 - 2\cos\theta$ and $r = \cos\theta$. Sketch both curves.

4.2 Other Coordinate Systems

In this chapter, we've introduced just one of many different coordinate systems that people have used over the centuries. Sometimes a problem can't be solved until the correct coordinate system is chosen. Problem 4.4 showed you how to graph the coordinate transformation given by polar coordinates. The following problem shows you how to graph in a different coordinate system.

Problem 4.20 Consider the coordinate transformation $T(a, \omega) = (a\cos\omega, a^2\sin\omega)$.

1. Let $a = 3$ and then graph the curve $\vec{T}(3, \omega) = (3\cos\omega, 9\sin\omega)$ for $\omega \in [0, 2\pi]$.

   See Sage. Click on the link to see how to check your answer in Sage.

2. Let $\theta = \frac{\pi}{4}$ and then, on the same axes as above, add the graph of $\vec{T}\left(a, \frac{\pi}{4}\right) = \left(a\frac{\sqrt{2}}{2}, a^2\frac{\sqrt{2}}{2}\right)$ for $a \in [0, 4]$.

   See Sage. Notice that you can add the two plots together to superimpose them on each other.

3. To the same axes as above, add the graphs of $\vec{T}(1, \omega), \vec{T}(2, \omega), \vec{T}(4, \omega)$ for $\omega \in [0, 2\pi]$ and $\vec{T}(a, 0), \vec{T}(a, \pi/2), \vec{T}(a, -\pi/6)$ for $a \in [0, 4]$.

   Use Sage to check your answer.

[Hint: when you're done, you should have a bunch of parabolas and ellipses.]

In 3 dimensions, the most common coordinate systems are cylindrical and spherical. The equations for these coordinate systems are in the table below.

Cylindrical Coordinates	Spherical Coordinates
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$
$z = z$	$z = \rho\cos\phi$

Problem 4.21 Let $P = (x, y, z)$ be a point in space. This point lies on a cylinder of radius r, where the cylinder has the z axis as its axis of symmetry. The height of the point is z units up from the xy plane. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the ray $\vec{OQ}$ and the x-axis is θ. Construct a graph in 3D of this information, and use it to develop the equations for cylindrical coordinates given above.

See page 893.

Problem 4.22 Let $P = (x, y, z)$ be a point in space. This point lies on a sphere of radius ρ ("rho"), where the sphere's center is at the origin $O = (0, 0, 0)$. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the ray $\vec{OQ}$ and the x-axis is θ, and is called the azimuth angle. The angle between the ray $\vec{OP}$ and the z axis is ϕ ("phi"), and is called the inclination angle, polar angle, or zenith angle. Construct a graph in 3D of this information, and use it to develop the equations for spherical coordinates given above.

See page 897.

There is some disagreement between different fields about the notation for spherical coordinates. In some fields (like physics), ϕ represents the azimuth angle and θ represents the inclination angle. In some fields, like geography, instead of the inclination angle, the *elevation* angle is given—the angle from the xy-plane (lines of lattitude are from the elevation angle). Additionally, sometimes the coordinates are written in a different order. You should always check the notation for spherical coordinates before communicating using them.

See the Wikipedia or MathWorld for a discussion of conventions in different disciplines.

4.3 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to Brainhoney and download the quiz. Once you have taken the quiz, you can upload your work back to brainhoney and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 5

Functions

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Describe uses for, and construct graphs of, space curves and parametric surfaces. Find derivatives of space curves, and use this to find velocity, acceleration, and find equations of tangent lines.
2. Describe uses for, and construct graphs of, functions of several variables. For functions of the form $z = f(x, y)$, this includes both 3D surface plots and 2D level curve plots. For functions of the form $w = f(x, y, z)$, construct plots of level surfaces.
3. Describe uses for, and construct graphs of, vector fields and transformations.
4. If you are given a description of a vector field, curve, or surface (instead of a function or parametrization), explain how to obtain a function for the vector field, or a parametrization for the curve or surface.

You'll have a chance to teach your examples to your peers prior to the exam.

5.1 Function Terminology

A function is a set of instructions involving two sets (called the domain and codomain). A function assigns to each element of the domain D exactly one element in the codomain R. We'll often refer to the codomain R as the target space. We'll write

$$f\colon D \to R$$

when we want to remind ourselves of the domain and target space. In this class, we will study what happens when the domain and target space are subsets of $\mathbb{R}^n$ (Euclidean n-space). In particular, we will study functions of the form

$$f\colon \mathbb{R}^n \to \mathbb{R}^m,$$

when m and n are 3 or less. The value of n is the dimension of the input vector (or number of inputs). The number m is the dimension of the output vector (or number of outputs). Our goal is to understand uses for each type of function, and be able to construct graphs to represent the function.

We will focus most of our time this semester on two- and three-dimensional problems. However, many problems in the real world require a higher number of dimensions. When you hear the word "dimension", it does not always represent a physical dimension, such as length, width, or height. If a quantity depends on 30 different measurements, then the problem involves 30 dimensions. As a quick illustration, the formula for the distance between two points depends on 6 numbers, so distance is really a 6-dimensional problem. As another example, if a piece of equipment has a color, temperature, age, and cost, we can think of that piece of equipment being represented by a point in four-dimensional space (where the coordinate axes represent color, temperature, age, and cost).

Problem 5.1 A pebble falls from a 64 ft tall building. Its height (in ft) above the ground t seconds after it drops is given by the function $y = f(t) = 64 - 16t^2$. What are n and m when we write this function in the form $f\colon \mathbb{R}^n \to \mathbb{R}^m$? Construct a graph of this function. How many dimensions do you need to graph this function?

See Sage or Wolfram Alpha. Follow the links to Sage or Wolfram Alpha in all the problems below to see how to get the computer to graph the function.

5.2 Parametric Curves: $\vec{f}\colon \mathbb{R} \to \mathbb{R}^m$

Problem 5.2 A horse runs around an elliptical track. Its position at time t is given by the function $\vec{r}(t) = (2\cos t, 3\sin t)$. We could alternatively write this as $x = 2\cos t, y = 3\sin t$.

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. Construct a graph of this function.
3. Next to a few points on your graph, include the time t at which the horse is at this point on the graph. Include an arrow for the horse's direction.
4. How many dimensions do you need to graph this function?

See Sage or Wolfram Alpha. See also Chapter 3 of this problem set. There's a lot more practice of this idea in 11.1. You'll also find more practice in 13.1: 1-8.

Notice in the problem above that we placed a vector symbol above the function name, as in $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$. When the target space (codomain) is 2-dimensional or larger, we place a vector above the function name to remind us that the output is more than just a number.

Problem 5.3 Consider the pebble from problem 5.1. The pebble's height was given by $y = 64 - 16t^2$. The pebble also has some horizontal velocity (it's moving at 3 ft/s to the right). If we let the origin be the base of the 64 ft building, then the position of the pebble at time t is given by $\vec{r}(t) = (3t, 64 - 16t^2)$.

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. At what time does the pebble hit the ground (the height reaches zero)? Construct a graph of the pebble's path from when it leaves the top of the building till when it hits the ground.
3. Find the pebble's velocity and acceleration vectors at $t = 1$? Draw these vectors on your graph with their base at the pebble's position at $t = 1$.
4. At what speed is the pebble moving when it hits the ground?

See Sage or Wolfram Alpha. The text has more practice in 13.1: 1-8.

See Section 3.2.1 and Definition 3.9.

In the next problem, we keep the input as just a single number t, but the output is now a vector in $\mathbb{R}^3$.

Problem 5.4 A jet begins spiraling upwards to gain height. The position of the jet after t seconds is modeled by the equation $\vec{r}(t) = (2\cos t, 2\sin t, t)$. We could alternatively write this as $x = 2\cos t$, $y = 2\sin t$, $z = t$.

See Sage or Wolfram Alpha. The text has more practice in 13.1: 9-14.

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. Construct a graph of this function by picking several values of t and plotting the resulting points $(2\cos t, 2\sin t, t)$.
3. Next to a few points on your graph, include the time t at which the jet is at this point on the graph. Include an arrow for the jet's direction.
4. How many dimensions do you need to graph this function?

In all the problems above, you should have noticed that in order to draw a function (provided you include arrows for direction, or use an animation to represent "time"), you can determine how many dimensions you need to graph a function by just summing the dimensions of the domain and codomain. This is true in general.

Problem 5.5 Use the same set up as problem 5.4, namely

$$\vec{r}(t) = (2\cos t, 2\sin t, t).$$

See Section 3.2.1 and Definition 3.9.
The text has more practice in 13.1: 19-22.

You'll need a graph of this function to complete this problem.

1. Find the first and second derivative of $\vec{r}(t)$.
2. Compute the velocity and acceleration vectors at $t = \pi/2$. Place these vectors on your graph with their tails at the point corresponding to $t = \pi/2$.
3. Give an equation of the tangent line to this curve at $t = \pi/2$.

5.3 Parametric Surfaces: $\vec{f}\colon \mathbb{R}^2 \to \mathbb{R}^3$

We now increase the number of inputs from 1 to 2. This will allow us to graph many space curves at the same time.

Problem 5.6 The jet from problem 5.4 is actually accompanied by several jets flying side by side. As all the jets fly, they leave a smoke trail behind them (it's an air show). The smoke from one jet spreads outwards to mix with the neighboring jet, so that it looks like the jets are leaving a rather wide sheet of smoke behind them as they fly. The position of two of the many other jets is given by $\vec{r}_3(t) = (3\cos t, 3\sin t, t)$ and $\vec{r}_4(t) = (4\cos t, 4\sin t, t)$. A function which represents the smoke stream is $\vec{r}(a,t) = (a\cos t, a\sin t, t)$ for $0 \le t \le 4\pi$ and $2 \le a \le 4$.

See Sage or Wolfram Alpha.

1. What are n and m when we write the function $\vec{r}(a,t) = (a\cos t, a\sin t, t)$ in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. Start by graphing the position of the three jets $\vec{r}(2,t) = (2\cos t, 2\sin t, t)$, $\vec{r}(3,t) = (3\cos t, 3\sin t, t)$ and $\vec{r}(4,t) = (4\cos t, 4\sin t, t)$.
3. Let $t = 0$ and graph the curve $r(a,0) = (a,0,0)$ for $a \in [2,4]$. Then repeat this for $t = \pi/2, \pi, 3\pi/2$.
4. Describe the resulting surface.

The function above is called a parametric surface. Parametric surfaces are formed by joining together many parametric space curves. Most of 3D computer animation is done using parametric surfaces. Woody's entire body in *Toy Story* is a collection of parametric surfaces. Car companies create computer models of vehicles using parametric surfaces, and then use those parametric surfaces to study collisions. Often the mathematics behind these models is hidden in the software program, but parametric surfaces are at the heart of just about every 3D computer model.

Problem 5.7 Consider the parametric surface $\vec{r}(u,v) = (u\cos v, u\sin v, u^2)$ for $0 \leq u \leq 3$ and $0 \leq v \leq 2\pi$. Construct a graph of this function. To do so, let u equal a constant (such as 1, 2, 3) and then graph the resulting space curve. Then let v equal a constant (such as 0, $\pi/2$, etc.) and graph the resulting space curve until you can visualize the surface. [Hint: Think satellite dish.]

See Sage or Wolfram Alpha.

5.4 Functions of Several Variables: $f\colon \mathbb{R}^n \to \mathbb{R}$

In this section we'll focus on functions of the form $f\colon \mathbb{R}^2 \to \mathbb{R}^1$ and $f\colon \mathbb{R}^3 \to \mathbb{R}^1$; we'll keep the output as a real number. In the next problem, you should notice that the input is a vector (x, y) and the output is a number z. There are two ways to graph functions of this type. The next two problems show you how.

Problem 5.8 A computer chip has been disconnected from electricity and sitting in cold storage for quite some time. The chip is connected to power, and a few moments later the temperature (in Celsius) at various points (x, y) on the chip is measured. From these measurements, statistics is used to create a temperature function $z = f(x, y)$ to model the temperature at any point on the chip. Suppose that this chip's temperature function is given by the equation $z = f(x, y) = 9 - x^2 - y^2$. We'll be creating a 3D model of this function in this problem, so you'll want to place all your graphs on the same x, y, z axes.

See Sage or Wolfram Alpha.

1. What is the temperature at $(0, 0)$, $(1, 2)$, and $(-4, 3)$? See 14.1: 1-4.
2. If you let $y = 0$, construct a graph of the temperature $z = f(x, 0) = 9 - x^2 - 0^2$, or just $z = 9 - x^2$. In the xz plane (where $y = 0$) draw this upside down parabola.
3. Now let $x = 0$. Draw the resulting parabola in the yz plane.
4. Now let $z = 0$. Draw the resulting curve in the xy plane.
5. Once you've drawn a curve in each of the three coordinate planes, it's useful to pick an input variable (either x or y) and let it equal various constants. So now let $x = 1$ and draw the resulting parabola in the plane $x = 1$. Then repeat this for $x = 2$.
6. Describe the shape. Add any extra features to your graph to convey the 3D image you are constructing. See 14.1: 37-48.

Problem 5.9 We'll be using the same function $z = f(x, y) = 9 - x^2 - y^2$ as the previous problem. However, this time we'll construct a graph of the function by only studying places where the temperature is constant. We'll create a graph in 2D of the surface (similar to a topographical map).

See Sage or Wolfram Alpha.

1. Which points in the plane have zero temperature? Just let $z = 0$ in $z = 9 - x^2 - y^2$. Plot the corresponding points in the xy-plane, and write $z = 0$ next to this curve. This curve is called a level curve. As long as you stay on this curve, your temperature will remain level, it will not increase nor decrease.

   See 14.1: 13-16 and 31-36.

2. Which points in the plane have temperature $z = 5$? Add this level curve to your 2D plot and write $z = 5$ next to it.

3. Repeat the above for $z = 8$, $z = 9$, and $z = 1$. What's wrong with letting $z = 10$?

   See 14.1: 37-48.

4. Using your 2D plot, construct a 3D image of the function by lifting each level curve to its corresponding height.

Definition 5.1. A level curve of a function $z = f(x, y)$ is a curve in the xy-plane found by setting the output z equal to a constant. Symbolically, a level curve of $f(x, y)$ is the curve $c = f(x, y)$ for some constant c. A 2D plot consisting of several level curves is called a contour plot of $z = f(x, y)$.

Problem 5.10 Consider the function $f(x, y) = x - y^2$.

See Sage or Wolfram Alpha.More practice is in 14.1: 37-48.

1. Construct a 3D surface plot of f. [So just graph in 3D the curves given by $x = 0$ and $y = 0$ and then try setting x or y equal to some other constants, like $x = 1$, $x = 2$, $y = 1$, $y = 2$, etc.]

2. Construct a contour plot of f. [So just graph in 2D the curves given by setting z equal to a few constants, like $z = 0$, $z = 1$, $z = -4$, etc.]

3. Which level curve passes through the point $(2, 2)$? Draw this level curve on your contour plot.

   See 14.1: 49-52.

Notice that when we graphed the previous two functions (of the form $z = f(x, y)$) we could either construct a 3D surface plot, or we could reduce the dimension by 1 and construct a 2D contour plot by letting the output z equal various constants. The next function is of the form $w = f(x, y, z)$, so it has 3 inputs and 1 output. We could write $f: \mathbb{R}^3 \to \mathbb{R}^1$. We would need 4 dimensions to graph this function, but graphing in 4D is not an easy task. Instead, we'll reduce the dimension and create plots in 3D to describe the level surfaces of the function.

Problem 5.11 Suppose that an explosion occurs at the origin $(0, 0, 0)$. Heat from the explosion starts to radiate outwards. Suppose that a few moments after the explosion, the temperature at any point in space is given by $w = T(x, y, z) = 100 - x^2 - y^2 - z^2$.

See Sage. Wolfram Alpha currently does not support drawing level surfaces. You could also use Mathematica or Wolfram Demonstrations.
You can access more problems on drawing level surfaces in 12.6:1-44 or 14.1:53-60.

1. Which points in space have a temperature of 99? To answer this, replace $T(x, y, z)$ by 99 to get $99 = 100 - x^2 - y^2 - z^2$. Use algebra to simplify this to $x^2 + y^2 + z^2 = 1$. Draw this object.

2. Which points in space have a temperature of 96? of 84? Draw the surfaces.

3. What is your temperature at $(3, 0, -4)$? Draw the level surface that passes through $(3, 0, -4)$.

4. The 4 surfaces you drew above are called level surfaces. If you walk along a level surface, what happens to your temperature?

5. As you move outwards, away from the origin, what happens to your temperature?

Problem 5.12 Consider the function $w = f(x, y, z) = x^2 + z^2$. This function has an input y, but notice that changing the input y does not change the output of the function.

See Sage.

1. Draw a graph of the level surface $w = 4$. [When $y = 0$ you can draw one curve. When $y = 1$, you should draw the same curve. When $y = 2$, again you draw the same curve. This kind of graph is called a cylinder, and is important in manufacturing where you extrude an object through a hole.]
2. Graph the surface $9 = x^2 + z^2$ (so the level surface $w = 9$).
3. Graph the surface $16 = x^2 + z^2$.

Most of our examples of function of the form $w = f(x, y, z)$ can be drawn by using our knowledge about conic sections. We can graph ellipses and hyperbolas if there are only two variables. So the key idea is to set one of the variables equal to a constant and then graph the resulting curve. Repeat this with a few variables and a few constants, and you'll know what the surface is. Sometimes when you set a specific variable equal to a constant, you'll get an ellipse. If this occurs, try setting that variable equal to other constants, as ellipses are generally the easiest curves to draw.

Problem 5.13 Consider the function $w = f(x, y, z) = x^2 - y^2 + z^2$.

See Sage. Remember you can find more practice in 12.6:1-44 or 14.1: 53-64.

We'll have a few people present this problem.

1. Draw a graph of the level surface $w = 1$. [You need to graph $1 = x^2 - y^2 + z^2$. Let $x = 0$ and draw the resulting curve. Then let $y = 0$ and draw the resulting curve. Let either x or y equal some more constants (whichever gave you an ellipse), and then draw the resulting ellipses.]
2. Graph the level surface $w = 4$. [Divide both sides by 4 (to get a 1 on the left) and the repeat the previous part.]
3. Graph the level surface $w = -1$. [Try dividing both sides by a number to get a 1 on the left. If $y = 0$ doesn't help, try $y = 1$ or $y = 2$.]
4. Graph the level surface that passes through the point $(3, 5, 4)$. [Hint: what is $f(3, 5, 4)$?]

5.4.1 Vertor Fields and Transformations: $\vec{f} \colon \mathbb{R}^n \to \mathbb{R}^n$

We've covered the following types of functions in the problems above.

- $y = f(x)$ or $f \colon \mathbb{R} \to \mathbb{R}$ (functions of a single variable)
- $\vec{r}(t) = (x, y)$ or $f \colon \mathbb{R} \to \mathbb{R}^2$ (parametric curves)
- $\vec{r}(t) = (x, y, z)$ or $f \colon \mathbb{R} \to \mathbb{R}^3$ (space curves)
- $\vec{r}(u, v) = (x, y, z)$ or $f \colon \mathbb{R}^2 \to \mathbb{R}^3$ (parametric surfaces)
- $z = f(x, y)$ or $f \colon \mathbb{R}^2 \to \mathbb{R}$ (functions of two variables)
- $z = f(x, y, z)$ or $f \colon \mathbb{R}^3 \to \mathbb{R}$ (functions of three variables)

We will finish this section by considering vector fields and transformations.

- $\vec{F}(x,y) = (M,N)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (vector fields in the plane)
- $\vec{F}(x,y,z) = (M,N,P)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ (vector fields in space)
- $\vec{T}(u,v) = (x,y)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (2D transformation)
- $\vec{T}(u,v,w) = (x,y,z)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ (3D transformation)

Notice that in all cases, the dimension of the input and output are the same. The difference between vector fields and transformations has to do with the application. We've already seen examples of transformations with polar, cylindrical, and spherical coordinates.

Problem 5.14 Consider the spherical coordinates transformation

$$\vec{T}(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi),$$

which could also be written as

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi. \end{aligned}$$

Graphing this transformation requires $3+3=6$ dimensions. In this problem we'll construct parts of this graph by graphing various surfaces. We did something similar for the polar coordinate transformation in problem 4.4.

Recall that ϕ ("phi") is the angle down from the z axis, θ ("theta") is the angle counterclockwise from the x-axis in the xy-plane, and ρ ("rho") is the distance from the origin. Review problem 4.22 if you need a refresher.

1. Let $\rho = 2$ and graph the resulting surface. What do you get if $\rho = 3$? See Sage or Wolfram Alpha.
2. Let $\phi = \pi/4$ and graph the resulting surface. What do you get if $\phi = \pi/2$? See Sage or Wolfram Alpha.
3. Let $\theta = \pi/4$ and graph the resulting surface. What do you get if $\theta = \pi/2$?

We now focus on vector fields.

Problem 5.15 Consider the vector field $\vec{F}(x,y) = (2x+y, x+2y)$. In this problem, you will construct a graph of this vector field by hand.

See Sage or Wolfram Alpha. The computer will shrink the largest vector down in size so it does not overlap any of the others, and then reduce the size of all the vectors accordingly. See 16.2: 39-44 for more practice.

1. Compute $\vec{F}(1,0)$. Then draw the vector $F(1,0)$ with its base at $(1,0)$.
2. Compute $\vec{F}(1,1)$. Then draw the vector $F(1,1)$ with its base at $(1,1)$.
3. Repeat the above process for the points $(0,1)$, $(-1,1)$, $(-1,0)$, $(-1,-1)$, $(0,-1)$, and $(1,-1)$. Remember, at each point draw a vector.

Problem 5.16: Spin field Consider the vector field $\vec{F}(x,y) = (-y,x)$. Construct a graph of this vector field. Remember, the key to plotting a vector field is "at the point (x,y), draw the vector $\vec{F}(x,y)$ with its base at (x,y)." Plot at least 8 vectors (a few in each quadrant), so we can see what this field is doing.

Use the links above to see the computer plot this. See 16.2: 39-44 for more practice.

Sage can also help us visualize 3d vector fields, like $\vec{F}(x,y,z) = (y,z,x)$.

5.5 Constructing Functions

We now know how to draw a vector field provided someone tells us the equation. How do we obtain an equation of a vector field? The following problem will help you develop the gravitational vector field.

Problem 5.17: Radial fields Do the following:

Use Sage to plot your vector fields. See 16.2: 39-44 for more practice.

1. Let $P = (x, y, z)$ be a point in space. At the point P, let $\vec{F}(x, y, z)$ be the vector which points from P to the origin. Give a formula for $\vec{F}(x, y, z)$.
2. Give an equation of the vector field where at each point P in the plane, the vector $\vec{F}_2(P)$ is a unit vector that points towards the origin.
3. Give an equation of the vector field where at each point P in the plane, the vector $\vec{F}_3(P)$ is a vector of length 7 that points towards the origin.
4. Give an equation of the vector field where at each point P in the plane, the vector $\vec{G}(P)$ points towards the origin, and has a magnitude equal to $1/d^2$ where d is the distance to the origin.

If someone gives us parametric equations for a curve in the plane, we know how to draw the curve. How do we obtain parametric equations of a given curve? In problem 5.2, we were given the parametric equation for the path of a horse, namely $x = 2\cos t, y = 3\sin t$ or $\vec{r}(t) = (2\cos t, 3\sin t)$. From those equations, we drew the path of the horse, and could have written a Cartesian equation for the path. How do we work this in reverse, namely if we had only been given the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, could we have obtained parametric equations $\vec{r}(t) = (x(t), y(t))$ for the curve?

Problem 5.18 Give a parametrization of the top half of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so $y \geq 0$. You can write your parametrization in the vector form $\vec{r}(t) = (?, ?)$, or in the parametric form $x =?,\ y =?$. Include bounds for t. [Hint: Review 5.2.]

Use Sage or Wolfram Alpha to visualize your parameterizations.

Problem 5.19 Give a parametrization of the straight line from $(a, 0)$ to $(0, b)$. You can write your parametrization in the vector form $\vec{r}(t) = (?, ?)$, or in the parametric form $x =?,\ y =?$. Remember to include bounds for t. [Hint: Review 2.9 and 3.16.]

Problem 5.20 Give a parametrization of the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$. Remember the bounds for t.

Problem 5.21 Give a parametrization of the function $y = f(x)$ for $x \in [a, b]$. You can write your parametrization in the vector form $\vec{r}(t) = (?, ?)$, or in the parametric form $x =?,\ y =?$. Include bounds for t.

If someone gives us parametric equations for a surface, we can draw the surface. This is what we did in problems 5.6 and 5.7. How do we work backwards and obtain parametric equations for a given surface? This requires that we write an equation for x, y, and z in terms of two input variables (see 5.6 and 5.7 for examples). In vector form, we need a function $\vec{r}\colon \mathbb{R}^2 \to \mathbb{R}^3$. We can often use a coordinate transformation $\vec{T}\colon \mathbb{R}^3 \to \mathbb{R}^3$ to obtain a parametrization of a surface. The next three problems show how to do this.

Problem 5.22 Consider the surface $z = 9 - x^2 - y^2$ plotted in problem 5.8.

Use Sage or Wolfram Alpha to plot your parametrization. See 16.5: 1-16 for more practice.

1. Using the rectangular coordinate transformation $\vec{T}(x, y, z) = (x, y, z)$, give a parametrization $\vec{r}_1 : \mathbb{R}^2 \to \mathbb{R}^3$ of the surface. [Hint: Use the surface equation to eliminate the input variable z in T.]
2. What bounds must you place on x and y to obtain the portion of the surface above the plane $z = 0$?
3. If $z = f(x, y)$ is any surface, give a parametrization of the surface (i.e., $x =?, y =?, z =?$ or $\vec{r}(?, ?) = (?, ?, ?)$.)

Problem 5.23 Again consider the surface $z = 9 - x^2 - y^2$.

Use Sage or Wolfram Alpha to plot your parametrization with your bounds (see 5.22 for examples). See 16.5: 1-16 for more practice.

1. Using cylindrical coordinates, $\vec{T}(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$, obtain a parametrization $\vec{f}(r, \theta) = (?, ?, ?)$ of the surface using the input variables r and θ.
2. What bounds must you place on r and θ to obtain the portion of the surface above the plane $z = 0$?

Problem 5.24 Recall the spherical coordinate transformation

We did very similar things in problem 5.14. marginparSee 16.5: 1-16 for more practice.

$$\vec{T}(\rho, \phi, \theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi).$$

This is a function of the form $\vec{T} : \mathbb{R}^3 \to \mathbb{R}^3$. If we hold one of the three inputs constant, then we have a function of the form $\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$, which is a parametric surface.

Use Sage or Wolfram Alpha to plot each parametrization (see 5.22 for examples).

1. Give a parametrization of the sphere of radius 2, using ϕ and θ as your input variables.
2. What bounds should you place on ϕ and θ if you want to hit each point on the sphere exactly once?
3. What bounds should you place on ϕ and θ if you only want the portion of the sphere above the plane $z = 1$?

Sometimes you'll have to invent your own coordinate system when constructing parametric equations for a surface. If you notice that there are lots of circles parallel to one of the coordinate planes, try using a modified version of cylindrical coordinates. Instead of circles in the xy plane ($x = r\cos\theta, y = r\sin\theta, z = z$), maybe you need circles in the yz-plane ($x = x, y = r\sin\theta, z = r\sin\theta$) or the xz plane. Just look for lots of circles, and then construct your parametrization accordingly.

Problem 5.25 Find parametric equations for the surface $x^2 + z^2 = 9$. [Hint: read the paragraph above.]

Use Sage or Wolfram Alpha to plot each parametrization (see 5.22 for examples).

1. What bounds should you use to obtain the portion of the surface between $y = -2$ and $y = 3$?
2. What bounds should you use to obtain the portion of the surface above $z = 0$?
3. What bounds should you use to obtain the portion of the surface with $x \geq 0$ and $y \in [2, 5]$?

5.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to Brainhoney and download the quiz. Once you have taken the quiz, you can upload your work back to brainhoney and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 6

Derivatives

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Find limits, and be able to explain when a function does not have a limit by considering different approaches.
2. Compute partial derivatives. Explain how to obtain the total derivative from the partial derivatives (using a matrix).
3. Find equations of tangent lines and tangent planes to surfaces. We'll do this three ways.
4. Find derivatives of composite functions, using the chain rule (matrix multiplication).

You'll have a chance to teach your examples to your peers prior to the exam.

6.1 Limits

In first-semester calculus, you learned how to compute limits of functions. We need to define limits before proceeding. One possible definition of a limit follows.

Definition 6.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We write $\lim_{x\to c} f(x) = L$ if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|f(x) - L| < \epsilon$.

This formal definition is studied extensively in upper division math classes. We're looking at it here because we need to compare it with the formal definition of limits in higher dimensions. The only difference: just put vector symbols above the input x and the output $f(x)$.

Definition 6.2. Let $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ be a function. We write $\lim_{\vec{x}\to\vec{c}} \vec{f}(\vec{x}) = \vec{L}$ if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |\vec{x} - \vec{c}| < \delta$ implies $|\vec{f}(\vec{x}) - \vec{L}| < \epsilon$.

We'll find that throughout this unit, the key difference between first-semester calculus and this course is that we replace input and output of functions with vectors.

Problem 6.1 For the function $f(x, y) = z$, we can write f in the vector notation $\vec{y} = \vec{f}(\vec{x})$ if we let $\vec{x} = (x, y)$ and $\vec{y} = (z)$. Notice that $\vec{x}$ is a vector of inputs, and $\vec{y}$ is a vector of outputs. For each of the functions below, state what $\vec{x}$ and $\vec{y}$ should be so that the function can be written in the form $\vec{y} = \vec{f}(\vec{x})$.

The point to this problem is to help you learn to recognize the dimensions of the domain and codomain of the function. If we write $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$, then $\vec{x}$ is a vector in $\mathbb{R}^n$ with n components, and $\vec{y}$ is a vector in $\mathbb{R}^m$ with m components.

1. $f(x, y, z) = w$
2. $\vec{r}(t) = (x, y, z)$
3. $\vec{r}(u, v) = (x, y, z)$
4. $\vec{F}(x, y) = (M, N)$
5. $\vec{F}(\rho, \phi, \theta) = (x, y, z)$

You learned to work with limits in first-semester calculus without needing the formal definitions above. The following problem has you review some of the limit techniques from first-semester calculus.

Problem 6.2 Compute each of the following limits, or state why the limit does not exist. Do these problems without using L'Hopital's rule, as there is not a good substitute for L'Hopital's rule in higher dimensions.

See 14.2: 1-30 for more practice.

1. $\lim_{x\to 2} x^2 - 3x + 5$
2. $\lim_{x\to 3} \dfrac{x^2 - 9}{x - 3}$
3. $\lim_{x\to 0} \dfrac{x}{|x|}$ [Hint: graph the function.]

Some of the techniques you used in single-variable calculus give us immediate techniques for handling multivariable functions.

Problem 6.3 Do the following limits:

1. $\lim_{(x,y)\to(2,1)} 9 - x^2 - y^2$
2. $\lim_{(x,y)\to(4,4)} \dfrac{x - y}{x^2 - y^2}$

You should have observed that all the limits above existed, except for the $x/|x|$ limit. You can show that limit does not exist by considering what happens from the left, and comparing it to what happens on the right. In first-semester calculus you used the following theorem extensively.

> If $y = f(x)$ is a function defined on some open interval containing c, then $\lim_{x\to c} f(x)$ exists if and only if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$.

A limit exists precisely when the limits from every direction exists, and all directional limits are equal. In first-semester calculus, this required that you check two directions (left and right). This theorem generalizes to higher dimensions, but it becomes much more difficult to apply. The following problem will show you why.

Problem 6.4 Consider the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Our goal is to determine if the function has a limit at $(0, 0)$.

You may want to look at a graph in Sage or Wolfram Alpha (try using the "contour lines" option). As you compute each limit, make sure you understand what that limit means in the graph.

1. In the xy-plane, how many ways are there to approach the point $(0,0)$? Give a few examples.

2. One approach to the origin is to travel along the x-axis (so $y=0$). Using this approach, compute the limit

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0}\frac{x^2-0^2}{x^2-0^2} = ?.$$

3. Another approach to the origin is to travel along the y-axis (so let $x=0$). Compute the limit along this approach, namely

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^2-y^2}{x^2+y^2}.$$

4. Another approach is to travel along the line $y=x$. What is the limit at $(0,0)$ along this approach?

5. Does this function have a limit at $(0,0)$? Explain.

See 14.2: 41-50 for more practice.

The theorem from first-semester calculus generalizes as follows.

> If $\vec{y}=\vec{f}(\vec{x})$ is a function defined on some open region containing $\vec{c}$, then $\lim_{\vec{x}\to\vec{c}}\vec{f}(\vec{x})$ exists if and only if the limit exists along every possible approach to $\vec{c}$ and all these limits are equal.

There's a fundamental problem with using this theorem to determine if a limit exists. Once the domain is 2-dimensional or higher, there are infinitely many ways to approach a point. There is no longer just a left and right side. So you can prove a limit exists, provided you can check infinitely many cases. That's the problem—checking infinitely many cases takes a really long time. The theorem can also be used to show that a limit does not exist. All you have to do is find two approaches with different limits.

Problem 6.5 Consider the function $f(x,y)=\dfrac{xy}{x^2+y^2}$. Does this function have a limit at $(0,0)$? Examine the function at $(0,0)$ by considering multiple approaches (feel free to use the same approaches as in the problem above).

See Sage.
See 14.2: 41-50 for more practice.

Problem This is an alternate way to solve the problem above. Consider the function $f(x,y)=\frac{xy}{x^2+y^2}$. Does this function have a limit at $(0,0)$? Compute the limit at $(0,0)$ along the approaches $y=mx$ (this takes care of every line through the origin, except $x=0$). Then compute the limit at $(0,0)$ along the approach $y=x^2$.

You might use the Sage tool above to investigate

Problem: Challenge Give an example of a function $f(x,y)$ so that the limit at $(0,0)$ along every straight line $y=mx$ exists and equals 0. However, show that the function has no limit at $(0,0)$ by considering an approach that is not a straight line.

6.2 Partial Derivatives

Recall from first-semester calculus the following definition of the derivative.

Definition 6.3. We define the derivative of a function f at x to be the limit

$$f'(x) = \frac{d}{dx}[f(x)] = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. Whether you write f' or $\frac{df}{dx}$ does not matter, as they both represent the same thing. The notation $\frac{df}{dx}$ leads to the differential notation $dy = f'dx$, which we will use to generalize the derivative to all dimensions.

Before discussing the derivative of a function in higher dimensions, we first define partial derivatives. A matrix of partial derivatives will make up the total derivative.

Definition 6.4: Partial Derivative. Let f be a function. The partial derivative of f with respect to x is the regular derivative of f, provided we hold all input variables constant except x. If $f = f(x, y, z)$, we write any of

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[f] = f_x = D_x f = \lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

to mean the partial of f with resepect to x. The partial of f with respect to y, written $\frac{\partial f}{\partial y} = f_y$, is the regular derivative of f, provided we hold all input variables constant except y. A similar definition holds for partials with respect to any variable.

Problem 6.6 Find the indicated partial derivatives.

See 14.3: 1-40 for more practice. I strongly suggest you practice a lot of this type of problem until you can compute partial derivatives with ease.

1. For $f(x, y) = x^2 + 2xy + 3y^2$ find $\frac{\partial f}{\partial x}$ and f_y.
2. For $f(x, y, z) = x^2y^3z^4$, find f_x, $\frac{\partial f}{\partial y}$ and $D_z f$.
3. For $\vec{r}(u, v) = (u, v, v\cos(uv))$, find $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$.
4. For $\vec{F}(x, y) = (-y, xe^{3y})$, find $\frac{\partial \vec{F}}{\partial x}$ and $\frac{\partial \vec{F}}{\partial y}$.

Since a partial derivative is a function, you can take partial derivatives of that function as well. If you want to first compute a partial with respect to x, and then with respect to y, you would write

$$f_{xy} = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}.$$

The shorthand notation f_{xy} is easiest to write, but in upper-level courses, we will use subscripts to mean other things. At that point, we'll use the fractional partial notation to avoid confusion.

Problem 6.7 Consider the function $f(x, y) = 3xy^3 + e^{x^2}$.

1. Compute the second partials $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial x \partial y}$.

2. For $f(x, y) = x^2 + 2xy + y^3$, compute both f_{xy} and f_{yx}.

3. Make a conjecture about a relationship between f_{xy} and f_{yx}.

4. Use your conjecture to quickly compute f_{xy} if

$$f(x, y) = \tan^2(\cos(x))(x^{49} + x)^{1000} + 3xy.$$

The next problem will help you visualize what a partial derivative means in the graph of a surface.

Problem 6.8 Consider the function $f(x, y) = 9 - x^2 - y^2$. Construct a 3D surface plot of f (see problem 5.8). We'll focus on the point $(2, 1)$. See Sage.

1. Let $y = 1$ and construct a graph in the xz plane of the curve $z = f(x, 1) = 9 - x^2 - 1^2$. Find an equation of the tangent line to this curve at $x = 2$. Write the equation in the form $(z - z_0) = m(x - x_0)$.

2. Let $x = 2$ and construct a graph in the yz plane of the curve $z = f(2, y) = 9 - 2^2 - y^2$. Find an equation of the tangent line to this curve at $y = 1$. Write the equation in the form $(z - z_0) = m(y - y_0)$.

3. Compute f_x and f_y and then evaluate each at $(2, 1)$. What does this have to do with the previous two parts?

4. (We'll answer the remaining parts of this problem in class together. If you complete them, we'll let you share with us your answer.) If the slope of a line $y = mx + b$ is m, then we know that an increase of 1 unit in the x direction will increase y by m units. Fill in the blanks, as they relate to the function $f(x, y) = 9 - x^2 - y^2$ and the lines above.

   - Increasing x by 1 unit when y does not change will cause z to increase by about ____ units.
   - Increasing y by 1 unit when x does not change will cause z to increase by about ____ units.

5. In the previous part, we said that z increased by *about* a certain amount. Why did we not say that z increases by *exactly* that amount?

Once we have partial derivatives, we can calculate tangent lines to a surface. This means we can also find normal vectors and tangent planes as well. Normal vectors to surfaces (i.e., vectors that are perpendicular to the surface) are extremely important in many areas, including physics, optics, and computer graphics.

Problem 6.9 Consider the function $f(x, y) = 9 - x^2 - y^2$ at the point $(2, 1)$. From the previous problem, we know that increasing x by 1 unit when y does not change will cause z to increase by about f_x units. In terms of vectors, we have $(\Delta x, \Delta y, \Delta z) = (1, 0, f_x)$ is a tangent vector to the surface. See Sage. See 14.6: 9-12 for more practice.

1. At the point $(2, 1)$, find a tangent vector to the surface in the x direction (so compute $f_x(2, 1)$ and put it in the vector $(1, 0, f_x)$). Then give a vector equation of the tangent line to f in the x direction.

2. At the point $(2, 1)$, find a tangent vector to the surface in the y direction. Then give a vector equation of the tangent line to f in the y direction.

3. Give an equation of the tangent plane to f at $(2, 1)$. [Hint: we've found equations of planes before—see problems 2.27 and 2.23. The cross product will come in handy.]

This next problem will help you see how parametric functions can simplify the process of finding tangent vectors and planes.

Problem 6.10 Again, consider the function $f(x, y) = 9 - x^2 - y^2$ at the point $(2, 1)$. A parametrization of this surface is $\vec{r}(x, y) = (x, y, 9 - x^2 - y^2)$. We'll use the parametrization to find an equation for the tangent plane at $(2, 1, 4)$.

See 16.5: 27-30 for more practice.

1. Compute $\frac{\partial \vec{r}}{\partial x}(2, 1)$. Then give a vector equation of the tangent line to f in the x direction.
2. Compute $\frac{\partial \vec{r}}{\partial y}(2, 1)$. Then give a vector equation of the tangent line to f in the y direction.
3. Give an equation of the tangent plane to f at $(2, 1)$. [Hint: See problem 2.27.]

Problem 6.11 Let $f(x, y) = x^2 + 4xy + y^2$. Give two vector equations of tangent lines to the surface at $(3, -1)$. Then give an equation of the tangent plane.

See Sage.
See 14.6: 9-12 for more practice.

The next problem helps you generalize what you did above to construct the general formula for the tangent plane and normal vector to a surface $z = f(x, y)$ at the point (a, b).

Problem 6.12 Recall that an equation of the tangent line to $y = f(x)$ at $x = c$ is $y - f(c) = f'(c)(x - c)$. Let $z = f(x, y)$ be a function whose partial derivatives exist.

Page 811 has the answer, but written in a slightly different form than you will get. In addition, they arrive at the solution in a completely different way.

1. Give two vectors tangent to the surface at $(x, y) = (a, b)$.
2. Give a normal vector to the surface at (a, b).
3. Give an equation of the tangent plane to the surface at (a, b).

The next problem generalizes the tangent plane and normal vector calculations above to work for a parametric surface $\vec{r}(u, v)$.

Problem 6.13 Consider the cone parametrized by $\vec{r}(u, v) = (u \cos v, u \sin v, u)$

See Sage.
See 16.5: 27-30 for more practice.

1. Give vector equations of two tangent lines to the surface at $(2, \pi/2)$ (so $u = 2$ and $v = \pi/2$).
2. Give a normal vector to the surface at $(2, \pi/2)$.
3. Give an equation of the tangent plane at $(2, \pi/2)$.

6.3 The Derivative

Remark 6.5. In problem 6.8, we learned the following for $z = f(x, y)$.

- Increasing x by 1 unit when y remains constant will cause z to increase by about f_x units.
- Increasing y by 1 unit when x remains constant will cause z to increase by about f_y units.

We will use these facts to introduce differential notation for functions of several variables, and then define the total derivative as a matrix of partial derivatives.

Problem 6.14 Fill in the blanks. For this example, consider the function $z = f(x, y)$.

- Remember that increasing x by 1 when $\Delta y = 0$ will cause z to increase by about f_x units. So increasing x by only $\frac{1}{2}$ when $\Delta y = 0$ will cause z to increase by about $\underline{\frac{1}{2}f_x}$. In a similar fashion, increasing y by $\frac{1}{10}$ when $\Delta x = 0$ will cause z to increase by about ______. Combining these two ideas, we find that if $\Delta x = \frac{1}{2}$ and $\Delta y = \frac{1}{10}$ then $\Delta z \approx$ ____________.
- Increasing x by dx when $\Delta y = 0$ will cause z to increase by about ______. Increasing y by dy when $\Delta x = 0$ will cause z to increase by about ______. So combining these two ideas, we find that if $\Delta x = dx$ and $\Delta y = dy$ then $\Delta z \approx$ ____________. [Hint: read the definition below to see if you're on the right track.]

Based on the answer from the previous problem, we define the following.

Definition 6.6. In first-semester calculus, if $y = f(x)$, we defined the differential of f to be

$$df = f'\,dx = \frac{df}{dx}\,dx,$$

where dx represents a change in x. We can think of this as "A tiny change in the output $f(x)$ is approximately the same as the derivative, multiplied by a small change in the input x."

Based on the defintion from first semester calculus, and the answer from the previous problem, if $z = f(x, y)$ we'll define the differential of f to be

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy,$$

or in short-hand notation $df = f_x dx + f_y dy$, where dx and dy are independent variables which represent small changes in x and y.

In the next problem, you will use differential notation to discover the derivative of a function in high dimensions. We would like to be able to think of differential notation in all dimensions in the same way we think about it in one dimension. Ideally we could say "A tiny change in the output $f(x, y)$ is approximately the same as the derivative $Df(x, y)$, multiplied by a small change in the inputs x and y."

Problem 6.15 In each problem below, your job is to find a matrix M so that the matrix product is the same as the corresponding differential notation.

See Section 1.3 to refresh on how to do matrix multiplication.

1. Let $f(x,y)=z$. Find a matrix M so that
$$df = f_x dx + f_y dy = M \begin{bmatrix} dx \\ dy \end{bmatrix}.$$

2. Let $f(x,y,z)=w$. Find a matrix M so that
$$df = f_x dx + f_y dy + f_z dz = M \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}.$$

3. Let $\vec r(t)=(x,y)$. Find a matrix M so that
$$d\vec r = \frac{d\vec r}{dt}dt = \begin{pmatrix} \frac{dx}{dt}dt \\ \frac{dy}{dt}dt \end{pmatrix} = M dt.$$

4. Let $\vec r(u,v)=(x,y,z)$. We will think of $\dfrac{\partial \vec r}{\partial u} = \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix}$ and $\dfrac{\partial \vec r}{\partial v} = \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix}$ as column vectors (single-column matrices). Find a matrix M so that
$$d\vec r = \frac{\partial \vec r}{\partial u}du + \frac{\partial \vec r}{\partial v}dv = M \begin{bmatrix} du \\ dv \end{bmatrix}.$$

Definition 6.7. The derivative of a function $\vec f : \mathbb{R}^n \to \mathbb{R}^m$, where $\vec y = \vec f(\vec x)$, is a matrix, written $D\vec f$ and read "the derivative of f". The columns of the matrix are the partial derivatives of the function. The order in which you list the input variables of f is precisely the order in which the partials occur in the columns of the matrix.

This matrix $D\vec f$ gives the best possible linear approximation to changes in the outputs, based upon changes in the inputs. We write the previous sentence symbolically as $d\vec y = D\vec f d\vec x$.

The derivative of a function, $D\vec f$, is given many names in the literature. It's called the total derivative, the matrix derivative, the Jacobian, the Jacobian matrix, and more. We'll just call $D\vec f$ the derivative.

In first-semester calculus, we wrote the derivative of $y=f(x)$ in differential notation as $dy = f' dx$. To generalize, we put a vector above each variable and change f' from a number (a one-by-one matrix) to a matrix. This results in the derivative of $\vec y = f(\vec x)$ being written in differential notation as $d\vec y = D\vec f d\vec x$. This more general differential notation is valid in all dimensions.

Remember, to find the derivative of a function, compute all the partial derivatives and then place them in the columns of the matrix. Every input variable gets a column. *Every input variable gets a column.* **Every input variable gets a column.**

Problem 6.16 For each function below, state the dimensions of the domain and codomain (numbers of inputs and outputs) and write the function in the form $f : \mathbb{R}^n \to R^m$ (figure out what n and m are). Then find the derivative (as a matrix). How does the number of rows and columns relate to n and m? Remember, every input variable gets a column.

This handwritten file has 6 problems, together with solutions, that you can use as extra practice.

I'll have 4 people present this one in class.

1. $f(x,y)=x^2+4xy+y^3$
2. $f(x,y,z)=x^2-yz^3$
3. $\vec r(t)=(\cos t, \sin t)$ (Remember to place vectors in columns.)

4. $\vec{r}(t) = (\cos t, \sin t, t)$
5. $\vec{r}(a,t) = (a\cos t, a\sin t, t)$
6. $\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta)$
7. $\vec{F}(x,y) = (2x+3y, 4x+5y)$
8. $\vec{F}(x,y,z) = (2x+3y-5z, 4x+5y+z^2, xyz)$

Once we have the multivariable (matrix) derivative, almost every idea from first-semester calculus can be generalized to all dimensions by just replacing f' with Df and putting vector symbols above the inputs and outputs. As a first example, let's examine how tangent lines generalize to tangent planes.

Remark 6.8. In problem 1.10, we saw that the differential notation $dy = f'dx$ allowed us to write an equation of the tangent line to $y = f(x) = x^2$ at $x = 3$. Here's a recap of what we did.

> The derivative is $f'(x) = 2x$ which at $x = 3$ equals $f'(3) = 6$. The graph of the function passes through the point $(3, f(3)) = (3, 9)$. If (x, y) is any point on the tangent line, then the change from $(3, 9)$ to (x, y) is given by $(dx, dy) = (x, y) - (3, 9) = (x-3, y-9)$. Differential notation then says $dy = f'(3)dx$, or in other words, $(y-9) = 6(x-3)$.

Tangent lines pop out instantly from differential notation. Tangent planes will "pop out" too, as well as tangent objects in any dimension.

Problem 6.17 Read the remark above. Then give an equation of the tangent plane to $f(x,y) = 9 - x^2 - y^2$ at $(2,1)$ by using differential notation. Try doing so without using the steps below, but rather just mimic what we did in the remark above (replacing inputs and outputs with vectors, and the derivative with the appropriate matrix). If you need the following steps, then use them. Compare with problem 6.9. See Sage.

1. Find $Df(x,y)$ and then $Df(2,1)$. You should have two matrices. Also Find the point $(2, 1, f(2,1))$ on the graph of the surface.
2. If (x,y,z) is any point on the tangent plane, find the change from $(2, 1, f(2,1))$ to (x,y,z) (subtract vectors). This change is (dx, dy, dz).
3. We'll use differential notation to finish this problem. We want to generalize $dy = f'dx$ (a change in outputs equals the derivative times a change in inputs), so we will use the notation $d\vec{y} = D\vec{f}d\vec{x}$. Recall the following:
   - The inputs to $z = f(x,y)$ are x and y, so the input vector is $\vec{x} = (x,y)$. The output is z, so the output vector is $\vec{y} = (z)$.
   - The change in inputs is (dx, dy). The change in outputs is (dz).
   - The differential notation $d\vec{y} = D\vec{f}\, d\vec{x}$ then, in this case, becomes $\begin{bmatrix} dz \end{bmatrix} = Df \begin{bmatrix} dx \\ dy \end{bmatrix}$. This means a very small change in the output z equals the derivative times very small changes in the inputs x and y.

   Now use the differential notation $\begin{bmatrix} dz \end{bmatrix} = Df \begin{bmatrix} dx \\ dy \end{bmatrix}$ to write a matrix equation of the tangent plane (use $Df(2,1)$ from part 1 and dx, dy, and dz from part 2). Then perform the matrix multiplication to get an equation of the tangent plane. Compare your answer with problem 6.9.

Problem 6.18 Suppose $z = f(x, y)$ has a derivative $Df(x, y)$. Use differential notation to give an equation of the tangent plane to the surface at $(x, y) = (a, b)$ (use the steps from the previous problem if needed). Multiply out any matrix products. What is a normal vector to the plane? Compare with problem 6.12.

6.4 The Chain Rule

Let's recall the chain rule from first-semester calculus.

Theorem 6.9 (The Chain Rule)**.** *Let x be a real number and f and g be functions of a single real variable. Suppose f is differentiable at $g(x)$ and g is differentiable at x. The derivative of $f \circ g$ at x is*

$$(f \circ g)'(x) = \frac{d}{dx}(f \circ g)(x) = f'(g(x)) \cdot g'(x).$$

Some people remember the theorem above as "the derivative of a composition is the derivative of the outside (evaluated at the inside) multiplied by the derivative of the inside." If $u = g(x)$, we sometimes write $\dfrac{df}{dx} = \dfrac{df}{du}\dfrac{du}{dx}$. The following problem is designed to help you master the notation.

Problem 6.19 Suppose we know that $f'(x) = \dfrac{\sin(x)}{2x^2 + 3}$ and $g(x) = \sqrt{x^2 + 1}$. Notice we don't know $f(x)$. This is actually quite common in real life, as we can often measure how something changes (a derivative) without knowing the actual function.

1. What is $f'(x)$ and $g'(x)$?
2. What is the difference between $f'(x)$ and $f'(g(x))$? State $f'(g(x))$.
3. Use the chain rule to compute $(f \circ g)'(x)$.

We now generalize to higher dimensions. If I want to write $\vec{f}(\vec{g}(\vec{x}))$, then $\vec{x}$ must be a vector in the domain of g. After computing $\vec{g}(\vec{x})$, we must get a vector that is in the domain of f.

Problem 6.20 Consider $f(x, y) = 9 - x^2 - y^2$ and $\vec{r}(t) = (2\cos t, 3\sin t)$. For this problem, imagine the following scenario. A horse is running around outside in the cold. The horse's position at time t is given by the elliptical path $\vec{r}(t)$. The temperature of the air at any point (x, y) is given by $T = f(x, y)$.

1. At time $t = 0$, what is the horse's position $\vec{r}(0)$, and what is the temperature $f(\vec{r}(0))$ at that position? Find the temperatures at $t = \pi/2$, $t = \pi$, and $t = 3\pi/2$ as well.
2. In the plane, draw the path of the horse for $t \in [0, 2\pi]$. Then, on the same 2D graph, include a contour plot of f. Make sure you include the level curves that pass through the points in part 1. (See 5.2 and 5.9 if you need help.) At the points addressed in part 1, write the temperature on the curve.

3. As the horse runs around, the temperature of the air around the horse is constantly changing. At which t does the temperature around the horse reach a maximum? A minimum? Explain, using your graph.

This idea will lead to a very important optimization technique, Lagrange multipliers, later in the semester.

4. As the horse moves past the point at $t = \pi/4$, is the temperature of the surrounding air increasing or decreasing? Use your graph to explain.

5. Draw the 3D surface plot of f. In the xy-plane of your 3D plot (so $z = 0$) add the path of the horse. In class, we'll project the path of the horse up into the 3D surface (give it a try yourself first).

Problem 6.21 Consider $f(x, y) = 9 - x^2 - y^2$ and $\vec{r}(t) = (2\cos t, 3\sin t)$, which means $x = 2\cos t$ and $y = 3\sin t$.

1. For the function $\vec{r}(t) = (x, y)$, the input is t and the outputs are x and y. So differential notation states that

   $$\begin{pmatrix} dx \\ dy \end{pmatrix} = D\vec{r}(t)\begin{pmatrix} dt \end{pmatrix}.$$

   Compute $D\vec{r}(t)$.

Try to always remember the following summary of differential notation: a change in the outputs equals the derivative times a change in the inputs.

2. For the function $T = f(x, y)$, the inputs are x and y, and the output is temperature T. Differential notation states that

   $$\begin{pmatrix} dT \end{pmatrix} = Df(x, y)\begin{pmatrix} dx \\ dy \end{pmatrix}.$$

   Compute $Df(x, y)$.

3. Now we want to find out how the temperature T changes with respect to time t. We already know

   $$\begin{pmatrix} dx \\ dy \end{pmatrix} = D\vec{r}(t)\begin{pmatrix} dt \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} dT \end{pmatrix} = Df(x, y)\begin{pmatrix} dx \\ dy \end{pmatrix}.$$

   Use these two differential facts to write a change in temperature dT in terms of a change in time dt.

4. Compute the matrix product $Df(x, y)D\vec{r}(t)$, and then substitute $x = 2\cos t$ and $y = 3\sin t$. You should have an expression of the form $dT = (?)dt$ where ? is some function of t.

5. What is df/dt (i.e., dT/dt) at $t = \pi/4$? Is it positive or negative? Compare with part 4 of the previous problem.

Problem 6.22 Consider $f(x, y) = 9 - x^2 - y^2$ and $\vec{r}(t) = (2\cos t, 3\sin t)$.

1. Writing $\vec{r}(t) = (2\cos t, 3\sin t)$ means $x = 2\cos t$ and $y = 3\sin t$. In $f(x, y)$, replace x and y with what they are in terms of t. This will give you f as a function of t.

2. Construct a graph of $f(t)$ (use software to draw this if you like). From your graph, at what time values do the maxima and minima occur?

3. Compute df/dt (the derivative as you did in first-semester calculus).

4. What is df/dt at $t = \pi/4$?

5. Compare your work with the previous problem.

The previous three problems all focused on exactly the same concept. The first looked at the concept graphically, showing what it means to write $(f \circ \vec{r})(t) = f(\vec{r}(t))$. The second tackled the problem by considering matrix derivatives. The third reduced the problem to first-semester calculus. In all three cases, we wanted to understand the following problem.

> If $z = f(x, y)$ is a function of x and y, and both x and y are functions of t, so in vector form we can write $\vec{r}(t) = (x(t), y(t))$, then find how quickly f changes as you change t. In other words, what is the derivative of f with respect to t. Notationally, we seek $\frac{df}{dt}$ which formally is written $\frac{d}{dt}[f(x(t), y(t))]$ or $\frac{d}{dt}[f(\vec{r}(t))]$.

The second problem above gave us an example of the multivariable chain rule.

Theorem 6.10 (The Chain Rule)**.** *Let $\vec{x}$ be a vector and $\vec{f}$ and $\vec{g}$ be functions so that the composition $\vec{f}(\vec{g}(\vec{x}))$ makes sense (the output of g can be used as an input to f). Suppose $\vec{f}$ is differentiable at $\vec{g}(\vec{x})$ and $\vec{g}$ is differentiable at $\vec{x}$. The derivative of $\vec{f} \circ \vec{g}$ at $\vec{x}$ is*

$$D(\vec{f} \circ \vec{g})(\vec{x}) = D\vec{f}(\vec{g}(\vec{x})) \cdot D\vec{g}(\vec{x}).$$

In other words, the derivative of a composition is equal to the derivative of the outside (evaluated at the inside), multiplied by the derivative of the inside.

This is exactly the same as the chain rule in first-semester calculus. The only difference is that now we have vectors above every variable and function, and we replaced the one-by-one matrices f' and g' with potentially larger matrices Df and Dg. If everything is written in vector notation, the chain rule in any dimensions is the same as the chain rule in one dimension.

Problem 6.23 Suppose $f(x, y) = x^2 + xy$ and $x = 2t + 3$ and $y = 3t^2 + 4$. See 14.4: 1-6 for more practice.

1. Rewrite the parametric equations $x = 2t + 3$ and $y = 3t^2 + 4$ in vector form, so we can apply the chain rule. This means you need to create a function $\vec{r}(t) = ($__________, __________$)$.

2. Compute the derivatives $Df(x, y)$ and $D\vec{r}(t)$.

3. The chain rule states that $D(f \circ \vec{r})(t) = Df(\vec{r}(t))D\vec{r}(t)$. What is the difference between $Df(x, y)$ and $Df(\vec{r}(t))$. [Hint: see problem 6.19.]

4. Use the chain rule to compute $D(f \circ \vec{r})(t)$. What is df/dt?

Problem 6.24 Suppose $f(x, y, z) = x + 2y + 3z^2$ and $x = u + v$, $y = 2u - 3v$, and $z = uv$. This means that changing u and v should cause f to change. Our goal is to find $\partial f/\partial u$ and $\partial f/\partial v$. Try doing this problem without looking at the steps below, but instead try to follow the patterns in the previous problem on your own. See 14.4: 7-12 for more practice.

1. Rewrite the equations for x, y, and z in vector form $\vec{r}(u, v) = (x, y, z)$.

2. Compute $Df(x, y, z)$ and $D\vec{r}(u, v)$.
3. Use the chain rule (matrix multiplication) to find $D(f \circ \vec{r})(u, v)$. Notice that since this composite function has 2 inputs, namely u and v, we should expect to get two columns when we are done.
4. What are $\partial f/\partial u$ and $\partial f/\partial v$? [Hint: remember, each input variable gets a column.]

Problem 6.25 Suppose $\vec{F}(s, t) = (2s + t, 3s - 4t, t)$ and $s = 3pq$ and $t = 2p + q^2$. This means that changing p and q should cause $\vec{F}$ to change. Our goal is to find $\partial\vec{F}/\partial p$ and $\partial\vec{F}/\partial q$. Note that since $\vec{F}$ is a vector-valued function, the two partial derivatives should be vectors. Try doing this problem without looking at the steps below, but instead try to follow the patterns in the previous problems on your own.

1. Rewrite the parametric equations for s and t in vector form.
2. Compute $D\vec{F}(s, t)$ and the derivative of your vector function from part 1.
3. Use the chain rule (matrix multiplication) to find the derivative of $\vec{F}$ with respect to p and q. How many columns should we expect to have when we are done multiplying matrices?
4. What are $\partial\vec{F}/\partial p$ and $\partial\vec{F}/\partial q$?

Problem 6.26 Suppose $w = f(x, y, z)$ and x, y, z are all function of one variable t (so $x = g(t), y = h(t), z = k(t)$). Find a general formula for dw/dt that involves partials of f and derivatives of x, y, and z. Try doing this problem without looking at the steps below, but instead try to follow the patterns in the previous problems.

See 14.4: 13-24 for more practice. Don't use the "branch diagram" in the book—use matrix multiplication instead. The branch diagram is just a way to express matrix multiplication without having to introduce matrices.

1. Rewrite the parametric equations for x, y, and z in vector form $\vec{r}(t) = (x, y, z)$.
2. Compute $Dw(x, y, z)$ and $D\vec{r}(t)$.
3. Multiply the matrices together to get $D(w \circ r)(t)$. The matrix should have one entry. State what dw/dt equals.

Problem 6.27 Suppose $z = f(s, t)$ and s and t are functions of u, v and w. Use the chain rule to give a general formula for $\partial z/\partial u$, $\partial z/\partial v$, and $\partial z/\partial w$.

You've now got the key ideas needed to use the chain rule in all dimensions. You'll find this shows up many places in upper-level math, physics, and engineering courses. The following problem will show you how you can use the general chain rule to get an extremely quick way to perform implicit differentiation from first-semester calculus.

Problem 6.28 Suppose $z = f(x, y)$. If z is held constant, this produces a level curve. As an example, if $f(x, y) = x^2 + 3xy - y^3$ then $5 = x^2 + 3xy - y^3$ is a level curve. Our goal in this problem is to find dy/dx in terms of partial derivatives of f.

See 14.4: 25-32 to practice using the formula you developed. To practice the idea developed in this problem, show that if $w = F(x, y, z)$ is held constant at $w = c$ and we assume that $z = f(x, y)$ depends on x and y, then $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$. This is done on page 798 at the bottom.

1. Suppose $x = x$ and $y = y(x)$, so y is a function of x. We can write this in parametric form as $\vec{r}(x) = (x, y(x))$. We now have $z = f(x, y)$ and $\vec{r}(x) = (x, y(x))$. Compute both $Df(x, y)$ and $D\vec{r}(x)$ symbolically. Don't use the function $f(x, y) = x^2 + 3xy - y^3$ until the last step.

2. Use the chain rule to compute $D(f(\vec{r}(x)))$. What is dz/dx (i.e., df/dx)?

3. Since z is held constant, we know that $dz/dx = 0$. Use this fact, together with part 2 to explain why $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$.

4. For the curve $5 = x^2 + 3xy - y^3$, use this formula to compute dy/dx.

6.5 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to Brainhoney and download the quiz. Once you have taken the quiz, you can upload your work back to brainhoney and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 7

Motion

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Develop formulas for the velocity and position of a projectile, if we neglect air resistance and consider only acceleration due to gravity. Show how to find the range, maximum height, and flight time of the projectile.

2. Develop the TNB frame for describing motion. Make sure you can explain why $\vec{T}$, $\vec{N}$, and $\vec{B}$ are all orthogonal unit vectors, and be able to perform the computations to find these three vectors.

3. Explain the concepts of curvature κ, radius of curvature ρ, center of curvature, and torsion τ. Make sure you can describe geometrically what theses quantities mean.

4. Find the tangential and normal components of acceleration. Show how to obtain the formulas $a_T = \frac{d}{dt}|\vec{v}|$ and $a_N = \kappa|\vec{v}|^2 = \frac{|\vec{v}|^2}{\rho}$, and explain what these equations physically imply.

You'll have a chance to teach your examples to your peers prior to the exam.

I have created a YouTube playlist to go along with this section. There are 11 videos, each 4-6 minutes long.

- YouTube playlist for 07 - Motion and The TNB Frame.
- PDF copy of the finished product (so you can follow along on paper).

To help you organize the information we study in this chapter, there's a table that includes all the vectors and scalars we will discuss at the end of the unit.

7.1 Projectile Motion

Suppose a projective is fired from a cannon with an initial speed v_0. The projectile leaves the cannon at an angle of α above the x-axis, and we'll use the y-axis to keep track of the height of the projectile. All the motion in this problem occurs with a plane, and we'll use x and y to represent motion in that plane. Our goal is to find the velocity $\vec{v}(t)$ and position $\vec{r}(t)$ of the projectile at any time t.

We need some assumptions prior to solving.

- Assume the only force acting on the object is the force due to gravity. We will neglect air resistance.
- The force due to gravity is the mass of the projectile multiplied by the acceleration of gravity. The mass of the object will not be important in our work here, though in future classes you may study how mass affects energy computations.
- The projectile is shot over a small enough range that we can assume gravity only pulls the object straight down.
- Most branches of science use the letter g to represent the magnitude of the vertical component of acceleration, so we can write the acceleration of the projectile as
$$\vec{a}(t) = (0, -g) = 0\mathbf{i} - g\mathbf{j}.$$
- Our text uses the approximations $g \approx 9.8$ m/s^2 or $g \approx 32$ ft/s^2.

To solve the next problem, you need to remember that acceleration is the derivative of velocity, and that velocity is the derivative of position. These facts hold true for vector-valued functions as well. Integration will help.

Problem 7.1 An object undergoes an acceleration of $\vec{a}(t) = (2t, -8)$. The initial velocity is $\vec{v}(0) = (4, 5)$ and the initial position is $\vec{r}(0) = (1, 3)$. Use this information to find the velocity and position at any time t. [Hint: Use integration to get velocity and position, but don't forget your arbitrary constants. You should be able to use the initial conditions to determine the constants.]

Problem 7.2 Suppose a projectile is fired from the point (x_0, y_0) with an initial velocity $\vec{v}(0) = (v_{x_0}, v_{y_0})$, and that gravity is the only force acting on the object. So the acceleration due to gravity is $\vec{a}(t) = (0, -g)$.

Watch a YouTube video. You can practice finding position from velocity and acceleration with problems 13.2: 11-18, and especially 13.2: 29.

1. Show that the velocity at any time t is $\vec{v}(t) = (v_{x_0}, -gt + v_{y_0})$.
2. Show that the position at any time t is $\vec{r}(t) = (v_{x_0}t + x_0, -\frac{1}{2}gt^2 + v_{y_0}t + y_0)$.
3. Give parametric equations $x = x(t)$ and $y = y(t)$ that give the horizontal and vertical position of the projectile at time t.

We make the following definitions for a projectile that starts at $(0, 0)$ and hits the ground at $(R, 0)$.

- The range is the horizontal distance R traveled by the projectile.
- The flight time is how long the projectile is in the air. It is the time t at which $\vec{r}(t) = (R, 0)$.
- The maximum height is the largest y value obtained by the projectile.

Problem 7.3 Answer the following questions. Assume that the projectile was fired from the origin.

Watch a YouTube video.

1. Explain why the flight time is $t = \dfrac{2v_{y_0}}{g}$? [Hint: How long does it take to reach maximum height? What should the velocity vector equal when the object has reached maximum height?]

2. Show that the maximum height is $y_{\max} = \dfrac{v_{y_0}^2}{2g}$. Then show that the range is $R = \dfrac{2v_{x_0}v_{y_0}}{g}$.

Problem 7.4 Use the results from the results from the previous problems. (So you can work on this problem, even if you couldn't finish the previous).

1. If the initial speed of the object is v_0, with a firing angle of α above the horizontal, rewrite v_{x_0} and v_{y_0} in terms of v_0 and α. [What's the difference between speed and velocity?]
2. Rewrite your equations for $\vec{v}(t)$ and $\vec{r}(t)$, so that they are in terms of v_0 and α instead of v_{x_0} and v_{y_0}.
3. Rewrite the equations for flight time, maximum height, and range so that they involve the speed v_0 and firing angle α.

This problem comes from your text. (See section 13.2.) Try it without reading the text. It's a fun application of the ideas above.

Problem 7.5 An archer stands at ground level and shoots an arrow at an object which is 90 feet away in the horizontal direction and 74 ft above ground. The arrow leaves the bow at about 6 ft above ground level (not the origin). The archer wants the arrow to hit the target at the peak of its parabolic path. For the purposes of this problem, Let $g = 32\text{ft/s}^2$. What initial speed v_0 and firing angle α are needed to achieve this result? [Hint: This is much easier to solve if you first find v_{x_0} and v_{y_0}, the horizontal and vertical components of the velocity. You may want to move the origin as well, so that you can use the formulas from above.]

This problem was created around the opening ceremony of the Barcelona Spain Olympics. Antonio Rebollo was the archer, but he didn't try to hit the flame at the peak of the flight. You can watch a YouTube video of the opening ceremony by following the link.
See 13.2: 19-28 for more practice.

7.2 Arc Length and the Unit Tangent Vector

In the next problem, you'll develop a formula for the arc length of a space curve (one input, 3 outputs). We've essentially already done this in chapters 3 and 4, but let's revisit the derivation once more.

Problem 7.6 A space ship travels through the galaxy. Let $\vec{r}(t) = (x, y, z)$ be the position of the space ship at time t, with the earth at the origin $(0, 0, 0)$.

- What are the velocity and speed of the space ship at time t? You answers should involve some derivatives (such as $\frac{dx}{dt}$).
- If the space ship travels for a really small time dt, then the speed is about constant. Since distance is speed times time, about how much distance (we'll call it ds) will the space ship travel in this short amount of time?
- As the ship travels from time $t = a$ to time $t = b$, explain why the distance traveled (the arc length of the path followed) is

$$s = \int_a^b |\vec{r}'(t)|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Watch a YouTube video.
Technically, we should write $\vec{r}(t) = (x(t), y(t), z(t))$. However, we already know that x, y, and z depend on t, hence we'll just leave the dependence on t off.

In all our work that follows, we want to consider space curves that have nice smooth paths. What does this mean? We want to be able to compute tangent vectors at any point, so we will require that a parametrization $\vec{r}$ be differentiable. We also don't want any cusps in our path (places where the direction of motion changes instantaneously). If the speed of an object ever reaches zero, then the object could stop moving, change direction, and then start moving instantly. We don't want this to happen, so we'll assume that the velocity is never zero.

Definition 7.1. Let $\vec{r}(t) = (x, y, z)$ be a parametrization of a space curve C. We say that $\vec{r}$ is smooth if $\vec{r}$ is differentiable, and the derivative is never the zero vector. Under these conditions, we'll say that C is a smooth curve.

Problem 7.7 Consider the helical space curve $\vec{r}(t) = (\cos t, \sin t, t)$. Find the length of this space curve for $t \in [0, 2\pi]$. Then find the length of the space curve from $t = 0$ to time $t = t$ (so after t seconds, what is the distance $s(t)$ traveled?).

Watch a YouTube Video.
See 13.3: 1-10 for more practice.

Problem 7.8 Let $\vec{r}(t) = (x, y, z)$ be a parametrization of a smooth space curve. Let $s(t) = \int_0^t \left|\frac{d\vec{r}}{d\tau}(\tau)\right| d\tau$. Explain why $\frac{ds}{dt}(t) = \left|\frac{d\vec{r}}{dt}(t)\right|$, the speed. [Hint: look up the fundamental theorem of calculus.] Then explain why $s(t)$ is an increasing function.

You can remember $\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|$ as follows. We use the differential ds to represents a change in distance, and dt represents a change in time. So the speed of an object is the change in distance ds over the change in time dt.

The quantity $s(t) = \int_0^t \left|\frac{d\vec{r}}{d\tau}(\tau)\right| d\tau$ is called the arc length parameter. It tells you how far you have traveled after t seconds. The fact that $s(t)$ is always an increasing function if the curve is smooth allows us to talk about taking derivatives with respect to the length traveled s instead of with respect to time t. The idea is to ask how much a curve changes if you increase length by 1 unit, instead of increasing time by 1 unit. We write

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|}.$$

Problem 7.9 Consider again the helical space curve $\vec{r}(t) = (\cos t, \sin t, t)$. We already have shown that $s(t) = t\sqrt{2}$. Solve for t in terms of s (so find the inverse of $s(t)$). You will now have a function of the form $t = t(s)$. Find the derivative (using the matrix chain rule) of $\vec{r}(t(s))$. In other words, what is $\frac{d\vec{r}}{ds}$? How are $\frac{d\vec{r}}{ds}$ and $\frac{d\vec{r}}{dt}$ related?

See 13.3: 11-14 for more practice.

Definition 7.2: Unit Tangent Vector. If $\vec{r}(t)$ is a parametrization of a space curve, then we define the unit tangent vector $\vec{T}(t)$ to be

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|}.$$

Problem 7.10 Suppose an object moves along the space curve given by $\vec{r}(t) = (a \cos t, a \sin t, bt)$. Find the object's velocity and speed. What is $\frac{d\vec{r}}{ds}$, the derivative of $\vec{r}$ with respect to arc length? State the unit tangent vector $\vec{T}(t)$.

See 13.3: 1-10 for more practice.

7.3 The TNB Frame

The unit tangent vector $\vec{T}$ provides us with a unit vector in the direction of motion. If we are moving along a straight line, then knowing $\vec{T}$ is sufficient to understanding the motion. However, if we veer off the straight line, then we would like to know in which direction we are turning (accelerating). This direction, called the normal direction, tells us the direction of acceleration. When you study dynamics (forces acting on moving objects), you'll find that knowing the tangent and normal directions are crucial. In our class, we only have time to develop equations for $\vec{T}$ and $\vec{N}$, as well as practice on a few examples.

In order to find $\vec{N}$, we first need to develop a crucial fact. This fact states that if a vector valued function has constant length, then the function is orthogonal to its derivative. Here's an example.

Problem 7.11 Consider $\vec{r}_1(t) = (\cos t, \sin t, 0)$ and $\vec{r}_2(t) = (\cos t, \sin t, t)$.

1. Show that $\vec{r}_1$ and $\dfrac{d\vec{r}_1}{dt}$ are orthogonal. Is $|\vec{r}_1|$ constant?
2. Show that $\vec{r}_2$ and $\dfrac{d\vec{r}_2}{dt}$ are not orthogonal. Is $|\vec{r}_2|$ constant?
3. Is the length of $\dfrac{d\vec{r}_2}{dt}$ constant? Are $\dfrac{d\vec{r}_2}{dt}$ and $\dfrac{d^2\vec{r}_2}{dt^2}$ orthogonal?

Theorem 7.3. *If a vector valued function $\vec{r}(t)$ has constant length, then the vector $\vec{r}$ and its derivative $\dfrac{d\vec{r}}{dt}$ are orthogonal for all t.*

Problem 7.12: Proof of Theorem 7.3 Prove the theorem above. Here are some hints [as an alternative to watching the YouTube video].

Watch a YouTube Video.

- We know that $\vec{r}(t)$ has constant length. This means $|\vec{r}| = c$ for some constant c.
- You need to get from a magnitude to the dot product. Look in your text for a way to relate magnitude to the dot product. See problem 2.15.
- After writing $|\vec{r}(t)| = c$ in terms of a dot product (squaring both sides may help), take the derivative of both sides. Apply the product rule to the dot product.

The above fact is so crucial, that we'll repeat what it says.

> If the vector $\vec{v}(t)$ has constant length, then the vector and its derivative $\frac{d\vec{v}}{dt}$ are orthogonal.

Problem 7.13 Let $\vec{r}$ be a smooth parametrization of a curve. How long is the *unit* tangent vector $\vec{T}(t)$? Explain why $\vec{T}$ is orthogonal to $\dfrac{d\vec{T}}{dt}$. Give a formula for computing a unit vector that is orthogonal to $\vec{T}(t)$.

Watch a YouTube Video.

Based on your answer above, we make the following definition of the principle unit normal vector. The key idea is that this vector points in the direction of normal acceleration.

Definition 7.4: Principle Unit Normal Vector. If $\vec{r}(t)$ is a parametrization of a space curve with unit tangent vector $\vec{T}(t)$, then we define the principle unit normal vector $\vec{N}(t)$ to be the vector

$$\vec{N}(t) = \frac{d\vec{T}/dt}{|d\vec{T}/dt|},$$

provided of course that $|d\vec{T}/dt| \neq 0$. From problem 7.13 we know that $\vec{T}$ and $\vec{N}$ are orthogonal.

Definition 7.5: Binormal Vector. If $\vec{r}$ is a parametrization of a smooth space curve with unit tangent vector $\vec{T}$ and principle unit normal vector $\vec{N}$, then we define the binormal vector $\vec{B}$ to be the cross product

$$\vec{B} = \vec{T} \times \vec{N}.$$

We now have the entire TNB frame. This gives us a moving collection of unit vectors that act like an xyz coordinate system. Many of you will use this frame a ton in your dynamics course. The TNB frame shows up in physical chemistry as well. A key fact to remember is that all three vectors are unit vectors, and they are each orthogonal to the other.

Problem 7.14 Answer the following questions (this will review your knowledge of the dot and cross products).

1. What is $\vec{T} \cdot \vec{N}$? Explain. Then explain why $\vec{T} \cdot \vec{B} = 0$ and $\vec{N} \cdot \vec{B} = 0$.
2. Both $\vec{T}$ and $\vec{N}$ are unit vectors. Why is $\vec{B}$ is a unit vector? [Think about the connection between the cross product and area.]
3. We defined $\vec{B} = \vec{T} \times \vec{N}$. This means that $\vec{N} \times \vec{T} = -\vec{B}$. Is $\vec{B} \times \vec{T}$ equal to $\vec{N}$ or $-\vec{N}$? Explain.

Problem 7.15 Consider the helix $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$. Find the unit tangent vector $\vec{T}(t)$, principle unit normal vector $\vec{N}(t)$, and the binormal vector $\vec{B}(t)$.

See 13.4: 9-16 and 13.5: 9-16 (the relevant parts) for more practice.

We've been working with helices in all the problems up to now because the velocity vectors have constant speed. Once the speed of the velocity vector is no longer constant, things get a lot messier. Ask me in class to show you what happens with the computations when you consider something like $r(t) = (t, t^2, t^3)$. Things get ugly really fast. Fortunately, when you're working with a curve that lies in a plane, there are some simplifications that occur.

Problem 7.16 Suppose you have already computed the unit tangent vector for a curve in the plane and found at a specific time it equals $\vec{T} = (a, b)$, which could easily be rewritten as $\vec{T} = (a, b, 0)$.

See 13.4: 7-8 for more practice, and perhaps a hint.

1. Find a nonzero vector that is orthogonal to $\vec{T} = (a, b)$.
2. If $\vec{r}(t) = (t, t^2)$, then we have $\frac{\vec{r}}{dt} = (1, 2t)$ and $\vec{T}(t) = \frac{(1,2t)}{\sqrt{1+4t^2}}$. Without computing any more derivatives, what is the principle unit normal vector $\vec{N}(t)$? Draw a picture of the curve, and then at $t = 1$ add to your picture the tangent and normal vectors.

3. What is $\vec{B}(t)$? We'll answer this in class if you are not sure.

Observation 7.6. From the problem above, we learn the following fact. If the tangent vector to a planar curve is $\vec{T}(t) = (a(t), b(t))$, then the principle unit normal vector is either $\vec{N}(t) = (-b(t), a(t))$ or $\vec{N}(t) = (b(t), -a(t))$. You just reverse the components, and then negate one of them. To determine which one to negate, draw a picture.

Problem 7.17 Consider the curve $\vec{r}(t) = (t^2, t)$. Compute $\vec{T}(t)$ and $\vec{N}(t)$ (to get $\vec{N}(t)$, make sure you use the previous observation). Draw the curve and on your graph include these vectors at $t = 1$.

Problem 7.18 Consider the curve $y = \sin x$, parametrized by $r(t) = (t, \sin t)$. Start by computing $\vec{T}(t)$.

See 13.4: 1-4 for more practice. Use the previous problems.

1. What are $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$ at $t = \pi/2$?
2. What are $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$ at $t = \pi/4$?
3. What are $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$ at $t = -\pi/4$?
4. What are $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$ at $t = 0$?

You've now developed the TNB frame for describing motion. Engineers will see this again when they study dynamics. Mathematicians who study differential geometry will use these ideas as well. Any time you want to analyze the forces acting on a moving object, the TNB frame may save the day. Chemists will encounter the TNB frame briefly when they study P-chem and the motion of subatomic particles.

7.4 Curvature and Torsion

We already know that $\vec{T} = \frac{d\vec{r}}{ds}$ has length 1. This means that if we move along the curve $\vec{r}$ using s as our parameter (not t), then we move along the curve at a constant speed of 1. The fact that we are moving at speed 1 means that we can study the properties of the curve without having to worry about our speed. We would like to know how sharp a corner is (which we'll call the curvature). To determine how sharp a corner is, we must forget about speed for a bit. If we encounter a really tight corner (so a rapid change in direction over a very short distance) we would expect $\frac{d\vec{T}}{ds}$ to be a fairly long vector. A small change in s results in a large change in T. However, if we were to move along this tight corner at a really slow speed, we would expect $\frac{d\vec{T}}{dt}$ to be a really small vector. A small change in t would not produce much change in T.

Problem 7.19 Suppose we are traveling along the space curve $\vec{r}$, and we know the unit tangent vector is $\vec{T}$.

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1. If we are moving along a straight line, then what is $\frac{d\vec{T}}{ds}$? Explain.
2. If we veer slightly off a straight line, should $\frac{d\vec{T}}{ds}$ be large or small? Why?

3. If we veer slightly off a straight line, and are moving extremely slow, should $\dfrac{d\vec{T}}{dt}$ be large or small? Explain.
4. If we veer slightly off a straight line, and are moving extremely fast, should $\dfrac{d\vec{T}}{dt}$ be large or small? Explain.
5. If we know $\dfrac{d\vec{T}}{ds}$ has length $\frac{1}{2}$, and our speed is 50, how long is $\dfrac{d\vec{T}}{dt}$? Explain. [Hint: remember that $\dfrac{d\vec{T}}{ds} = \dfrac{d\vec{T}/dt}{ds/dt}$, and we've seen ds/dt before.]

We will often be computing derivatives with respect to s, instead of t, because we want to determine physical properties about the curve. Moving really slowly around a tight corner won't produce a large tagent vector because our speed is slow. Similarly, moving quickly along a curve that hardly changes could produce a misleading large tangent vector. However, if we remove the speed from the problem, by taking a derivative with respect to s instead of t, then we'll learn how quickly the curve veers away from $\vec{T}$ as we increase in length. When we compute $\dfrac{d\vec{N}}{ds}$, we will find how rapidly $\vec{N}$ rotates away from the plane containing T and N (motion and acceleration). When we compute $\dfrac{d\vec{B}}{ds}$, we will find how rapidly $\vec{B}$ rotates. We'll show that both $\dfrac{d\vec{N}}{ds}$ and $\dfrac{d\vec{B}}{ds}$ cause a rotation of $\vec{N}$ and $\vec{B}$ about the tangent vector $\vec{T}$. The magnitude of this rotation, as $\vec{B}$ wraps around $\vec{T}$ counterclockwise, is called the torsion. Let's formally define curvature and torsion.

Definition 7.7: Curvature and Torsion. Let $\vec{r}(t)$ be a parametrization of a smooth curve C with unit tangent vector $\vec{T}(t)$. The curvature vector, written $\vec{\kappa}(t)$, is the derivative of $\vec{T}$ with respect to arc length, which means

$$\vec{\kappa}(t) = \frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt} = \frac{d\vec{T}/dt}{|d\vec{r}/dt|}.$$

The length of the curvature vector is the curvature, written $\kappa = |\vec{\kappa}|$. Notice that κ is a number.

The derivative of $\vec{B}$ with respect to s tells us how rapidly the plane containing $\vec{T}$ and $\vec{N}$ rotates. We'll define the torsion vector to be

Watch a YouTube Video.

$$\vec{\tau} = \frac{d\vec{B}}{ds} = \frac{d\vec{B}/dt}{ds/dt} = \frac{d\vec{B}/dt}{|d\vec{r}/dt|}.$$

The torsion τ, up to a sign, is the length of this vector. We say there is positive torsion if $\vec{\tau}$ causes a counterclockwise rotation about $\vec{T}$, which occurs precisely when $\vec{tau}$ and $\vec{N}$ point in opposite directions. We can summarize this is

$$\tau = \left|\frac{d\vec{B}}{ds}\right| \quad \text{or} \quad \tau = -\left|\frac{d\vec{B}}{ds}\right|,$$

where you choose "+" if $\vec{N}$ and $\vec{\tau}$ point in opposite directions.

Problem 7.20 Consider the helix $r(t) = (3\cos t, 3\sin t, 4t)$. In problem 7.15 we found $\vec{T}$, $\vec{N}$, and $\vec{B}$. Compute both $\vec{\kappa} = \dfrac{d\vec{T}}{ds}$ and $\vec{\tau} = \dfrac{d\vec{B}}{ds}$, and then give κ and τ.

See 13.4: 9-16 and 13.5: 9-16 (the relevant parts) for more practice.

Problem 7.21 Consider the helix $r(t) = (4\sin t, 4\cos t, 3t)$. Use a computer to find $\vec{T}$, $\vec{N}$, $\vec{B}$, $\vec{\kappa}$, and $\vec{\tau}$. Use your answers to then give κ and τ. (When you present on the board, just write down the 5 vectors, and then explain how you obtained κ and τ from these vectors.)

In both examples above, you should have noticed that $\vec{\tau}$ was either parallel to $\vec{N}$ or anti-parallel to $\vec{N}$. We'll now show this is always the case.

Problem 7.22 Suppose a curve $\vec{r}(t)$ has the frame $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$. Prove that $\dfrac{d\vec{B}}{ds}$ is either parallel to $\vec{N}$, or points opposite $\vec{N}$. Here are some steps.

Watch a YouTube Video.

- Why is $\dfrac{d\vec{B}}{ds}$ orthogonal to $\vec{B}$? [Hint: How long is $\vec{B}$? Use a key theorem from earlier.]
- We know $\vec{B} = \vec{T} \times \vec{N}$. Compute the derivative of both sides using the product rule. Explain why $\frac{d\vec{T}}{ds} \times \vec{N}$ cancels out. Then explain why $\dfrac{d\vec{B}}{ds}$ is orthogonal to $\vec{T}$.
- If $\dfrac{d\vec{B}}{ds}$ is orthogonal to both $\vec{B}$ and $\vec{T}$ why must it be either parallel or anti-parallel to $\vec{N}$?

When the curvature is nonzero, the curve bends away from the direction of motion. We could use a circle to approximate how great this bend is. A small change in direction would require a large circle. A large change in direction would require a small circle. What we want is to find a circle that best approximates the curve (kind of like a Taylor polynomial, only now we'll use a circle.) We want the circle to meet the curve $\vec{r}$ tangentially, and we want the curvature of the circle to match the curvature of the curve. The next problem shows you the relationship between the radius ρ of this circle and the curvature κ of the curve.

Problem 7.23 Consider the curve $\vec{r}(t) = (a\cos t, a\sin t)$.

1. Draw the curve, and state the radius ρ of the best approximating circle.
2. Find the curvature $\vec{\kappa}$ by performing a computation.
3. What relationship exists between ρ and κ? If the radius ρ were to increase, what would happen to κ?

Definition 7.8: Circle and Center of Curvature. When the curvature κ of a smooth curve is nonzero, we'll define the radius of curvature, written ρ, to be the reciprocal $\rho = \dfrac{1}{\kappa}$. The center of curvature is the center of this circle.

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Problem 7.24 Consider the curve $\vec{r}(t) = (t, \sin 3t)$. Find the radius and center of curvature at $t = \pi/6$ (*see the suggestion below). Draw the curve, and draw the circle of curvature at $t = \pi/6$. (You will have shown why the center of curvature is at $\vec{r} + \rho\vec{N}$.)

*The computations here can get pretty ugly. After getting the unit tangent vector $\vec{T}(t) = \dfrac{(1, 3\cos 3t)}{\sqrt{1 + 9\cos^2(3t)}}$, you will need to compute $d\vec{T}/dt$. Just use the quotient rule (don't try to simplify, rather just write out the big mess that comes from the quotient rule). Then immediately plug in $t = \pi/6$ into $d\vec{T}/dt$ before trying to find $\vec{\kappa}$ and ρ at $t = \pi/6$. Most of complication will disappear. Another option is to use problem 7.26.

Problem 7.25 Consider the helix $\vec{r}(t) = (t, \sin t, \cos t)$. Find the radius of curvature $t = \pi/2$. Draw the curve, and draw the circle of curvature at $t = \pi/2$. Then find the center of curvature at $t = \pi/2$. Guess the center of curvature at $t = \pi$?

Here's two final problem related to curvature. They provide a really easy way to compute the curvature of a function of the form $y = f(x)$, and of any curve in the plane. Coming up with the formulas is not necessarily easy, but using them is fairly quick. This formula gets used in dynamics, and shows up on the Fundamentals of Engineering exam (where you just have to use the formula, not prove where it comes from).

Problem 7.26 The function $y = f(x)$ can be given the parametrization $\vec{r}(x) = (x, f(x))$. Use this parametrization to show that the curvature is See 13.4: 5.

$$\kappa(x) = \frac{|f''(x)|}{(1 + (f')^2)^{3/2}}.$$

When a civil engineering team builds a road, they have to pay attention to the curvature of the road. If the curvature of the road is too large, accidents will happen and the civil engineering team will be liable. How do they make sure the curvature never gets to large? They use the circle of curvature. When they want to cause a road to turn, they'll find the center of curvature, send a surveyor out to the center, and then have the surveyor make sure that the road follows the circle of curvature for a short distance. They actually pace out the circle of curvature and then build the road along this circle for a hundred feet or so. Then, they recompute the radius of curvature (if they need the direction to change again), and pace out another circle. In this way, they can guarantee that the curvature never gets large. In the next section we'll see how curvature is directly related to normal acceleration (which is what causes semis to tip, and vehicles to slide off icy roads.)

7.5 Tangential and Normal Components of Acceleration

In this section, we'll show that you write the acceleration of an object moving along a curve $\vec{r}(t)$ with velocity $\vec{v}(t)$ as the sum

$$\vec{a}(t) = a_T\vec{T} + a_N\vec{N} = \frac{d}{dt}|\vec{v}(t)|\vec{T} + \kappa|\vec{v}|^2\vec{N}.$$

The scalars $a_T = \dfrac{d}{dt}|\vec{v}(t)|$ and $a_N = \kappa|\vec{v}|^2$ are called the tangential and normal components of acceleration. All we are doing is writing the vector $\vec{a}(t)$ as the sum of a vector parallel to $\vec{T}$ and a vector orthogonal to $\vec{T}$. Before we decompose the acceleration into its tangential and normal components, let's look at two examples to see what these facts physically represent.

Engineers often use the equivalent formula $a_N = \frac{|\vec{v}|^2}{\rho}$, as ρ is a physical distance that they can measure.

Problem 7.27 Consider the path of an object in projectile motion that has been fired from the origin. Draw a typical path followed by a projectile. The acceleration $\vec{a}(t) = (0, -g)$ acts straight down for any time t.

See 13.5: 17-20 for more practice.

- Pick a point on your path before the max height occurs. At that point, draw both $\vec{T}$, $\vec{a}$, and the projection of $\vec{a}$ onto $\vec{T}$. Is a_T positive or negative?
- At the point you chose above, is the speed of the projectile increasing or decreasing as it climbs higher? Why is it reasonable to believe $a_T = \frac{d}{dt}|\vec{v}(t)|$? Explain.
- Now pick a point after the projectile passes the peak. Then repeat the last two parts at this point.

Problem 7.28 Imagine that you are riding as a passenger on a road and encounter a series of switchbacks (so the road starts to zigzag up the mountain). Right before each bend in the road, you see a yellow sign that tells you a U-turn is coming up, and that you should reduce your speed from 45 mi/hr to 15 mi/hr. Assume the largest curvature along the turn is κ. Recall that $a_N = \kappa|\vec{v}|^2$. The engineers of the road designed the road so that if you are moving at 15 mi/hr, then the normal acceleration will be at most A units.

1. Suppose that your driver (Ben) ignores the suggestion to slow down to 15 mi/hr. He keeps going 45 mi/hr through the turn. Had he slowed down, the max acceleration would be A. You're traveling 3 times faster than suggested. What will your maximum normal acceleration be? [It's more than $3A$.]
2. You yell at Ben to slow down (you don't want to die). So Ben decides to only slow to 30 mi/hr. He figures this means you'll only feel twice as much acceleration as A. Explain why this line of reasoning is flawed.
3. Ben gets frustrated by the fact that he has to slow down. He complains about the engineers who designed the road, and says, "they should have just built a larger corner so I could keep going 45." How much larger should the radius of the circle be so that you can travel 45 mi/hr instead of 15 mi/hr, and still feel the same acceleration A?
4. Which will cause the normal acceleration to decrease more, halving your speed or halving the curvature (doubling the radius)?

Problem 7.29 Prove that $\vec{a}(t) = a_T\vec{T} + a_N\vec{N} = \dfrac{d}{dt}|\vec{v}|\vec{T} + \kappa|\vec{v}|^2\vec{N}$. Here's some hints.

Watch a YouTube Video.

- Rewrite the velocity $\vec{v}$ as a magnitude $|\vec{v}|$ times a direction $\vec{T}$.
- We know that $\vec{a}(t) = \frac{d}{dt}\vec{v}(t)$ (acceleration is the derivative of velocity). Take the derivative of $\vec{v} = |\vec{v}|\vec{T}$ by using the product rule (on the scalar product $|\vec{v}|\vec{T}$).

- You should encounter the quantity $d\vec{T}/dt$ somewhere in your product. Write this quantity as a magnitude times a direction. [We've seen $d\vec{T}/dt$ in much of our previous work. You'll need to prove that $d\vec{T}/dt = \kappa|\vec{v}|\vec{N}$.]

Problem 7.30 We now know that $\vec{a}(t) = a_T\vec{T} + a_N\vec{N} = \frac{d}{dt}|\vec{v}|\vec{T} + \kappa|\vec{v}|^2\vec{N}$. Use this to prove that the curvature can be obtained from the formula

$$\kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3}.$$

[Hint: cross both sides with $\vec{v}$, simplify, take the magnitude of each side, and solve for κ.]

Here's a table that summarizes some of the concepts we have discussed in this unit. The goal of this unit is to understand how the vectors in this table are related and why.

Unit Tangent Vector	$\vec{T}$	$\frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{\vec{r}'(t)}{\lvert\vec{r}'(t)\rvert}$
Curvature Vector	$\vec{\kappa}$	$\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt} = \frac{d\vec{T}/dt}{\lvert\vec{v}\rvert} = \frac{\vec{T}'(t)}{\lvert\vec{r}'(t)\rvert}$
Curvature (not a vector, but a scalar)	κ	$\left\lvert\frac{d\vec{T}}{ds}\right\rvert = \left\lvert\frac{d\vec{T}/dt}{ds/dt}\right\rvert = \frac{\left\lvert d\vec{T}/dt\right\rvert}{\lvert\vec{v}\rvert} = \frac{\lvert\vec{T}'(t)\rvert}{\lvert\vec{r}'(t)\rvert}$
Principal unit normal vector	$\vec{N}$	$\frac{d\vec{T}/dt}{\lvert d\vec{T}/dt\rvert} = \frac{\vec{T}'(t)}{\lvert\vec{T}'(t)\rvert} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} = \frac{1}{\kappa\lvert\vec{v}\rvert}\frac{d\vec{T}}{dt}$
Binormal vector	$\vec{B}$	$\vec{T}\times\vec{N}$
Radius of curvature	ρ	$1/\kappa$
Center of curvature at t		$\vec{r}(t) + \rho(t)\vec{N}(t)$
Torsion	τ	$\pm\left\lvert\frac{d\vec{B}}{ds}\right\rvert$ (pick the sign) or $-\frac{d\vec{B}}{ds}\cdot\vec{N}$
Tangential Component of acceleration	a_T	$\vec{a}\cdot\vec{T} = \frac{d}{dt}\lvert\vec{v}\rvert$
Normal Component of acceleration	a_N	$\vec{a}\cdot\vec{N} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa\lvert\vec{v}\rvert^2$

7.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to Brainhoney and download the quiz. Once you have taken the quiz, you can upload your work back to brainhoney and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 8

Line Integrals

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Describe how to integrate a function along a curve. Use line integrals to find the area of a sheet of metal with height $z = f(x, y)$ above a curve $\vec{r}(t) = (x, y)$ and the average value of a function along a curve.

2. Find the following geometric properties of a curve: centroid, mass, center of mass, inertia, and radii of gyration.

3. Compute the work (flow, circulation) and flux of a vector field along and across piecewise smooth curves.

4. Determine if a field is a gradient field (hence conservative), and use the fundamental theorem of line integrals to simplify work calculations.

You'll have a chance to teach your examples to your peers prior to the exam.

In this chapter, we generalize integrals along the x-axis from previous semesters in calculus to integrals along any curve.

8.1 Surface Area

In this section, we'll first generalize the concept of the integral. We'll approach everything from the point of view of area, though the applications are much more extensive. The first problem is a review problem from first-semester calculus. The second problem generalizes the idea to integrals along a curve (which we call a line integral). The third problem has you generalize your results. Let's start with a quick review of sigma notation.

Example 8.1. Recall that we write $\sum_{i=1}^{20} i^3$ as short hand for

$$\sum_{i=1}^{20} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + 20^3.$$

This notation is called sigma notation. It allows us to express really long sums in a very short space. We could also write

$$\sum_{i=30}^{4000} x_i = x_{30} + x_{31} + x_{32} + \cdots + x_{4000}$$

if we needed to add up the numbers starting at x_{30} and ending at x_{4000}. If we wanted to add the integers starting at 1 and ending at n, then we would write

$$1+2+3+\cdots+n=\sum_{i=1}^{n}.$$

This first problem is a step-by-step review of how we did integrals in first-semester calculus. It also asks you to write sums in sigma notation.

Problem 8.1 Consider the region in the xy plane that is below the function $f(x)=x^2+1$ and above the x-axis where $x\in[-1,2]$. Think of this region as a metal plate. We will find its surface area.

1. Draw the curve over the given bounds, and shade the region.

2. Now partition the interval $[-1,2]$ into 6 equally-spaced parts. On your graph, draw 6 rectangles to approximate the area under f. Use the right endpoint of each interval to determine the height of each rectangle. The width of each rectangle we call Δx. What is Δx in this example?

We could have used the left endpoint point, or the midpoint, or any other point in each partition. I chose the right endpoint to make sure we all had the same answers.

3. Recall from first semester calculus that we typically name the x-coordinates of the ends of our rectangles using the notation x_0, x_1, x_2, ... In this example, we have

$$x_0=-1,\ x_1=-\frac{1}{2},\ x_2=0,\ x_3=\frac{1}{2},\ x_4=1,\ x_5=\frac{3}{2},\ x_6=2.$$

The area of the first rectangle is $\Delta A_1=f(x_1)\Delta x$. The area of the second rectangle is $\Delta A_2=f(x_2)\Delta x$. The total combined area of the 6 rectangles you drew above is the sum

$$\begin{aligned}&\Delta A_1+\Delta A_2+\Delta A_3+\Delta A_4+\Delta A_5+\Delta A_6\\&\quad=f(x_1)\Delta x+f(x_2)\Delta x+f(x_3)\Delta x+f(x_4)\Delta x+f(x_5)\Delta x+f(x_6)\Delta x.\end{aligned}$$

Write this sum using sigma notation (i.e., using a $\sum$).

4. Instead of using 6 rectangles, let's now use n equally wide rectangles. Let $\Delta x=dx$ be the width of each rectangle, and the right endpoint of each segment we'll call $x_1,x_2,\ldots,x_n$. What is the area ΔA_i of the ith rectangle? The sum of these n little areas is approximately the total area under the curve, i.e.

$$A\approx\Delta A_1+\Delta A_2+\cdots+\Delta A_n.$$

Write this sum using sigma notation.

5. Explain why the area under f over the interval $[-1,2]$ is $A=\int_{-1}^{2}(x^2+1)dx$.

The next problem should mimic the steps in the previous problem. The only difference is that you are now integrating over a curve, not over in interval—this is the generalization from first-semester calculus that we are looking at now.

Problem 8.2 Consider the surface in space that is below the function $f(x,y)=9-x^2-y^2$ and above the curve C parametrized by $\vec{r}(t)=(2\cos t,3\sin t)$ for $t\in[0,2\pi]$. Think of this region as a metal plate that has been stood up with its base on C where the height above each spot is given by $z=f(x,y)$.

Watch a YouTube video. See Sage for a picture of this sheet.

1. Draw the curve C in the xy-plane.

See Problem 6.20

2. Now partition the curve into 6 parts, using equally spaced time intervals. Draw a straight line between the spots on the curve given by $\vec{r}(0)$, $\vec{r}(\pi/3)$, $\vec{r}(2\pi/3)$, etc. You should have 6 straight lines connecting points on an ellipse. The length of each segment you drew is called Δs (an approximation to arc length). If we drew lots of tiny segments, we would use $\Delta s = ds$ to represent this length. Why does $ds = \sqrt{(-2\sin t)^2 + (3\cos t)^2}dt$?

3. We'll call the t-coordinates of our partition t_0, t_1, t_2, ... We have

$$t_0 = 0,\ t_1 = \pi/3,\ t_2 = 2\pi/3,\ t_3 = \pi,\ t_4 = 4\pi/3,\ t_5 = 5\pi/3,\ t_6 = 2\pi.$$

We need the surface area of the sheet that lies above the ellipse, but under the function $f(x, y)$. Above each little straight segment of length Δs, we could approximate the area by assuming height of $f(x, y)$ along the entire segment is the same as the height above the right endpoint, i.e. using a height of $f(\vec{r}(t_i))$. In a 3D picture, add the surface, ellipse, and 6 rectangles. See the Sage link above.

The area of the first rectangle is $\Delta\sigma_1 = f(\vec{r}(t_1))\Delta s_1$. The area of the second rectangle is $\Delta\sigma_2 = f(\vec{r}(t_2))\Delta s_2$. Using our 6 rectangles, the total surface area of the sheet would approximately be the sum

We'll use σ (a lower-case "sigma") to stand for surface area.

$$\begin{aligned}&\Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3 + \cdots + \Delta\sigma_6\\ &\quad= f(\vec{r}(t_1))\Delta s_1 + f(\vec{r}(t_2))\Delta s_2 + f(\vec{r}(t_3))\Delta s_3 + \cdots + f(\vec{r}(t_6))\Delta s_6\end{aligned}$$

Write this sum using sigma notation.

4. Instead of using 6 rectangles, let's now use n rectangles, equally spaced by time. Let Δs_i be the width of the ith rectangle. Use the time values $t_1, t_2, \ldots, t_n$ to find the heights of the ith rectangle. What is the surface area $\Delta\sigma_i$ of the ith rectangle? The sum of these n little surface areas is approximately the total surface area under f above C, i.e we have

$$\sigma \approx \Delta\sigma_1 + \Delta\sigma_2 + \cdots + \Delta\sigma_n.$$

Write this sum using sigma notation.

5. Use your sum from the previous part to explain why the area of the metal sheet that lies above C and under f is given by the integral

$$\sigma = \int_C (9-x^2-y^2)ds = \int_0^{2\pi} (9-(2\cos t)^2-(3\sin t)^2)\sqrt{(-2\sin t)^2 + (3\cos t)^2}dt.$$

Your results from the problem above suggest the following definition.

Definition 8.2: Line Integral. Let f be a function and C be a piecewise smooth curve given by the parametrization $\vec{r}(t)$ for $t \in [a, b]$. We require that the composition $f(\vec{r}(t))$ be continuous for all $t \in [a, b]$. Then we define the line integral of f over C to be the integral

The line integral is also called the path integral, contour integral, or curve integral.

$$\int_C f ds = \int_a^b f(\vec{r}(t))\frac{ds}{dt}dt = \int_a^b f(\vec{r}(t))\left|\frac{d\vec{r}}{dt}\right|dt.$$

Notice that this definition suggests the following four steps. These four steps are the key to computing any line integral.

When we ask you to set up a line integral, it means that you should do steps 1–3, so that you get an integral with a single variable and with bounds that you could plug into a computer or do in Calculus 2.

Please compute all integrals we ask you to compute to get a numeric answer. Compute the

1. Start by getting a parametrization $\vec{r}(t)$ for $a \le t \le b$ of the curve C.

2. Find the speed by computing $\frac{d\vec{r}}{dt}$ and then $\left|\frac{d\vec{r}}{dt}\right|$.

3. Multiply f by the speed, and replace each x, y, z with what it equals in terms of t.

4. Integrate the product from the previous step.

Problem 8.3 Let $f(x,y,z) = x^2+y^2-2z$ and let C be two coils of the helix $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$, starting at $t = 0$. Remember that the parameterization means $x = 3\cos t$, $y = 3\sin t$, and $z = 4t$. Compute $\int_C f ds$. [You will have to find the end bound yourself. How much time passes to go around two coils?]

See 16.1: 9-32. Some problems give you a parametrization, some expect you to come up with one on your own.

Problem 8.4 Consider the function $f(x,y) = 3xy+2$. Let C be a circle of radius 4 centered at the origin. Compute $\int_C f ds$. [You'll have to come up with your own parameterization.]

To practice matching parameterizations to curves, try 16.1:1-8.

Problem 8.5 Let $f(x,y,z) = x^2+3yz$. Let C be the straight line segment from $(1,0,0)$ to $(0,4,5)$. Compute $\int_C f ds$.

If you've forgotten how to parametrize line segments, see 2.9.

Problem 8.6 Let $f(x,y) = x^2+y^2-25$. Let C be the portion of the parabola $y^2 = x$ between $(1,-1)$ and $(4,2)$. We want to compute $\int_C f ds$.

See 5.18 if you forgot how to parametrize plane curves.

1. Draw the curve C and the function $f(x,y)$ on the same 3D xyz axes.

2. Without computing the line integral $\int_C f ds$, determine if the integral should be positive or negative. Explain why this is so by looking at the values of $f(x,y)$ at points along the curve C. Is $f(x,y)$ positive, negative, or zero, at points along C?

3. Parametrize the curve and set up the line integral $\int_C f ds$. [Hint: if you let $y = t$, then $x =$? What bounds do you put on t?]

4. Use technology to compute $\int_C f ds$ to get a numeric answer. Was your answer the sign that you determined above?

8.2 Average Value

The concept of averaging values together has many applications. In first-semester calculus, we saw how to generalize the concept of averaging numbers together to get an average value of a function. We'll review both of these concepts. Later, we'll generalize average value to calculate centroids and center of mass.

Problem 8.7 Suppose a class takes a test and there are three scores of 70, five scores of 85, one score of 90, and two scores of 95. We will calculate the average class score, $\bar{s}$, four different ways to emphasize four ways of thinking about the averages. We are emphasizing the pattern of the calculations in this problem, rather than the final answer, so it is important to write out each calculation completely in the form $\bar{s} =$ ______ before calculating the number $\bar{s}$.

1. Compute the average by adding 11 numbers together and dividing by the number of scores. Write down the whole computation before doing any arithmetic. $\bar{s} = \frac{\sum \text{values}}{\text{number of values}}$

2. Compute the numerator of the fraction in the previous part by multiplying each score by how many times it occurs, rather than adding it in the sum that many times. Again, write down the calculation for $\bar{s}$ before doing any arithmetic. $\bar{s} = \frac{\sum(\text{value}\cdot\text{weight})}{\sum \text{weight}}$

3. Compute $\bar{s}$ by splitting up the fraction in the previous part into the sum of four numbers. This is called a "weighted average" because we are multiplying each score value by a weight. $\bar{s} = \sum(\text{value} \cdot (\% \text{ of stuff}))$

4. Another way of thinking about the average $\bar{s}$ is that $\bar{s}$ is the number so that if all 11 scores were $\bar{s}$, you'd have the same sum. Write this way of thinking about these computations by taking the formulas for $\bar{s}$ in the first two parts and multiplying both sides by the denominator. $(\text{number of values})\bar{s} = \sum \text{values}$ $(\sum \text{weight})\bar{s} = \sum(\text{value} \cdot \text{weight})$

In the next problem, we generalize the above ways of thinking about averages from a discrete situation to a continuous situation. You did this in first-semester calculus when you did average value using integrals.

Problem 8.8 Suppose the price of a stock is \$10 for one day. Then the price of the stock jumps to \$20 for two days. Our goal is to determine the average price of the stock over the three days.

1. Let $f(t) = \begin{cases} 10 & 0 < t < 1 \\ 20 & 1 < t < 3 \end{cases}$, the price of the stock for the three-day period. Draw the function f, and find the area under f where $t \in [0, 3]$.

2. Find a single constant $\bar{f}$ so that the areas under both $\bar{f}$ and f, above the interval $[0, 3]$, are the same numbers. [Hint: The area under $\bar{f}$ is just the area of a rectangle.]

3. We found a constant $\bar{f}$ so that the area under $\bar{f}$ matched the area under f. In other words, we solved the equation below for $\bar{f}$:
$$\int_a^b \bar{f} dx = \int_a^b f dx$$
Solve for $\bar{f}$ symbolically, without doing any of the integrals. This quantity is called the average value of f over $[a, b]$.

4. The formula for $\bar{f}$ in the previous part resembles at least one of the ways of calculating averages from Problem 8.7. Which ones and why?

Ask me in class about the "ant farm" approach to average value.

Problem 8.9 Let the curve C have the parametrization $\vec{r}(t) = (2\cos t, 3\sin t)$. Let f be the function $f(x, y) = 9 - x^2 - y^2$. Watch a YouTube video.

1. Draw the surface f in 3D. Add to your drawing the curve C in the xy plane. Then draw the sheet whose area is given by the integral $\int_C f ds$.

2. What's the maximum height and minimum height of the sheet? See problem 6.20.

3. We would like to find a constant height $\bar{f}$ so that the area under f above C, is the same as area under $\bar{f}$, above C. What integral gives us the area under $\bar{f}$ above C? What integral gives us the area under f above C? Explain why the average value of f along C is

$$\bar{f} = \frac{\int_C f ds}{\int_C ds}.$$

Connect this formula with the ways of thinking about averages from Problem 8.7.

Please read Isaiah 40:4 and Luke 3:5. These scriptures should help you remember how to find average value.

4. Use a computer to evaluate the integrals $\int_C f ds$ and $\int_C ds$, and then give an approximation to the average value of f along C. Is your average value between the maximum and minimum of f along C? Why should it be?

Problem 8.10 The temperature $T(x, y, z)$ at points on a wire helix C given by $\vec{r}(t) = (\sin t, 2t, \cos t)$ is known to be $T(x, y, z) = x^2 + y + z^2$. What are the temperatures at $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$ and $t = 2\pi$? You should notice the temperature is constantly changing. Make a guess as to what the average temperature is (share with the class why you made the guess you made—it's OK if you're wrong). Then compute the average temperature of the wire using the integral formula from the previous problem.

8.3 Work, Flow, Circulation, and Flux

We now look at an exciting application of line integrals. This application helps us study the transfer of energy (work), as well as understanding the flow of air along a wing (circulation) and the flow of a fluid across a surface (flux).

Let's start with a review of work. As an object moves through a vector field, energy transfer occurs. When an object falls from high place, potential energy is transfered to kinetic energy. The gravitational vector field is the field which does work. Prior to problem 2.19 on page 15, we made the following statements.

> If a force F acts through a displacement d, then the most basic definition of work is $W = Fd$, the product of the force and the displacement. This basic definition has a few assumptions.
>
> - The force F must act in the same direction as the displacement.
> - The force F must be constant throughout the displacement.
> - The displacement must be in a straight line.

We used the dot product to remove the first assumption, and we showed in problem 2.19 that the work is simply the dot product

$$W = \vec{F} \cdot \vec{r},$$

where $\vec{F}$ is a force acting through a displacement $\vec{r}$. We now remove the other two assumptions so that we can deal with variable forces acting on objects moving along somewhat arbitrary curves. We will use the unit tangent vector $\vec{T}$ to a curve $\vec{r}$ that was introduced in Definition 7.2. Recall that

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|}.$$

Problem 8.11: Work Let $\vec{F}(x,y) = (M,N)$ be a vector field, where M and N are functions of x and y. Let C be a curve parametrized by $\vec{r}(t) = (x,y)$, where x and y are functions of t and $t \in [a,b]$.

Watch a YouTube video.

1. Draw a random curve on your paper. Cut the curve $\vec{r}(t)$ into lots of little segments. Each little segment has a length, which we call ds. If your segments are really small, then $\vec{F}$ is almost constant on this segment. Explain why the work done by $\vec{F}$ along this tiny segment is approximately
$$d(\text{Work}) = \vec{F} \cdot (\vec{T}ds).$$
2. Explain why the work done by $\vec{F}$ along C is
$$W = \int_C \vec{F} \cdot \vec{T}ds = \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt}dt = \int_a^b M\frac{dx}{dt} + N\frac{dy}{dt}dt.$$
3. If you are familiar with the units of energy, complete the following. What are the units of $\vec{F}$, $\vec{T}$, ds, and dWork.

The work done by a vector field may show up in any of the following ways:

$$\begin{aligned} W &= \int_C \vec{F} \cdot \vec{T}\ ds \\ &= \int_C \vec{F} \cdot \frac{d\vec{r}}{ds}\ ds \\ &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt}\ dt \\ &= \int_C Mdx + Ndy \\ &= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt}\right) dt. \end{aligned}$$

When working with a vector field in space, we often use the notation $\vec{F}(x,y,z) = (M,N,P)$, so we often write work as $\int_C Mdx + Ndy + Pdz$.

Notice that only two integrals above have the bounds a and b. These two integrals are the actual formula used to compute the integral. The others are just symbolic ways to remember the integral.

Definition 8.3: Flow and Circulation. If the vector field $\vec{F}$ represents the velocity field of a fluid, such as airflow along a wing (so units are m/s), then the work integral is often called flow. If the start and stop point for the curve are the same, then we'll call the the work integral *circulation*. In this case, we'll often add a circle to the integral, as in $\oint_C \vec{F} \cdot d\vec{r}$, to emphasize that the integral is along a closed curve. In most cases, we'll be computing circulation along curves in the counterclockwise direction. If we want to emphasize the direction we are going along a closed curve, we'll use an arrow on the small circle on the integral sign.

Definition 8.4: Simple Closed Curve. If C is a smooth curve, and the start and end points of C are the same, we call C a closed curve. If the closed curve does not intersect itself, we call the curve a simple closed curve.

Problem 8.12 Let $\vec{F}(x,y) = (M,N)$ be a vector field. Let C be a simple closed curve parametrized by $\vec{r}(t) = (x(t), y(t))$. Then $d\vec{r} = (dx, dy)$.

1. Draw a simple closed curve to represent $\vec{r}$ (just draw any simple closed curve you like). We know that the circulation (work) along C is given by the integral $\int_C (M,N)\cdot(dx,dy)$. Pick a spot on your curve, and draw a tangent vector to represent (dx,dy) so that you traverse the curve using a counterclockwise orientation.
2. What is the angle between (dx,dy) and $(-dy,dx)$? What is the angle between (dx,dy) and $(dy,-dx)$? [Hint: see Problems 2.12 and 2.13.]
3. Which vector, $(-dy,dx)$ or $(dy,-dx)$, points towards the outside of the curve if we are going around the curve counterclockwise? Pick a few points on your curve and explain why $(dy,-dx)$ always points towards the outside the curve.

When we compute the work done by a vector field along a curve, we are focusing on how much of the vector field points in the direction of motion. The integral $\oint_C \vec{F}\cdot\vec{T}ds$ measures the flow along a curve. If we let $\vec{n}$ be the outward pointing normal vector to the curve, then the integral $\oint_C \vec{F}\cdot\vec{n}ds$ measures the flow outward across a curve. We'll define this outward flow as "Flux."

Definition 8.5: Flux. Let $\vec{F}(x,y)=(M,N)$ be a vector field. Let C be a simple closed curve parametrized by $\vec{r}(t)=(x,y)$, and oriented in the counterclockwise direction. We know the circulation of $\vec{F}$ along C (the flow of $\vec{F}$ along C) is the integral $\oint_C \vec{F}\cdot\vec{T}\,ds=\oint_C (M,N)\cdot(dx,dy)$. The outward flux of $\vec{F}$ across C is the line integral

Watch a YouTube video.

$$\text{Flux}=\Phi=\oint_C \vec{F}\cdot\vec{n}\,ds=\oint_C (M,N)\cdot(dy,-dx)=\oint_C Mdy-Ndx.$$

The flux of $\vec{F}$ measures the outward flow of $\vec{F}$ across C instead of along C. The vector $\vec{n}ds=(dy,-dx)$ is correct if the curve is oriented in the counterclockwise direction (which was shown in the previous problem).

We're now prepared to compute both work (circulation, flow) and flux. The next 4 problems ask you to do so. The most common way to remember these, provided the vector field is $\vec{F}(x,y)=(M(x,y),N(x,y))$, is

$$\text{Work}=\int_C Mdx+Ndy \qquad \text{Flux}=\int_C Mdy-Ndx.$$

Problem 8.13 Consider the rotational field $\vec{F}=(-y,x)$ and the circle C of radius 5 parametrized by $\vec{r}(t)=(5\cos t,5\sin t)$ for $t\in[0,2\pi]$.

If you haven't yet, please watch the YouTube videos for work and flux.

1. Draw the curve C and vector field $\vec{F}$ on the same axes.
2. Compute the circulation (work) of $\vec{F}$ along C.
3. Compute the outward flux of $\vec{F}$ along C.
4. Can you explain why one of these integrals must be zero, and the other must be positive? We'll answer this in class if you are unable.

Problem 8.14 Consider the radial field $\vec{F}=(2x,2y)$ and curve C parametrized by $\vec{r}(t)=(3\cos t,3\sin t)$ for $t\in[0,2\pi]$.

1. Draw the curve C and vector field $\vec{F}$ on the same axes.

2. Compute the circulation (work) of $\vec{F}$ along C.

3. Compute the outward flux of $\vec{F}$ along C.

4. Can you explain why one of these integrals must be zero, and the other must be positive? We'll answer this in class if you are unable.

From the previous two problems you might ask, "Are there vector fields where the work and flux can both be nonzero?" The next problems answer this in the affirmative. The previous two problems just dealt with a rotation field (where the vector field only rotates things, does not push in or out, so zero flux) and a radial field (which only pushes out, so no circulation).

Problem 8.15 Let $\vec{F} = (-y, x + y)$ and C be the triangle with vertices $(2, 0)$, $(0, 2)$, and $(0, 0)$.

Watch a YouTube video. Also, see Sage for a picture.

1. Look at a drawing of C and the vector field (see margin for the Sage link). If we go counterclockwise around the triangle, for each side of the triangle, guess the signs of the counterclockwise circulation and the flux (positive, negative, zero).

2. Find the counterclockwise circulation (work) done by $\vec{F}$ along C. You'll have three separate calculations, one for each side. You'll need to parametrize three line segments.

Problem 8.16 Consider the vector field $\vec{F} = (2x - y, x)$. Let C be the curve that starts at $(-2, 0)$, follows a straight line to $(1, 3)$, and then back to $(-2, 0)$ along the parabola $y = 4 - x^2$.

See Sage.

1. Look at a drawing of C and the vector field (see margin for the Sage link). If we go counterclockwise around C, for each part of C, guess the signs of the counterclockwise circulation and the flux (positive, negative, zero).

2. Find the flux of $\vec{F}$ across C. There are two curves to parametrize. Make sure you traverse along the curves in the correct direction.

8.4 Physical Properties

A number of physical properties of real-world objects can be calculated using the concepts of averages and line integrals. We explore some of these in this section. Additionally, many of these concepts and calculations are used in statistics.

8.4.1 Centroids

Definition 8.6: Centroid. Let C be a curve. If we look at all of the x-coordinates of the points on C, the "center" x-coordinate, $\bar{x}$, is the average of all these x-coordinates. Likewise, we can talk about the averages of all of the y coordinates or z coordinates of points on the function ($\bar{y}$ or $\bar{z}$, respectively). The *centroid* of an object is the geometric center $(\bar{x}, \bar{y}, \bar{z})$, the point with coordinates that are the average x, y, and z coordinates.

Problem 8.17: Centroid Notice the word "average" in the definition of the centroid. Use the concept of average value to explain why the coordinates of the centroid are

Watch a YouTube video.

Formulas for the centroid.

$$\bar{x} = \frac{\int_C xds}{\int_C ds}, \quad \bar{y} = \frac{\int_C yds}{\int_C ds}, \quad \text{and} \quad \bar{z} = \frac{\int_C zds}{\int_C ds}.$$

Notice that the denominator in each case is just the arc length $s = \int_C ds$.

Problem 8.18 Let C be the semicircular arc $\vec{r}(t) = (a\cos t, a\sin t)$ for $t \in [0, \pi]$. Without doing any computations, make an educated guess for the centroid $(\bar{x}, \bar{y})$ of this arc. Then compute the integrals given in problem 8.17 to find the actual centroid. Share with the class your guess, even if it was incorrect.

8.4.2 Mass and Center of Mass

Density is generally a mass per unit volume. However, when talking about a curve or wire, as in this chapter, it's simpler to let density be the mass per unit length. Sometimes an object is made out of a composite material, and the density of the object is different at different places in the object. For example, we might have a straight wire where one end is aluminum and the other end is copper. In the middle, the wire slowly transitions from being all aluminum to all copper. The centroid is the midpoint of the wire. However, since copper has a higher density than aluminum, the balance point (the center of mass) would not be at the midpoint of the wire, but would be closer to the denser and heavier copper end. In this section, we'll develop formulas for the mass and center of mass of such a wire. Such composite materials are engineered all the time (though probably not our example wire). In future mechanical engineering courses, you would learn how to determine the density δ (mass per unit length) at each point on such a composite wire.

Problem 8.19: Mass Suppose a wire C has the parameterization $\vec{r}(t)$ for $t \in [a, b]$. Suppose the wire's density at a point (x, y, z) on the wire is given by the function $\delta(x, y, z)$. You'll learn to calculate this function in a future class. For the purposes of our class, we'll just assume we know what $\delta(x, y, z)$ is.

Watch a YouTube video.

1. Consider a small portion of the curve at $t = t_0$ of length ds. Explain why the mass of the small portion of the curve is $dm = \delta(\vec{r}(t_0))ds$.

2. Explain why the mass m of an object is given by the formulas below (explain why each equals sign is true):

$$m = \int_C dm = \int_C \delta ds = \int_a^b \delta(\vec{r}(t)) \left|\frac{d\vec{r}}{dt}\right| dt.$$

Problem 8.20 A wire lies along the straight segment from $(0, 2, 0)$ to $(1, 1, 3)$. The wire's density (mass per unit length) at a point (x, y, z) is $\delta(x, y, z) = x + y + z$.

1. Is the wire heavier at $(0, 2, 0)$ or at $(1, 1, 3)$?

2. What is the total mass of the wire? [You'll need to parameterize the line as your first step—see Problem 2.9 if you need a refresher.]

The center of mass of an object is the point where the object balances. In order to calculate the x-coordinate of the center of mass, we average the x-coordinates, but we weight each x-coordinate with its mass. Similarly, we can calculate the y and z coordinates of the center of mass.

Wikipedia has some interesting applications of center of mass.

The next problem helps us reason about the center of mass of a collection of objects. Calculating the center of mass of a collection of objects is important, for example, in astronomy when you want to calculate how two bodies orbit each other.

Problem 8.21 Suppose two objects are positioned at the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. Our goal in this problem is to understand the difference between the centroid and the center of mass.

1. Find the centroid of two objects.
2. Suppose both objects have the same mass of 2 kg. Find the center of mass by averaging the x, y, and z coordinates, weighted by how much mass is at each coordinate.
3. If the mass of the object at point P_1 is 2 kg, and the mass of the object at point P_2 is 3 kg, will the center of mass be closer to P_1 or P_2? Give a physical reason for your answer before doing any computations. Then find the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the two points. [Hint: you should get $\bar{x} = \frac{2x_1+3x_2}{2+3}$.]

Problem 8.22 This problem reinforces what you just did with two points in the previous problem. However, it now involves two people on a seesaw. Ignore the mass of the seesaw in your work below (pretend it's an extremely light seesaw, so its mass is negligible compared to the masses of the people).

See Wikipedia for a seesaw picture.

1. My daughter and her friend are sitting on a seesaw. Both girls have the same mass of 30 kg. My wife stands about 1 m behind my daughter. We'll measure distance in this problem from my wife's perspective. We can think of my daughter as a point mass located at $(1\text{m}, 0)$ whose mass is 30 kg. Suppose her friend is located at $(5\text{m}, 0)$. Suppose the kids are sitting just right so that the seesaw is perfectly balanced. That means the the center of mass of the girls is precisely at the pivot point of the seesaw. Find the distance from my wife to the pivot point by finding the center of mass of the two girls.
2. My daughter's friend has to leave, so I plan to take her place on the seesaw. My mass is 100 kg. Her friend was sitting at the point $(5, 0)$. I would like to sit at the point $(a, 0)$ so that the seesaw is perfectly balanced. Without doing any computations, is $a > 5$ or $a < 5$? Explain.
3. Suppose I sit at the spot $(x, 0)$ (perhaps causing my daughter or I to have a highly unbalanced ride). Find the center of mass of the two points $(1, 0)$ and $(x, 0)$ whose masses are 30 and 100, respectively (units are meters and kilograms).
4. Where should I sit so that the seesaw is perfectly balanced (what is a)?

Problem 8.23: Center of mass In problem 8.21, we focused on a system with two points (x_1, y_1) and (x_2, y_2) with masses m_1 and m_2. The center of mass in the x direction is given by

Watch a YouTube video.

$$\bar{x} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2}$$

1. If we consider a system with 3 points, what formula gives the center of mass in the x direction?

2. Consider a system with n points labeled (x_1, y_1, z_1), (x_2, y_2, z_2), $\ldots$, (x_n, y_n, z_n), having masses m_1, m_2, $\ldots$, m_n respectively. Give a formula for the center of mass in the y direction (the x and z directions are similar).

3. Suppose now that we have a wire located along a curve C. The density of the wire is known to be $\delta(x, y, z)$ (which could be different at different points on the curve). Imagine cutting the wire into a thousand or more tiny chunks. Each chunk would be centered at some point (x_i, y_i, z_i) and have length ds_i. Explain why the mass of each little chunk is $dm_i \approx \delta ds_i$.

4. Give a formula for the center of mass in the y direction of the thousands of points (x_i, y_i, z_i), each with mass dm_i. [This should almost be an exact copy of the second part.] Then explain why

$$\bar{y} = \frac{\int_C y dm}{\int_C dm} = \frac{\int_C y \delta ds}{\int_C \delta ds}.$$

For quick reference, the formulas for the centroid of a wire along C are

$$\bar{x} = \frac{\int_C x ds}{\int_C ds}, \quad \bar{y} = \frac{\int_C y ds}{\int_C ds}, \quad \text{and} \quad \bar{z} = \frac{\int_C z ds}{\int_C ds}. \quad \text{(Centroid)}$$

If the wire has density δ, then the formulas for the center of mass are

$$\bar{x} = \frac{\int_C x dm}{\int_C dm}, \quad \bar{y} = \frac{\int_C y dm}{\int_C dm}, \quad \text{and} \quad \bar{z} = \frac{\int_C z dm}{\int_C dm}, \quad \text{(Center of mass)}$$

where $dm = \delta ds$. Notice that the denominator in each case is just the mass $m = \int_C dm$.

We'll often use the notation $(\bar{x}, \bar{y}, \bar{z})$ to talk about both the centroid and the center of mass. If no density is given in a problem, then $(\bar{x}, \bar{y}, \bar{z})$ is the centroid. If a density is provided, then $(\bar{x}, \bar{y}, \bar{z})$ refers to the center of mass. If the density is constant, it doesn't matter (the centroid and center of mass are the same, which is what the seesaw problem showed).

The quantity $\int_C x dm$ is sometimes called the first moment of mass about the yz-plane (so $x = 0$). Notationally, some people write $M_{yz} = \int_C x ds$. Similarly, we could write $M_{xz} = \int_C y dm$ and $M_{xy} = \int_C z dm$. With this notation, we could write the center of mass formulas as

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right).$$

Problem 8.24 Suppose a wire with density $\delta(x, y) = x^2 + y$ lies along the curve C which is the upper half of a circle around the origin with radius 7.

1. Parametrize C (find $\vec{r}(t)$ and the domain for t).

2. Where is the wire heavier, at $(7, 0)$ or at $(0, 7)$?

3. In problem 8.18, we showed that the centroid of the wire is $(\bar{x}, \bar{y}) = \left(0, \frac{2(7)}{\pi})\right)$. We now seek the center of mass. Before computing, will $\bar{x}$ change? Will $\bar{y}$ change? How will each change? Explain.

4. Set up the integrals needed to find the center of mass. Then use technology to compute the integrals. Give an exact answer (involving fractions), rather than a numerical approximation.

Problem 8.25 The quantity $M_{yz} = \int_C x dm$ is sometimes called the first moment of mass about the yz-plane (the plane $x = 0$). It adds up the weighted distances from the plane $x = 0$. One way to view the center of mass is to ask yourself the following question.

> The mass m of a curve C is known. If you could place all the mass at one single spot, called $(\bar{x}, \bar{y}, \bar{z})$, what should $\bar{x}$ be so that the first moment of mass about the yz-plane does not change.

We want the moment $\int_C \bar{x} dm$ (all mass at one point) and the moment $\int_C x dm$ (the mass is spread across infinitely many points) to be exactly the same. Use this idea to solve for $\bar{x}$ in the equation

$$\int_C \bar{x} dm = \int_C x dm.$$

Then similarly obtain $\bar{y}$ and $\bar{z}$. [Hint: the number $\bar{x}$ is a constant, whereas x is not. Does $\int 2f dx = 2 \int f dx$?]

8.4.3 Inertia and Radii of Gyration

Some of you may have already had a physics class, in which you learned that the kinetic energy of an object with mass m moving at speed v is

$$KE = \frac{1}{2} m v^2.$$

One of the main reasons we are studying mass, center of mass, centroids, etc., is so that we can understand energy. The transfer of energy (for example from kinetic to electrical and then back from electrical to kinetic) is one of the most important ideas in modern innovations. Our goal in this unit it to help us understand rotational kinetic energy. We'll show that the kinetic energy of an object that is rotating about a line L, and has an angular velocity of ω radians per second about the line, is precisely

$$KE = \frac{1}{2} I \omega^2,$$

where I is the (second) moment of inertia. The moment of inertia can be obtained by integrating $I = \int_C (d)^2 dm$ where d is the radius of rotation about L, i.e. the distance from a point (x, y, z) to the axis of rotation L. If the line L is one of the coordinate axes, then we obtain the key formulas

$$I_x = \int_C (y^2 + z^2) dm, \quad I_y = \int_C (x^2 + z^2) dm, \quad I_z = \int_C (x^2 + y^2) dm.$$

If you have never worked with kinetic energy before, you may skip the next problem and then just practice using these formulas.

Problem 8.26 Suppose that an object, whose mass is m, is attached to a string (whose mass is so small we'll ignore it). The object is rotated about a point, where the angular velocity is ω radians per second. The length of the string (distance from the point to the center of rotation) is d. Watch a YouTube video.

1. We know kinetic energy is $KE = \frac{1}{2}mv^2$. If a string is being rotated with with angular velocity ω, why is the velocity of an object, that is located d units away from the axis of rotation, equal to $v = d\omega$? Show that the kinetic energy of this object in rotational motion is $KE = \frac{1}{2}(d^2m)\omega^2$. The quantity $I = d^2m$ is called the moment of inertia. This problem only applies if you have a single point.

2. Suppose the point $P = (x, y, z)$, which has mass m, is attached to a negligible mass string. The point is rotated about the x-axis with angular velocity ω. Find the kinetic energy, using the results from the previous problem. [So what's the distance from (x, y, z) to the x-axis.]

3. We can think of a curve as thousands of points (x, y, z), each with mass $dm = \delta ds$. As we rotate an entire curve about the x-axis with angular velocity ω, each little piece contributes small amount of kinetic energy, which we'll call dKE. Explain why $dKE = \frac{1}{2}(y^2 + z^2)\omega^2 dm$.

4. Explain why the kinetic energy of the curve (when rotated about the x axis) is
$$KE = \frac{1}{2}\left(\int_C (y^2 + z^2)dm\right)\omega^2 = \frac{1}{2}I_x\omega^2.$$

5. If we rotated about the y-axis instead, how does this formula change?

Problem 8.27 A wire follows the helix $\vec{r}(t) = (3\cos t, 4t, 3\sin t)$ for $t \in [0, 4\pi]$. The density is $\delta(x, y, z) = x^2 + y + 2z^2$. Set up formulas to compute I_x, I_y, and I_z. Use software to compute the integrals. In your presentation, show us the set up you used, and then just give us the numerical solutions.

In problem 8.25, we showed how to find the center of mass by replacing the variable distance x in $\int_C xdm$ with the constant distance $\bar{x}$, and then solving for $\bar{x}$ in the equation $\int_C \bar{x}dm = \int_C xdm$. The idea is simple; if all the mass were located at one spot, what would that spot have to be for the moment of mass to be the same. The radii of gyration are obtained in the exact same manner. They can be thought of as a rotational center of mass.

Problem 8.28: Radii of Gyration Suppose a wire lies on the curve C and has density δ. The inertia about a line L we know is $I_x = \int_C d^2dm$, where d is the radius of rotation (distance to the line L). What constant radius R should we replace the variable radius d with so that $\int_C d^2dm = \int_C R^2dm$. Explain how to obtain the radii of gyration about the x axis.

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You only needed to show how to obtain the radius of gyration about the x axis. All three radii of gyration are found using the formulas

$$R_x = \sqrt{\frac{\int_C (y^2 + z^2)dm}{\int_C dm}}, \quad R_y = \sqrt{\frac{\int_C (x^2 + z^2)dm}{\int_C dm}}, \text{ and } R_z = \sqrt{\frac{\int_C (x^2 + y^2)dm}{\int_C dm}}.$$

For the remaining 2 problems, you are asked to review the key ideas in this section. You have to obtain a parametrization of the curve, and then just set up the appropriate integrals.

Problem 8.29 Consider the curve $y = 4-x^2$ for $x \in [-1, 2]$, with $\delta(x, y) = y$. Set up integral formulas which would give (1) the x coordinate $\bar{x}$ of the centroid, (2) the y coordinate $\bar{y}$ of the center of mass, (3) the moment of inertia I_x about the x-axis, and (4) the radius of gyration R_y about the y axis.

Problem 8.30 Consider a straight wire which lies on the line segment between $(-2, 1, 0)$ and $(0, -1, 2)$. The density of the wire is known to be $\delta(x, y, z) = x + y + z + 2$. Set up integral formulas which would give (1) the x coordinate $\bar{x}$ of the centroid, (2) the z coordinate $\bar{z}$ of the center of mass, (3) the moment of inertia I_y about the y-axis, and (4) the radius of gyration R_x about the x-axis.

8.5 The Fundamental Theorem of Line Integrals

In this final section we'll return to the concept of work. Many vector fields are actually the derivative of a function. When this occurs, computing work along a curve is extremely easy. All you have to know is the endpoints of the curve, and the function f whose derivative gives you the vector field. This function is called a potential for a vector field. Once we are comfortable finding potentials, we'll show that the work done by such a vector field is the difference in the potential at the end points. This makes finding work extremely fast.

Definition 8.7: Gradients and Potentials. Let $\vec{F}$ be a vector field. A potential for the vector field is a function f whose derivative equals $\vec{F}$. So if $Df = \vec{F}$, then we say that f is a potential for $\vec{F}$. When we want to emphasize that the derivative of f is a vector field, we call Df the gradient of f and write $Df = \vec{\nabla} f$. If $\vec{F}$ has a potential, then we say that $\vec{F}$ is a gradient field.

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The symbol $\vec{\nabla} f$ is read "the gradient of f" or "del f."

We'll quickly see that if a vector field has a potential, then the work done by the vector field is the difference in the potential. If you've ever dealt with kinetic and potential energy, then you hopefully recall that the change in kinetic energy is precisely the difference in potential energy. This is the reason we use the word "potential."

Problem 8.31 Let's practice finding gradients and potentials.

Watch a YouTube Video.

1. Let $f(x, y) = x^2 + 3xy + 2y^2$. Find the gradient of f, i.e. find $Df(x, y)$. Then compute $D^2 f(x, y)$ (you should get a square matrix). What are f_{xy} and f_{yx}?

2. Consider the vector field $\vec{F}(x, y) = (2x + y, x + 4y)$. Find the derivative of $\vec{F}(x, y)$ (it should be a square matrix). Then find a function $f(x, y)$ whose gradient is $\vec{F}$ (i.e. $Df = \vec{F}$). What are f_{xy} and f_{yx}?

3. Consider the vector field $\vec{F}(x, y) = (2x + y, 3x + 4y)$. Find the derivative of $\vec{F}$. Why is there no function $f(x, y)$ so that $Df(x, y) = \vec{F}(x, y)$? [Hint: what would f_{xy} and f_{yx} have to equal?]

See problem 6.7.

Based on your observations in the previous problem, we have the following key theorem.

Theorem 8.8. *Let $\vec{F}$ be a vector field that is everywhere continuously differentiable. Then $\vec{F}$ has a potential if and only if the derivative $D\vec{F}$ is a symmetric matrix. We say that a matrix is symmetric if interchanging the rows and columns results in the same matrix (so if you replace row 1 with column 1, and row 2 with column 2, etc., then you obtain the same matrix).*

Problem 8.32 For each of the following vector fields, find a potential, or explain why none exists.

If you haven't yet, please watch this YouTube video.

1. $\vec{F}(x, y) = (2x - y, 3x + 2y)$
2. $\vec{F}(x, y) = (2x + 4y, 4x + 3y)$
3. $\vec{F}(x, y) = (2x + 4xy, 2x^2 + y)$
4. $\vec{F}(x, y, z) = (x + 2y + 3z, 2x + 3y + 4z, 2x + 3y + 4z)$
5. $\vec{F}(x, y, z) = (x + 2y + 3z, 2x + 3y + 4z, 3x + 4y + 5z)$
6. $\vec{F}(x, y, z) = (x + yz, xz + z, xy + y)$
7. $\vec{F}(x, y) = \left(\dfrac{x}{1 + x^2} + \arctan(y), \dfrac{x}{1 + y^2}\right)$

If a vector field has a potential, then there is an extremely simple way to compute work. To see this, we must first review the fundamental theorem of calculus. The second half of the fundamental theorem of calculus states,

> If f is continuous on $[a, b]$ and F is an anti-derivative of f, then $F(b) - F(a) = \int_a^b f(x)dx$.

If we replace f with f', then an anti-derivative of f' is f, and we can write,

> If f is continuously differentiable on $[a, b]$, then $f(b) - f(a) = \int_a^b f'(x)dx$.

This last version is the version we now generalize.

Theorem 8.9 (The Fundamental Theorem of Line Integrals)**.** *Suppose f is a continuously differentiable function, defined along some open region containing the smooth curve C. Let $\vec{r}(t)$ be a parametrization of the curve C for $t \in [a, b]$. Then we have*

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$$f(\vec{r}(b)) - f(\vec{r}(a)) = \int_a^b Df(\vec{r}(t))D\vec{r}(t)\ dt.$$

Notice that if $\vec{F}$ is a vector field, and has a potential f, which means $\vec{F} = Df$, then we could rephrase this theorem as follows.

> Suppose $\vec{F}$ is a a vector field that is continuous along some open region containing the curve C. Suppose $\vec{F}$ has a potential f. Let A and B be the start and end points of the smooth curve C. Then the work done by $\vec{F}$ along C depends only on the start and end points, and is precisely
>
> $$f(B) - f(A) = \int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy.$$
>
> The word done by $\vec{F}$ is the difference in potential.

If you are familiar with kinetic energy, then you should notice a key idea here. Work is a transfer of energy. As an object falls, energy is transferred from potential energy to kinetic energy. The total kinetic energy at the end of a fall is precisely equal to the difference between the potential energy at the top of the fall and the potential energy at the bottom of the fall (neglecting air resistance). So work (the transfer of energy) is exactly the difference in potential energy.

Problem 8.33: Proof of Fundamental Theorem Suppose $f(x, y)$ is continuously differentiable, and suppose that $\vec{r}(t)$ for $t \in [a, b]$ is a parametrization of a smooth curve C. Prove that $f(\vec{r}(b)) - f(\vec{r}(a)) = \int_a^b Df(\vec{r}(t))D\vec{r}(t)\, dt$. [Let $g(t) = f(\vec{r}(t))$. Why does $g(b) - g(a) = \int_a^b g'(t)dt$? Use the chain rule (matrix form) to compute $g'(t)$. Then just substitute things back in.]

The proof of the fundamental theorem of line integrals is quite short. All you need is the fundamental theorem of calculus, together with the chain rule (6.10).

Problem 8.34 For each vector field and curve below, find the work done by $\vec{F}$ along C. In other words, compute the integral $\int_C M dx + N dy$ or $\int_C M dx + N dy + P dz$. [Hint: if you parametrize the curve, then you've done the problem the HARD way. You don't need any parameterizations.]

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1. Let $\vec{F}(x, y) = (2x + y, x + 4y)$ and C be the parabolic path $y = 9 - x^2$ for x from -3 to 2. See Sage.
2. Let $\vec{F}(x, y, z) = (2x + yz, 2z + xz, 2y + xy)$ and C be the straight segment from $(2, -5, 0)$ to $(1, 2, 3)$. See Sage.

Problem 8.35 Let $\vec{F} = (x, z, y)$. Let C_1 be the curve which starts at $(1, 0, 0)$ and follows a helical path $(\cos t, \sin t, t)$ to $(1, 0, 2\pi)$. Let C_2 be the curve which starts at $(1, 0, 2\pi)$ and follows a straight line path to $(2, 4, 3)$. Let C_3 be any smooth curve that starts at $(2, 4, 3)$ and ends at $(0, 1, 2)$.

See Sage—C_1 and C_2 are in blue, and several possible C_3 are shown in red.

- Find the work done by $\vec{F}$ along each path C_1, C_2, C_3.
- Find the work done by $\vec{F}$ along the path C which follows C_1, then C_2, then C_3.
- If C is any path that can be broken up into finitely many smooth sub-paths, and C starts at $(1, 0, 0)$ and ends at $(0, 1, 2)$, what is the work done by $\vec{F}$ along C?

If you are parameterizing the curves, you're doing this the really hard way. Are you using the potential of the vector field?

Definition 8.10. We say that a vector field is conservative if the integral $\int_C \vec{F} \cdot d\vec{r}$ does not depend on the path C. We say that a curve C is piecewise smooth if it can be broken up into finitely many smooth curves.

Problem 8.36 The gravitational vector field is directly related to the radial field $\vec{F} = \dfrac{(-x, -y, -z)}{(x^2 + y^2 + z^2)^{3/2}}$. Find a potential for $\vec{F}$, and then compute the work done by an object that moves from $(1, 2, -2)$ to $(0, -3, 4)$ along ANY path that avoids the origin.

Problem 8.37 Suppose $\vec{F}$ is a gradient field. Let C be a piecewise smooth closed curve. What is $\int_C \vec{F} \cdot d\vec{r}$? Explain.

Surface Area	$\sigma = \int_C d\sigma = \int_C f ds = \int_a^b f \left\lvert \frac{d\vec{r}}{dt} \right\rvert dt$
Average Value	$\bar{f} = \frac{\int f ds}{\int ds}$
Work, Flow, Circulation	$W = \int_C d\text{Work} = \int_C (\vec{F} \cdot \vec{T}) ds = \int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy$
Flux	$\text{Flux} = \int_C d\text{Flux} = \int_C \vec{F} \cdot \vec{n} ds = \oint_C M dy - N dx$
Mass	$m = \int_C dm = \int_C \delta ds$
Centroid	$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\int x ds}{\int_C ds}, \frac{\int y ds}{\int_C ds}, \frac{\int z ds}{\int_C ds} \right)$
Center of Mass	$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\int x dm}{\int_C dm}, \frac{\int y dm}{\int_C dm}, \frac{\int z dm}{\int_C dm} \right)$
(Second) Moment of Inertia	$I_x = \int (y^2 + z^2) dm$, $I_y = \int (x^2 + z^2) dm$, $I_z = \int (x^2 + y^2) dm$
Radius of Gyration	$R = \sqrt{I/m}$
Fund. Thm of Line Int.	$f(B) - f(A) = \int_C \vec{\nabla} f \cdot d\vec{r}$

Table 8.1: A summary of the ideas in this unit.

8.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to Brainhoney and download the quiz. Once you have taken the quiz, you can upload your work back to brainhoney and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 9

Optimization

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Explain the properties of the gradient, its relation to level curves and level surfaces, and how it can be used to find directional derivatives.
2. Find equations of tangent planes using the gradient and level surfaces. Use the derivative (tangent planes) to approximate functions, and use this in real world application problems.
3. Explain the second derivative test in terms of eigenvalues. Use the second derivative test to optimize functions of several variables.
4. Use Lagrange multipliers to optimize a function subject to constraints.

You'll have a chance to teach your examples to your peers prior to the exam.

9.1 The Gradient

Recall from the previous unit that the derivative Df of a function $f : \mathbb{R}^n \to \mathbb{R}$ (one output dimension) is called the gradient of f, and written $\vec{\nabla} f$, when we want to emphasize that the derivative is a vector field.

Problem 9.1 Consider the functions $f(x, y) = 9 - x^2 - y^2$, $g(x, y) = 2x - y$, and $h(x, y) = \sin x \cos y$.

You'll want a computer to help you construct the graphs, particularly h. Please use the Mathematica introduction in Brainhoney. You could use Wolfram Alpha (use the links in the function chapter if you forgot how to graph).

See Sage. You can modify these commands to help in the plots below too.

1. Compute $\vec{\nabla} f(x, y)$. Then draw both $\vec{\nabla} f$ and several level curves of f on the same axes.
2. Compute $\vec{\nabla} g(x, y)$. Then draw both $\vec{\nabla} g$ and several level curves of g on the same axes.
3. Compute $\vec{\nabla} h(x, y)$. Then draw both $\vec{\nabla} h$ and several level curves of h on the same axes.
4. What relationships do you see between the gradient vector field and level curves?

When you present in class, be prepared to provide rough sketches of the level curves and gradients of each function.

The next few problems will focus on explaining why the relationships you saw are always true.

Problem 9.2 Suppose $\vec{r}(t)$ is a level curve of $f(x, y)$.

1. Suppose you know that at $t = 0$, the value of f at $\vec{r}(0)$ is 7. What is the value of f at $\vec{r}(1)$? [What does it mean to be on a level curve?]
2. As you move along the level curve $\vec{r}$, how much does f change? Use this to tell the class what $\frac{df}{dt}$ must equal.
3. At points along the level curve $\vec{r}$, we have the composite function $f(\vec{r}(t))$. Compute the derivative $\frac{df}{dt}$ using the chain rule.
4. Use your work from the previous parts to explain why the gradient always meets the level curve at a 90° angle. We say that the gradient is *normal* to level curves (i.e., a gradient vector is orthogonal to the tangent vector of the curve).

In Section 6.3, we extended differential notation from $dy = f'dx$ to $d\vec{y} = D\vec{f}d\vec{x}$. The key idea is that a small change in the output variables is approximated by the product of the derivative and a small change in the input variables. As a quick refresher, if we have the function $z = f(x, y)$, then differential notation states that

$$dz = \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}.$$

Problem 9.3 Suppose the temperature at a point in the plane is given by the function $T(x, y) = x^2 - xy - y^2$ degrees Fahrenheit. A particle is at $P = (2, 3)$.

1. Use differentials to estimate the change in temperature if the particle moves 1 unit in the direction of $\vec{u} = (3, 4)$. [Hint: Find a unit vector in that direction.]
2. What is the actual change in temperature if the particle moves 1 unit in the direction of $\vec{u} = (3, 4)$?
3. Use differentials to estimate the change in temperature if the particle moves about .2 units in the direction of $\vec{u} = (3, 4)$.

We can define partial derivatives solely in terms of differential notation. We can define derivatives in any direction in terms of differential notation.

Problem 9.4 Suppose that $z = f(x, y)$ is a differentiable function (so the derivative is the matrix $\begin{bmatrix} f_x & f_y \end{bmatrix}$). Remember to use differential notation in this problem.

1. If $(dx, dy) = (1, 0)$, which means we've moved one unit in the x direction while holding y constant, what is dz?
2. If $(dx, dy) = (0, 1)$, which means we've moved one unit in the y direction while holding x constant, what is dz?
3. Consider the direction $\vec{u} = (2, 3)$. Find a unit vector in the direction of $\vec{u}$. If we move one unit in the direction of $\vec{u}$, what is dz? [It's all right to leave you answer as a dot product.]

Definition 9.1. The directional derivative of f in the direction of the unit vector $\vec{u}$ at a point P is defined to be

$$D_{\vec{u}}f(P) = Df(P)\vec{u} = \vec{\nabla} f \cdot \vec{u}.$$

We dot the gradient of f with the direction vector $\vec{u}$. The partial derivative of f with respect to x is precisely the directional derivative of f in the $(1, 0)$ direction. Similarly, the partial derivative of f with respect to y is precisely the directional derivative of f in the $(0, 1)$ direction. This definition extends to higher dimensions.

Note that in the definition above, we require the vector $\vec{u}$ to be a unit vector. If you are asked to find a directional derivative in some direction, make sure you start by finding a unit vector in that direction. We want to deal with unit vectors because when we say something has a slope of m units, we want to say "The function rises m units if we run 1 unit."

Problem 9.5 Consider the function $f(x, y) = 9 - x^2 - y^2$.

1. Draw several level curves of f.
2. At the point $P = (2, 1)$, place a dot on your graph. Then draw a unit vector based at P that points in the direction $\vec{u} = (3, 4)$ [not to the point $(3, 4)$, but in the direction $\vec{u} = (3, 4)$]. If you were to move in the direction $(3, 4)$, starting from the point $(2, 1)$, would the value of f increase or decrease?
3. Find the slope of f at $P = (2, 1)$ in the direction $\vec{u} = (3, 4)$ by finding the directional derivative. This should agree with your previous answer.
4. If you stand at $Q = (-2, 3)$ and move in the direction $\vec{v} = (1, -1)$, will f increase or decrease? Find the directional derivative of f in the direction $\vec{v} = (1, -1)$ at the point $Q = (-2, 3)$.

Problem 9.6 Recall that the directional derivative of $\vec{f}$ in the direction $\vec{u}$ is the dot product $\vec{\nabla} f \cdot \vec{u} = |\vec{\nabla} f||\vec{u}| \cos\theta$. In this problem, you'll explain why the gradient points in the direction of greatest increase.

1. Why is the directional derivative of $\vec{f}$ the largest when $\vec{u}$ points in the exact same direction as $\vec{\nabla} f$? [Hint: What angle maximizes the cosine function?]
2. When $\vec{u}$ points in the same direction as $\vec{\nabla} f$, show that $D_{\vec{u}}f = |\vec{\nabla} f|$. In other words, explain why the length of the gradient is precisely the slope of f in the direction of greatest increase (the slope in the steepest direction).
3. Which direction points in the direction of greatest decrease?

Problem 9.7 Suppose you are looking at a topographical map (see Wikipedia for an example). On this topographical map, each contour line represents 100 ft in elevation. You notice in one section of the map that the contour lines are really close together, and they start to form circles around a spot on the graph. You notice in another section of the map that the contour lines are spaced quite far apart. Let $f(x, y)$ be the elevation of the land, so that the topographical map is just a contour plot of f.

1. Where is the slope of the terrain larger, in the section with closely packed contour lines, or the section with contour lines that are spread out. In which section will the gradient be a longer vector?

2. At the very top of a mountain, or the very bottom of a valley, will the gradient be a long vector or a small vector? How do you locate a peak in a topographical map?

3. Create you own topographical map to illustrate the ideas above. Just make sure your map has a section with some contours that are closely packed together, and some that are far apart, as well as a contour that intersects itself. Then on your topographical map, please add a few gradient vectors, where you emphasize which ones are long, and which ones are short. Show us how to find a peak, as well as what the gradient vector would be at the peak.

If you're stuck, look at a contour plot of $f(x,y) = (x+1)^3 - 3(x+1)^2 - y^2 + 2$ in Sage. Then make your own example.

Theorem 9.2. *Let f be a continuously differentiable function, with $\vec{r}$ a level curve of the function.*

- *The gradient is always normal to level curves, meaning $\vec{\nabla} f \cdot \dfrac{d\vec{r}}{dt} = 0$.*
- *The gradient points in the direction of greatest increase.*
- *The directional derivative of f in the direction of the gradient is the length of the gradient. Symbolically, we write $D_{\vec{\nabla} f} f = |\vec{\nabla} f|$.*
- *At a maximum or minimum, the gradient is the zero vector.*

The next few problems have you practice using differentials, and then obtain tangent lines and planes to curves and surfaces using differentials.

Problem 9.8 The volume of a cylindrical can is $V(r,h) = \pi r^2 h$. Any manufacturing process has imperfections, and so building a cylindrical can with designed dimensions (r,h) will result in a can with dimensions $(r + dr, h + dh)$.

1. Compute both DV (the derivative of V) and dV (the differential of V).

2. If the can is tall and slender (h is big, r is small), which will cause a larger change in volume: an error in r or an error in h? Use dV to explain your answer.

3. If the can is short and wide (like a tuna can), which will cause a larger change in volume: an error in r or an error in h? Use dV to explain your answer.

Problem 9.9 Consider the function $f(x,y) = x^2 + y^2$. Consider the level curve C given by $f(x,y) = 25$. Our goal is to find an equation of the tangent line to C at $P = (3,-4)$.

1. Draw C. Compute $\vec{\nabla} f$ and add to your graph the vector $\vec{\nabla} f(P)$.

2. We know the point $P = (3,-4)$ is on the tangent line. Let $Q = (x,y)$ represent another point on the tangent line. Add to your graph the point Q and the vector $\vec{PQ} = (x-3, y+4)$.

3. Why are $\vec{\nabla} f(P)$ and $\vec{PQ}$ orthogonal? Use this fact to write an equation of the tangent line.

4. What is a normal vector to the line?

The previous problem had you give an equation of the tangent line to a level curve, by using differential notation. The next problems asks you to repeat this idea and give an equation of a tangent plane to a level surface.

Problem 9.10 Consider the function $f(x, y, z) = x^2 + y^2 + z^2$. Consider the level surface S given by $f(x, y, z) = 9$. Our goal is to find an equation of the tangent plane to S at $P = (1, 2, -2)$.

1. Draw S.
2. Compute $\vec{\nabla} f$. Add to your graph the vector $\vec{\nabla} f(P)$, with its base at P.
3. We know the point $P = (1, 2, -2)$ is on the tangent plane. Let $Q = (x, y, z)$ be any other point on the tangent plane. What is the component form of the vector $\vec{PQ}$?
4. Why are $\vec{\nabla} f(P)$ and $\vec{PQ}$ orthogonal? Use this fact to write an equation of the tangent plane.
5. What is a normal vector to the plane?

Problem 9.11 Find an equation of the tangent plane to the hyperboloid of one sheet $1 = x^2 - y^2 + z^2$ at the point $(-3, 3, 1)$.

Problem 9.12 The two surfaces $x^2 + y^2 + z^2 = 14$ and $3x + 4y - z = -1$ intersect in a curve C. Draw both surfaces, and show us the curve C. Then, at the point $(2, -1, 3)$, find an equation of the tangent line to this curve. [Hint: The line is in both tangent planes, so it is orthogonal to both normal vectors. The cross product gets you a vector that is orthogonal to two vectors.]

9.2 The Second Derivative Test

We start with a review problems from first-semester calculus.

Problem 9.13 Let $f(x) = x^3 - 3x^2$. Find the critical values of f by solving $f'(x) = 0$. Determine if each critical value leads to a local maximum or local minimum by computing the second derivative. State the local maxima/minima of f. Sketch the function using the information you discovered.

We now generalize the second derivative test to all dimensions. We've already seen that the second derivative of a function such as $z = f(x, y)$ is a square matrix. The second derivative test relied on understanding if a function was concave up or concave down. We need a way to examine the concavity of f as we approach a point (x, y) from any of the infinitely many directions. Such a method exists, and leads to an eigenvalue/eigenvector problem. I'm assuming that most of you have never heard the word "eigenvalue." We could spend an entire semester just studying eigenvectors. We'd need a few weeks to discover what they are from a problem-based approach. Instead, here is an example of how to find eigenvalues and eigenvectors.

Definition 9.3. Let A be a square matrix, so in 2D we have $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The identity matrix I is a square matrix with 1's on the diagonal and zeros everywhere else, so in 2D we have $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The eigenvalues of A are the solutions λ to the equation $|A - \lambda I| = 0$. Remember that $|A|$ means, "Compute the determinant of A." So in 2D, we need to find the value λ so that

$$\left|\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0.$$

This definition extends to any square matrix. In 3D, the eigenvalues are the solutions to the equation

$$\left|\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = \begin{vmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{vmatrix} = 0.$$

An eigenvector of A corresponding to λ is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda x$.

As you continue taking more upper level science courses (in physics, engineering, mathematics, chemistry, and more) you'll soon see that eigenvalues and eigenvectors play a huge role. You'll start to see them in most of your classes. For now, we'll use them without proof to apply the second derivative test. In class, make sure you ask me to show you pictures with each problem we do, so we can see how eigenvalues and eigenvectors appear in surfaces.

Theorem 9.4 (The Second Derivative Test)**.** *Let $f(x, y)$ be a function so that all the second partial derivatives exist and are continuous. The second derivative of f, written D^2f and sometimes called the Hessian of f, is a square matrix. Let λ_1 be the largest eigenvalue of D^2f, and λ_2 be the smallest eigenvalue. Then λ_1 is the largest possible second derivative obtained in any direction. Similarly, the smallest possible second derivative obtained in any direction is λ_2. The eigenvectors give the directions in which these extreme second derivatives are obtained. The second derivative test states the following.*

Suppose (a, b) is a critical point of f, meaning $Df(a, b) = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

- *If all the eigenvalues of $D^2f(a, b)$ are positive, then in every direction the function is concave upwards at (a, b) which means the function has a local minimum at (a, b).*
- *If all the eigenvalues of $D^2f(a, b)$ are negative, then in every direction the function is concave downwards at (a, b). This means the function has a local maximum at (a, b).*
- *If the smallest eigenvalue of $D^2f(a, b)$ is negative, and the largest eigenvalue of $D^2f(a, b)$ is positive, then in one direction the function is concave upwards, and in another the function is concave downwards. The point (a, b) is called a saddle point.*
- *If the largest or smallest eigenvalue of f equals 0, then the second derivative tests yields no information.*

Example 9.5. Consider the function $f(x, y) = x^2 - 2x + xy + y^2$. The first and second derivatives are

$$Df(x, y) = \begin{bmatrix} 2x - 2 + y, x + 2y \end{bmatrix} \quad \text{and} \quad D^2f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The first derivative is zero (the zero matrix) when both $2x - 2 + y = 0$ and $x + 2y = 0$. We need to solve the system of equations $2x + y = 2$ and $x + 2y = 0$. Double the second equation, and then subtract it from the first to obtain $0x - 3y = 2$, or $y = -2/3$. The second equation says that $x = -2y$, or that $x = 4/3$. So the only critical point is $(4/3, -2/3)$.

We find the eigenvalues of $D^2 f(4/3, -2/3)$ by solving the equation

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 1 = 0.$$

In this example, the second derivative is constant, so the point $(4/3, -2/3)$ did not change the matrix. In general, the point will affect your matrix. See Sage to see a graph which shows the eigenvectors in which the largest and smallest second derivatives occur.

Expanding the left hand side gives us $4 - 4\lambda + \lambda^2 - 1 = 0$. Simplifying and factoring gives us $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$. This means the eigenvalues are $\lambda = 1$ and $\lambda = 3$. Since both numbers are positive, the function is concave upwards in every direction. The critical point $(4/3, -2/3)$ corresponds to a local minimum of the function. The local minimum is the output $f(4/3, -2/3) = (4/3)^2 - 2(4/3) + (4/3)(-2/3) + (-2/3)^2$.

A graph of f is provided on the right. The red vector $(1, 1)$ points in the direction in which the second derivative is the largest value 3. The red vector $(-1, 1)$ points in the direction in which the second derivative is the smallest value 1. These vectors are called eigenvectors, and you can learn much more about them, in particular how to find them, in a linear algebra course. For this course, we just need to be able to find eigenvalues.

Problem 9.14 Consider the function $f(x, y) = x^2 + 4xy + y^2$.

See 14.7 for more practice.

1. Find the critical points of f by finding when $Df(x, y)$ is the zero matrix.
2. Find the eigenvalues of $D^2 f$ at any critical points.
3. Label each critical point as a local maximum, local minimum, or saddle point, and state the value of f at the critical point.

Problem 9.15 Consider the function $f(x, y) = x^3 - 3x + y^2 - 4y$.

1. Find the critical points of f by finding when $Df(x, y)$ is the zero matrix.
2. Find the eigenvalues of $D^2 f$ at any critical points. [Hint: First compute $D^2 f$. Since there are two critical points, evaluate the second derivative at each point to obtain 2 different matrices. Then find the eigenvalues of each matrix.]
3. Label each critical point as a local maximum, local minimum, or saddle point, and state the value of f at the critical point.

Problem 9.16 Consider the function $f(x, y) = x^3 + 3xy + y^3$.

1. Find the critical points of f by finding when $Df(x, y)$ is the zero matrix.
2. Find the eigenvalues of $D^2 f$ at any critical points.
3. Label each critical point as a local maximum, local minimum, or saddle point, and state the value of f at the critical point.

You now have the tools needed to find optimal solutions to problems in any dimension. Here's a silly problem that demonstrates how we can use what we've just learned.

Problem 9.17 For my daughter's birthday, she has asked for a Barbie princess cake. I purchased a metal pan that's roughly in the shape of a paraboloid $z = f(x, y) = 9 - x^2 - y^2$ for $z \geq 0$. To surprise her, I want to hide a present inside the cake. The present is a bunch of small candy that can pretty much fill a box of any size. I'd like to know how large (biggest volume) of a rectangular box I can fit under the cake, so that when we start cutting the cake, she'll find her surprise present. The box will start at $z = 0$ and the corners of the box (located at $(x, \pm y)$ and $(-x, \pm y)$) will touch the surface of the cake $z = 9 - x^2 - y^2$.

1. What is the function $V(x, y)$ that we are trying to maximize?
2. If you find all the critical points of V, you'll discover there are 9. However, only one of these critical points makes sense in the context of this problem. Find that critical point.
3. Use the second derivative test to prove that the critical point yields a maximum volume.
4. What are the dimensions of the box? What's the volume of the box?

The only thing left for me is to now determine how much candy I should buy to fill the box. I'll take care of that.

In this problem, we'll derive the version of the second derivative test that is found in most multivariate calculus texts. The test given below only works for functions of the form $f : \mathbb{R}^2 \to \mathbb{R}$. The eigenvalue test you have been practicing will work with a function of the form $f : \mathbb{R}^n \to \mathbb{R}$, for any natural number n.

Problem 9.18: Optional Suppose that $f(x, y)$ has a critical point at (a, b).

1. Find a general formula for the eigenvalues of $D^2 f(a, b)$. Your answer will be in terms of the second partials of f.
2. Let $D = f_{xx} f_{yy} - f_{xy}^2$.
   - If $D < 0$, explain why f has a saddle point at (a, b).
   - If $D = 0$, explain why the second derivative test fails.
   - If $D > 0$, explain why f has either a maximum or minimum at (a, b).
   - If $D > 0$, and $f_x(a, b) > 0$, does f have a local max or local min at (a, b). Explain.
3. The only critical point of $f(x, y) = x^2 + 3xy + 2y^2$ is at $(0, 0)$. Does this point correspond to a local maximum, local minimum, or saddle point? Give the eigenvalues (which should come instantly out of part 1). Find D, from part 2, to answer the question.

9.3 Lagrange Multipliers

The last problem was an example of an optimization problem where we wish to optimize a function (the volume of a box) subject to a constraint (the box has to fit inside a cake). If you are economics student, this section may be the key reason why you were asked to take multivariate calculus. In the business world, we often want to optimize something (profit, revenue, cost, utility, etc.) subject to some constraint (a limited budget, a demand curve, warehouse space, employee hours, etc.). An aerospace engineer will build the best wing that can withstand given forces. Everywhere in the engineering world, we often seek to create the "best" thing possible, subject to some outside constraints. Lagrange discovered an extremely useful method for answering this question, and today we call it "Lagrange Multipliers."

Rather than introduce Cobb-Douglass production functions (from economics) or sheer-stress calculations (from engineering), we'll work with simple examples that illustrate the key points. Sometimes silly examples carry the message across just as well.

Problem 9.19 Suppose an ant walks around the circle $g(x, y) = x^2 + y^2 = 1$. As the ant walks around the circle, the temperature is $f(x, y) = x^2 + y + 4$. Our goal is to find the maximum and minimum temperatures reached by the ant as it walks around the circle. We want to optimize $f(x, y)$ subject to the constraint $g(x, y) = 1$.

1. Draw the circle $g(x, y) = 1$. Then, on the same set of axes, draw several level curves of f. The level curves $f = 3, 4, 5, 6$ are a good start. Then add more (maybe at each 1/4th). If you make a careful, accurate graph, it will help a lot below.
2. Based solely on your graph, where does the minimum temperature occur? What is the minimum temperature?
3. If the ant is at the point $(0, 1)$, and it moves left, will the temperature rise or fall? What if the ant moves right?
4. On your graph, place a dot(s) where you believe the ant reaches a maximum temperature (it may occur at more than one spot). Explain why you believe this is the spot where the maximum temperature occurs. What about the level curves tells you that these spots should be a maximum.
5. Draw the gradient of f at the places where the minimum and maximum temperatures occur. Also draw the gradient of g at these spots. How are the gradients of f and g related at these spots?

Theorem 9.6 (Lagrange Multipliers)**.** *Suppose f and g are continuously differentiable functions. Suppose that we want to find the maximum and minimum values of f subject to the constraint $g(x, y) = c$ (where c is some constant). Then if a maximum or minimum occurs, it must occur at a spot where the gradient of f and the gradient of g point in the same, or opposite, directions. So the gradient of g must be a multiple of the gradient of f. To find the maximum and minimum values (if they exist), we just solve the system of equations that result from*

$$\vec{\nabla} f = \lambda \vec{\nabla} g, \quad \textit{and} \quad g(x, y) = c$$

where λ is the proportionality constant. The maximum and minimum values will be among the solutions of this system of equations.

Problem 9.20 Suppose an ant walks around the circle $x^2 + y^2 = 1$. As the ant walks around the circle, the temperature is $T(x, y) = x^2 + y + 4$. Our goal is to find the maximum and minimum temperatures T reached by the ant as it walks around the circle.

1. What function $f(x, y)$ do we wish to optimize? What is the constraint $g(x, y) = c$?

2. Find the gradient of f and the gradient of g. Then solve the system of equations that you get from the equations

$$\vec{\nabla} f = \lambda \vec{\nabla} g, \qquad x^2 + y^2 = 1.$$

You should obtain 4 ordered pairs (x, y).

3. At each ordered pair, find the temperature. What is the maximum temperature obtained? What is the minimum temperature obtained.

The most common error on this problem is to divide both sides of an equation by x, which could be zero. If you do this, you'll only get 2 ordered pairs.

Problem 9.21 Consider the curve $xy^2 = 54$ (draw it). The distance from each point on this curve to the origin is a function that must have a minimum value. Find a point (a, b) on the curve that is closest to the origin.

[The distance to the origin is $d(x, y) = \sqrt{x^2 + y^2}$. This distance is minimized when $f(x, y) = x^2 + y^2$ is minimized. So use $f(x, y) = x^2 + y^2$ as the function you wish to minimize. What's the constraint $g(x, y) = c$?]

See 14.8 for more practice.

Problem 9.22 Find the dimensions of the rectangular box with maximum volume that can be inscribed inside the ellipsoid

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{5^2} = 1.$$

[What is the function f you wish to optimize? What is the constraint $g = c$? Try solving each equation for λ so you can eliminate it from the problem.]

Problem 9.23 Repeat problem 9.17, but this time use Lagrange multipliers. Find the dimensions of the rectangular box of maximum volume that fits underneath the surface $z = f(x, y) = 9 - x^2 - y^2$ for $z \geq 0$.

[Hint: Let $f(x, y, z) = (2x)(2x)(z)$ and $g(x, y, z) = z + x^2 + y^2 = 9$. You'll get a system of 4 equations with 4 unknowns (x, y, z, λ). Try solving each equation for lambda. You know x, y, z can't be zero or negative, so ignore those possible cases.]

9.4 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to Brainhoney and download the quiz. Once you have taken the quiz, you can upload your work back to brainhoney and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 10

Double Integrals

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Explain how to setup and compute a double integral. Show how to interchange the bounds of integration.
2. For planar regions, find area, mass, centroids, center of mass, moments of inertia, and radii of gyration.
3. Explain how to change coordinate systems in integration, in particular to polar coordinates. Explain what the Jacobian is, and show how to use it.
4. Explain how to use Green's theorem to compute flow along and flux across a curve.

You'll have a chance to teach your examples to your peers prior to the exam.

10.1 Double Integrals and Applications

Before we introduce integration, let's practice using inequalities to describe regions in the plane. In first semester calculus, we often use the inequalities $a \leq x \leq b$ and $g(x) \leq y \leq f(x)$ to describe the region above g below f for x between a and b. We trapped x between two constants, and y between two functions. Sometimes we wrote $c \leq y \leq d$ where $g(y) \leq x \leq f(y)$ to describe the region to the right of g and left of f for y between c and d. We need to practice writing inequalities in this form, as these inequalities will provide us the bounds of integration for double integrals.

Problem 10.1 Consider the region R in the xy-plane that is below the line $y = x + 2$, above the line $y = 2$, and left of the line $x = 5$. We can describe this region by saying for each x with $0 \leq x \leq 5$, we want y to satisfy $2 \leq y \leq x + 2$. In set builder notation, we write

$$R = \{(x, y) \mid 0 \leq x \leq 5, 2 \leq y \leq x + 2\}.$$

1. Describe the region R by saying for each y with $c \leq y \leq d$, we want x to satisfy $a(y) \leq x \leq b(y)$. In other words, find constants c and d, and functions $a(y)$ and $b(y)$, so that for each y between c and d, the x values must be between the functions $a(y)$ and $b(y)$.

2. Write your last answer in the set builder notation

$$R = \{(x, y) \mid c \le y \le d, a(y) \le x \le b(y)\}.$$

Problem 10.2 For each region R below, draw the region and give a set of inequalities of the form $a \le x \le b, c(x) \le y \le d(x)$ or $c < y < d, a(y) \le x \le b(y)$. In class, we'll give whichever one you did not.

1. The region R is above the line $x + y = 1$ and inside the circle $x^2 + y^2 = 1$.
2. The region R is below the line $y = 8$, above the curve $y = x^2$, and to the right of the y-axis.
3. The region R bounded by $2x + y = 3$, $y = x$, and $x = 0$.

We now introduce double integrals. Just as single integrals gave us the area under a function over an interval, double integrals will give us the volume under a function, above a region in the plane. We'll introduce double integrals by looking at cross sections of a solid.

Problem 10.3 Consider the solid domain D in space that is beneath the surface $f(x, y) = 9 - x^2 - y^2$ and above the xy-plane, where the x values satisfy $x \ge 0$. The region is half of a parabolic solid. Our goal in this problem is to find the volume of the solid D.

See Sage.

1. Draw the solid D.
2. The plane $y = 0$ intersects the solid, resulting in a planar region above the xy plane and below the parabola $z = 9 - x^2$. The plane $y = 2$ intersects the solid in a region below the parabola $z = 9 - x^2 - (2)^2$. The plane $y = y_i$ intersects the solid in the parabola $y = 9 - x^2 - y_i^2$ for each y_i between -3 and 3. When we slice the solid along the plane $y = y_i$, we obtain a cross section of the surface. Explain why the area of each of these cross sections is $\displaystyle\int_0^{\sqrt{9-y_i^2}} (9 - x^2 - y_i^2)dx$.
3. Imagine now that you cut the surface into 6 pieces, using the plane $y = y_i$ for each y_i in $\{-3, -2, -1, 0, 1, 2, 3\}$. Let $y_0 = -3$, $y_1 = -2$, …, $y_6 = 3$. The change in y between each point is $\Delta y = 1$. In the plane $y = y_i$, we know the area under the surface is $\int_0^{\sqrt{9-y_i^2}} (9 - x^2 - y_i^2)dx$. If we multiply this area by the thickness $\Delta y = 1$, we obtain the volume of a solid (think $dV = (A)dy$). Draw these solids corresponding to $y_3 = 0$ and $y_4 = 1$ in your picture. Then explain why an approximation to the volume of the entire solid D is
$$\sum_{i=1}^{6} \left(\int_0^{\sqrt{9-y_i^2}} (9 - x^2 - y_i^2)dx \right) \Delta y.$$
4. Explain why the volume of D equals $\displaystyle\int_{-3}^{3} \left(\int_0^{\sqrt{9-y^2}} (9 - x^2 - y^2)dx \right) dy$.

The integral above is called an iterated integral because you first compute the inside integral and then you compute the outside integral (you iteratively integrate). Often the parenthesis are not written because we know that the inside integral should be performed first without writing the parenthesis. We could also explicitly emphasize which variables go with each bound by writing

$$\int_{y=-3}^{y=3} \left(\int_{x=0}^{x=\sqrt{9-y^2}} 9 - x^2 - y^2 dx \right) dy.$$

Problem 10.4 The bounds of the integral $\int_{-3}^{3} \left(\int_{0}^{\sqrt{9-y^2}} (9 - x^2 - y^2) dx \right) dy$ describe a region R in the plane, namely $-3 \leq y \leq 3$ and $0 \leq x \leq \sqrt{9-y^2}$. Draw this region R in the plane. Then give bounds to describe the region alternately by first stating constants which trap x (so $a \leq x \leq b$) and then functions which trap y (so $c(x) \leq y \leq d(x)$). Use these new bounds to write an iterated integral

$$\int_{x=a}^{x=b} \left(\int_{y=c(x)}^{y=d(x)} 9 - x^2 - y^2 dx \right) dy$$

that gives the exact same volume of the solid D from the previous problem.

Ask me in class to show you how the answer to the previous problem could have been obtained by considering cross sections of the original solid

Let R be some region in the plane. If we let $dA = dxdy = dydx$, then we can write a little bit of volume as $dV = fdA = fdxdy = fdydx$. Adding up little bits of volume gives us the double integral

$$V = \iint_R f dA,$$

which equals either iterated integral we've been setting up above.

Problem 10.5 Consider the iterated integral $\int_0^3 \int_x^3 e^{y^2} dydx$. Write the bounds as two inequalities ($0 \leq x \leq 3$ and $? \leq y \leq ?$). Draw and shade the region R described by these two inequalities. Then swap the order of integration by reversing the order of your inequalities (so trap y between 2 constants and x between 2 functions). Finally, compute the new integral by hand (you'll need a u-substitution).

Problem 10.6 Consider the region R in the plane that is trapped between the curves $y = 2x$ and $y = x^2$. We would like to compute $\iint_R x dA$ over this region R. Set up both iterated integrals. Then compute one of them.

In the line integral chapter, we introduce the ideas of average value, centroid, center of mass, moment of inertia, and radius of gyration. We now extend those ideas to regions in the plane, in exactly the same way. For example, the average value formula in the line integral section was $\bar{f} = \dfrac{\int_C f dx}{\int_C ds}$. For double integrals, we just change ds to dA, and add an integral. This gives the formula

$\bar{f} = \dfrac{\iint_R f dA}{\iint_R dA}$. The same substitution works on all the integrals from before. We now have $dm = \delta dA$ instead of $dm = \delta ds$. We obtained arc length by computing $s = \int_C ds$ (add up little bits of arc length). We can compute area by using $A = \iint_R dA$ (add up little bits of area).

Problem 10.7 Consider the rectangular region R in the xy-plane described by $\{(x, y) \mid 2 \leq x \leq 11, 3 \leq y \leq 7\}$.

1. Set up an integral formula which would give $\bar{y}$ for the centroid of R. Then evaluate the integral.
2. State $\bar{x}$ from geometric reasoning.
3. Set up an integral to give the moment of inertia about the y-axis if $\delta = 5$. Note that $z = 0$ in the xy-plane.
4. Set up an integral to give the R_x if the density is $\delta(x, y) = xy^2$.

Problem 10.8 Consider the region in the plane that is bounded by the curves $x = y^2 - 3$ and $x = y - 1$. A metal plate occupies this region in space, and its temperature function on the plate is give by the function $T(x, y) = 2x + y$. Find the average temperature of the metal plate.

Problem 10.9 Consider the region R that is the circular disc which is inside the circle $(x - 2)^2 + (y + 1)^2 = 9$. The centroid is clearly $(2, -1)$, and the area is $A = \pi(3)^2 = 9\pi$. We can use these fact to simplify many integrals that require integrating over the region R.

1. Compute $\iint_R 3dA = 3 \iint_R dA$. [How can area help you?]
2. Explain why $\iint_R xdA = \bar{x}A$ for any region R, and then compute $\iint_R xdA$ for the circular disc. [You don't need to set up any integrals at all.]
3. Compute the integral $\iint_R 5x + 2ydA$ by using centroid and area facts.

Problem 10.10 Consider the region R in the xy-plane that is formed from two rectangular regions. The first region R_1 satisfies $x \in [-2, 2]$ and $y \in [0, 7]$. The second region R_2 satisfies $x \in [-5, 5]$ and $y \in [7, 10]$. Find the centroids of R_1, R_2 and then finally R.

Problem 10.11 Let R be the region in the plane with $a \leq x \leq b$ and $g(x) \leq y \leq f(x)$. Let A be the area of R.

1. Set up an iterated integral to compute the area of R. Then compute the inside integral. You should obtain a familiar formula from first-semester calculus.
2. Set up an iterated integral formula to compute $\bar{x}$ for the centroid. By computing the inside integral, show why $\bar{x} = \dfrac{1}{A}\displaystyle\int_a^b x(f - g)dx$.
3. If the density depends only on x, so $\delta = \delta(x)$, set up an iterated integral formula to compute $\bar{y}$ for the center of mass. Explain why

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}(f^2 - g^2)\delta(x)dx.$$

When you use double integrals to find centroids, the formulas for the centroid are the same for both $\bar{x}$ and $\bar{y}$. In other courses, you may see the formulas on the left, because the ideas will be presented without requiring knowledge of double integrals. Integrating the inside integral from the double integral formula gives the single variable formulas.

10.2 Switching Coordinates: The Jacobian

We now want to explore how to perform u-substitution in high dimensions. Let's start with a review from first semester calculus.

Problem 10.12 Consider the integral $\int_{-1}^{4} e^{-3x}dx$.

1. Let $u = -3x$. Solve for x and then compute dx.
2. Explain why $\int_{-1}^{4} e^{-3x}dx = \int_{3}^{-12} e^{u}\left(-\frac{1}{3}\right)du$.
3. Explain why $\int_{-1}^{4} e^{-3x}dx = \int_{-12}^{3} e^{u}\left|-\frac{1}{3}\right|du$.
4. If the u-values are between -3 and 2, what would the x-values be between? How does the length of the u interval $[-3, 2]$ relate to the length of the corresponding x interval?

In the problem above, we used a change of coordinates $u = -3x$, or $x = -1/3u$. By taking derivatives, we found that $dx = -\frac{1}{3}du$. The negative means that the orientation of the interval was reversed. The fraction $\frac{1}{3}$ tells us that lengths dx using x coordinates will be 1/3rd as long as lengths du using u coordinates. When we write $dx = \frac{dx}{du}du$, the number $\frac{dx}{du}$ is called the Jacobian of x with respect to u. The Jacobian tells us how lengths are altered when we change coordinate systems. We now generalize this to polar coordinates. Before we're done with this section, we'll generalize the Jacobian to any change of coordinates.

Theorem 10.1. *If we use the polar coordinate transformation $x = r\cos\theta, y = r\sin\theta$, then we can switch from (x, y) coordinates to (r, θ) coordinates if we use*

$$dxdy = |r|drd\theta.$$

The number $|r|$ is called the Jacobian of x and y with respect to r and θ. If we require all bounds for r to be nonnegative, we can ignore the absolute value. If R_{xy} is a region in the xy plane that corresponds to the region $R_{r\theta}$ in the $r\theta$ plane (where $r \geq 0$), then we can write

$$\iint_{R_{xy}} f(x, y)dxdy = \iint_{R_{r\theta}} f(r\cos\theta, r\sin\theta)r\ drd\theta.$$

We'll prove later why the Jacobian is $|r|$. For now, we need some practice using this idea. We start by describing regions using inequalities on r and θ. Ask me in class to give you an informal picture approach that explains why $dxdy = rdrd\theta$.

Problem 10.13 For each region R below, draw the region in the xy-plane. Then give a set of inequalities of the form $a \leq r \leq b, \alpha(r) \leq \theta \leq \beta(r)$ or $\alpha < \theta < \beta, a(\theta) \leq r \leq b(\theta)$. For example, if the region is the inside of the circle $x^2 + y^2 = 9$, then we could write $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 3$.

1. The region R is the quarter circle in the first quadrant inside the circle $x^2 + y^2 = 25$.
2. The region R is below $y = \sqrt{9 - x^2}$, above $y = x$, and to the right of $x = 0$.

3. The region R is the triangular region below $y = \sqrt{3}x$, above the x-axis, and to the left of $x = 1$.

Problem 10.14 Consider the opening problem for this unit. We want to find the volume under $f(x, y) = 9 - x^2 - y^2$ where $x \geq 0$ and $z \geq 0$. We obtained the integral formula

See Sage.

$$\iint_R f dA = \int_{y=-3}^{y=3} \int_{x=0}^{x=\sqrt{9-y^2}} 9 - x^2 - y^2 dx dy.$$

1. Write bounds for the region R by giving bounds for r and θ.
2. Rewrite the double integral as an iterated integral with bounds for r and θ. Don't forget the Jacobian (as $dxdy = rdrd\theta$).
3. Compute the integral in the previous part by hand. [Suggestion: you'll want to simplify $9 - x^2 - y^2$ to $9 - r^2$ before integrating.]

Problem 10.15 Find the centroid of a semicircular disc of radius a ($y \geq 0$). Actually compute any integrals.

After doing this, in class we'll set up the integral formulas needed to find R_y, the radius of gyration about the y-axis, assuming the density is $\delta(x, y) = x^2 + y^2$.

Problem 10.16 Compute the integral $\displaystyle\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1 + x^2 + y^2)^2} dy dx$. [Hint: try switching coordinate systems.]

We're now ready to define the Jacobian of any transformation.

Definition 10.2. Suppose $\vec{T}(u, v) = (x(u, v), y(u, v))$ is a differentiable coordinate transformation. To find the Jacobian of this transformation, we first find the derivative of $\vec{T}$. This is a square matrix, so it has a determinant, which should give us information about area. As the determinant may be positive or negative, we then take the absolute value to obtain the Jacobian. So the Jacobian of the transformation $\vec{T}$ is the absolute value of the determinant of the derivative. Notationally we write

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = |\det(D\vec{T}(u, v))|.$$

For a tongue twister, say "the absolute value of the determinant of the derivative" ten times really fast.

Problem 10.17 Find the Jacobian of the polar coordinate transformation $x = r \cos\theta$ and $y = r \sin\theta$ (so $\vec{T}(r, \theta) = (r \cos\theta, r \sin\theta)$).

Problem 10.18 Consider the transformation $u = x + 2y$ and $v = 2x - y$.

1. Solve for x and y in terms of u and v. Then compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.
2. We were give u and v in terms of x and y, so we could have directly computed $\frac{\partial(u,v)}{\partial(x,y)}$. Do so now.

3. Make a conjecture about the relationship between $\frac{\partial(x,y)}{\partial(u,v)}$ and $\frac{\partial(u,v)}{\partial(x,y)}$.

Theorem 10.3. *Suppose that f is integrable over a region R_{xy} in the xy plane. Suppose that $\vec{T}(u,v) = (x(u,v), y(u,v))$ is a coordinate transformation that has the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$. Suppose the region R_{uv} in the uv-plane corresponds to the region R_{xy} in the xy-plane. Provided the Jacobian is nonzero except possibly on regions with zero area, we can then write*

$$\iint_{R_{xy}} f(x,y)dxdy = \iint_{R_{uv}} f(x(u,v), y(u,v))\frac{\partial(x,y)}{\partial(u,v)}dudv.$$

We can remember this in differential form as

$$dxdy = \frac{\partial(x,y)}{\partial(u,v)}dudv.$$

Let's use this to rapidly find the area inside of an ellipse.

Problem 10.19 Consider the region R inside the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. We'll consider the change of coordinates given by $u = (x/a)$ and $v = (y/b)$.

1. Draw the region R in the xy-plane. After substituting $u = x/a$ and $v = y/b$, draw the region R_{uv} in the uv-plane. You should have a circle. What is the area inside this circle in the uv-plane?

2. Solve for x and y, and then compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$. Show how to get the same result from directly computing $\dfrac{\partial(u,v)}{\partial(x,y)}$.

3. We know the area in the xy-plane of the ellipse is $\iint_{R_{xy}} dxdy$. Use the previous theorem to switch to an integral over the region R_{uv}. Then evaluate this integral by using facts about area so prove that the area in the xy plane is πab. [Hint: you don't actually have to set up any bounds, rather just reduce this to an area integral over the region R_{uv}.]

Problem 10.20 Let R be the region in the plane bounded by the curves $x + 2y = 1$, $x + 2y = 4$, $2x - y = 0$, and $2x - y = 8$. We want to compute the integral $\iint_R x dxdy$. Draw the region R in the xy-plane. Use the change of coordinates $u = x + 2y$ and $v = 2x - y$ to evaluate this integral. Make sure you provide a sketch of the region R_{uv} in the uv-plane (it should be a rectangle). [Hints: what are the bounds for u and v? You'll want to solve for x and y in terms of u and v, and then you'll need a Jacobian.]

Problem 10.21 Use the transformation $u = 3x + 2y$ and $v = x + 4y$ to evaluate the integral

This is problem 7 in section 15.8.

$$\iint_R (3x^2 + 14xy + 8y^2)dxdy = \iint_R (3x + 2y)(x + 4y)dxdy$$

for the region R that is bounded by the lines $y = -(3/2)x + 1$, $y = -(3/2)x + 3$, $y = -(1/4)x$, and $y = (-1/4)x + 1$.

10.3 Green's Theorem

Now that we have double integrals, it's time to make some of our circulation and flux problems from the line integral section get extremely simple. We'll start by defining the circulation density and flux density for a vector field $\vec{F}(x,y) = \langle M, N\rangle$ in the plane.

Definition 10.4: Circulation Density and Flux Density (Divergence). Let $\vec{F}(x,y) = \langle M, N\rangle$ be a continuously differentiable vector field. At the point (x,y) in the plane, create a circle C_a of radius a centered at (x,y), where the area inside of C_a is $A_a = \pi a^2$. The quotient $\frac{1}{A_a}\oint_{C_a} \vec{F}\cdot\vec{T}ds$ is a circulation per area. The quotient $\frac{1}{A_a}\oint_{C_a} \vec{F}\cdot\vec{n}ds$ is a flux per area.

- The circulation density of $\vec{F}$ at (x,y) we define to be
$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = N_x - M_y = \lim_{a\to 0}\frac{1}{A_a}\oint_{C_a}\vec{F}\cdot d\vec{r} = \lim_{a\to 0}\frac{1}{A_a}\oint_{C_a} Mdx + Ndy.$$

We will not prove that the partial derivative expressions $N_x - M_y$ and $M_x + N_y$ are actually equal to the limits given here. That is best left to an advanced course.

- The divergence, or flux density, of $\vec{F}$ at (x,y) we define to be
$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = M_x + N_y = \lim_{a\to 0}\frac{1}{A_a}\oint_{C_a}\vec{F}\cdot\vec{n}ds = \lim_{a\to 0}\frac{1}{A_a}\oint_{C_a} Mdy - Ndx.$$

In the definitions above, we could have replaced the circle C_a with a square of side lengths a centered at (x,y) with interior area A_a. Alternately, we could have chosen any collection of curves C_a which "shrink nicely" to (x,y) and have area A_a inside. Regardless of which curves you chose, it can be shown that

$$N_x - M_y = \lim_{a\to 0}\frac{1}{A_a}\oint_{C_a}\vec{F}\cdot\vec{T}ds \quad\text{and}\quad M_x + N_y = \lim_{a\to 0}\frac{1}{A_a}\oint_{C_a}\vec{F}\cdot\vec{n}ds.$$

To understand what the circulation and flux density mean in a physical sense, think of $\vec{F}$ as the velocity field of some gas.

- The circulation density tells us the rate at which the vector field $\vec{F}$ causes objects to rotate around points. If circulation density is positive, then particles near (x,y) would tend to circulate around the point in a counterclockwise direction. The larger the circulation density, the faster the rotation. The velocity field of a gas could have some regions where the gas is swirling clockwise, and some regions where the gas is swirling counterclockwise.
- The divergence, or flux density, tells us the rate at which the vector field causes object to either flee from (x,y) or come towards (x,y). For the velocity field of a gas, the gas is expanding at points where the divergence is positive, and contracting at points where the divergence is negative.

We are now ready to state Green's Theorem. Ask me in class to give an informal proof as to why this theorem is valid.

Theorem 10.5 (Green's Theorem)**.** *Let $\vec{F}(x,y) = (M,N)$ be a continuously differentiable vector field, which is defined on an open region in the plane that contains a simple closed curve C and the region R inside the curve C. Then we can compute the counterclockwise circulation of $\vec{F}$ along C, and the outward flux of $\vec{F}$ across C by using the double integrals*

$$\oint_C \vec{F}\cdot\vec{T}ds = \iint_R N_x - M_y dA \quad\textit{and}\quad \oint_C \vec{F}\cdot\vec{n}ds = \iint_R M_x + N_y dA.$$

Let's now use this theorem to rapidly find circulation (work) and flux.

Problem 10.22 Consider the vector field $\vec{F} = (2x + 3y, 4x + 5y)$. Start by computing $N_x - M_y$ and $M_x + N_y$. If C is the boundary of the rectangle $2 \le x \le 7$ and $0 \le y \le 3$, find both the circulation and flux of $\vec{F}$ along C. You should be able to reduce the integrals to facts about area. [If you tried doing this without Green's theorem, you would have to parametrize 4 line segments, compute 4 integrals, and then sum the results.]

See 16.4 for more practice. Try doing a bunch of these, as they get really fast.

Problem 10.23 Consider the vector field $\vec{F} = (x^2 + y^2, 3x + 5y)$. Start by computing $N_x - M_y$ and $M_x + N_y$. If C is the circle $(x - 3)^2 + (y + 1)^2 = 4$ (oriented counterclockwise), then find both the circulation and flux of $\vec{F}$ along C. You should be able to reduce the integrals to facts about the area and centroid.

Problem 10.24 Repeat the previous problem, but change the curve C to the boundary of the triangular region R with vertexes at $(0, 0)$, $(3, 0)$, and $(3, 6)$. You can complete this problem without having to set up the bounds on any integrals, if you reduce the integrals to facts about area and centroids. You are welcome to look up the centroid of a triangular region without computing it.

10.4 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to Brainhoney and download the quiz. Once you have taken the quiz, you can upload your work back to brainhoney and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 11

Surface Integrals

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Explain how to setup surface integrals, and use them to compute surface area, average value, centroids, center of mass, moments of inertia, and radii of gyration.

2. Use surface integrals to compute flux across a surface, in a given direction.

3. Explain how to use Stokes's theorem to compute circulation.

You'll have a chance to teach your examples to your peers prior to the exam.

11.1 Surface Area and Surface Integrals

In first-semester calculus, we learned how to compute integrals $\int_a^b f dx$ along straight (flat) segments $[a, b]$. This semester, in the line integral unit, we learned how to change the segment to a curve, which allowed us to compute integrals $\int_C f ds$ along any curve C, instead of just along curves (segments) on the x-axis. The integral $\int_a^b dx = b - a$ gives the length of the segment $[a, b]$. The integral $\int_C ds$ gives the length s of the curve C.

In the double integral unit we learned how to compute double integrals $\iint_R f dA$ along flat regions R in the plane. We'll now learn how to change the flat region R into a curved surface S, and then compute integrals of the form $\iint_S f d\sigma$ along curved surfaces. The differential $d\sigma$ stands for a little bit of surface area. We already know that $\iint_R dA$ gives the area of R. We'll define $\iint_S d\sigma$ so that it gives the surface area of S.

Problem 11.1 Consider the surface S given by $z = 9 - x^2 - y^2$ (we've seen this surface many times). A parametrization of this surface is

$$\vec{r}(x, y) = (x, y, 9 - x^2 - y^2).$$

1. Draw the surface S. Add to your surface plot the parabolas given by $\vec{r}(x, 0)$, $\vec{r}(x, 1)$, and $\vec{r}(x, 2)$, as well as the parabolas given by $\vec{r}(0, y)$, $\vec{r}(1, y)$, and $\vec{r}(2, y)$. You should have an upside down paraboloid, with at least 6 different parabolas drawn on the surface. These parabolas should divide the surface up into a bunch of different patches. Our goal is to find the area of each patch, where each patch is almost like a parallelogram.

2. Find $\frac{\partial \vec{r}}{\partial x}$ and $\frac{\partial \vec{r}}{\partial y}$. At the point $(2,1)$, draw both vectors. These vectors form the edges of a parallelogram. Add that parallelogram to your picture.

3. Show that the area of a parallelogram whose edges are the vectors $\frac{\partial \vec{r}}{\partial x}(x,y)$ and $\frac{\partial \vec{r}}{\partial y}$ is $\sqrt{1+4x^2+4y^2}$. [Hint: think about the cross product.]

4. Find the area of the parallelogram whose edges are the vectors $\frac{\partial \vec{r}}{\partial x}dx$ and $\frac{\partial \vec{r}}{\partial y}dy$, where dx and dy are to be determined.

In the previous problem, you showed that the area of the parallelogram with edges given by $\frac{\partial \vec{r}}{\partial x}dx$ and $\frac{\partial \vec{r}}{\partial y}dy$ is

$$d\sigma = \left|\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}\right| dxdy.$$

This little bit of area approximates the area of a tiny patch on the surface. If we add all these areas up, we should obtain the surface area.

Definition 11.1. Let S be a surface. Let $\vec{r}(u,v) = (x,y,z)$ be a parametrization of the surface, where the bounds on u and v form a region R in the uv plane. Then the surface area element (representing a little bit of surface) is

$$d\sigma = \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dudv = |\vec{r}_u \times \vec{r}_v|\, dudv.$$

The surface integral of a continuous function $f(x,y,z)$ along the surface S is

$$\iint_S f(x,y,z)d\sigma = \iint_R f(\vec{r}(u,v)) \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dudv.$$

If we let $f = 1$, then the surface area of S is simply

$$\sigma = \iint_S d\sigma = \iint_R \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dudv.$$

This definition tells us how to compute any surface integral. The steps are almost identical to the line integral steps.

1. Start by getting a parametrization $\vec{r}$ of the surface S where the bounds form a region R.

2. Find a little bit of surface area by computing $d\sigma = \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dudv$.

3. Multiply f by $d\sigma$, and replace each x, y, z with what they equals from the parametrization.

4. Integrate the previous function along R, your parameterization's bounds.

Problem 11.2 Consider the surface S given by $z = 9 - x^2 - y^2$, for $z \geq 0$. A parametrization of this surface is

$$\vec{r}(x,y) = (x, y, 9 - x^2 - y^2), \quad \text{where } 9 - x^2 - y^2 \geq 0.$$

1. Give a set of inequalities for x and y that describe the region R over which we need to integrate. The inequalities you give should be in a form that you can use them as the bounds of a double integral.

2. Find $d\sigma = |\vec{r}_x \times \vec{r}_y|\, dxdy$.
3. Set up the surface integral $\iint_S d\sigma$ as an iterated double integral over R.
4. Convert the integral above to an integral in polar coordinates (don't forget the Jacobian).

Problem 11.3 Consider the surface S given by $z = 9 - x^2 - y^2$, for $z \geq 0$. A different parametrization of this surface is

$$\vec{r}(r, \theta) = (r\cos\theta, r\sin\theta, 9 - r^2), \quad \text{where } 9 - r^2 \geq 0.$$

1. Give a set of inequalities for r and θ that describe the region $R_{r\theta}$ over which we need to integrate. The inequalities you give should be in a form that you can use them as the bounds of a double integral.
2. Find $d\sigma = |\vec{r}_r \times \vec{r}_\theta|\, drd\theta$.
3. Set up the surface integral $\iint_S d\sigma$ as an iterated double integral over $R_{r\theta}$.

Problem 11.4 Find, actually compute, the surface area of the surface S given by $z = 9 - x^2 - y^2$, for $z \geq 0$. Do this by computing any of the integrals from the previous two problems.

Problem 11.5 If a surface S is parametrized by $\vec{r}(x, y) = (x, y, f(x, y))$, show that $d\sigma = \sqrt{1 + f_x^2 + f_y^2}\; dxdy$ (compute a cross product). If $\vec{r}(x, z) = (x, f(x, z), z)$, what does $d\sigma$ equal (compute a cross product - you should see a pattern)? Use the pattern you've discovered to quickly compute $d\sigma$ for the surface $x = 4 - y^2 - z^2$, and then set up an iterated double integral that would give the surface area of S for $x \geq 0$.

Problem 11.6 Consider the sphere $x^2 + y^2 + z^2 = a^2$. We'll find $d\sigma$ using two different parameterizations.

1. If you use the rectangular parametrization $\vec{r}(x, y) = (x, y, \sqrt{a^2 - x^2 - y^2})$, what is $d\sigma$? [Hint, use the previous problem.] Why can this parametrization only be use if the surface has positive z-values?
2. If you use the spherical parametrization

   You'll want to memorize this result.

   $$\vec{r}(\phi, \theta) = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi),$$

   show that

   $$d\sigma = (a^2|\sin\phi|)d\phi d\theta = (a^2\sin\phi)d\phi d\theta,$$

   where we can ignore the absolute values if we require $0 \leq \phi \leq \pi$. Along the way, you'll show that

   $$\vec{r}_\phi \times \vec{r}_\theta = a^2\sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

We can compute average value, centroids, center of mass, moments of inertia, and radii of gyration as before. We just replace dA with $d\sigma$, and all the formulas are the same.

Problem 11.7 Consider the hemisphere $x^2 + y^2 + z^2 = a^2$ for $z \geq 0$.

1. Set up a formula that would give $\bar{z}$ for the centroid of the hemisphere.
2. Compute the two integrals in your formula. By doubling the bottom integral, you'll have also shown that the surface area of a sphere of radius a is $\sigma = 4\pi a^2$.
3. Set up an integral formula for R_z, the radius of gyration about the z axis, provided the density is constant.

11.1.1 Flux across a surface

We now want to look at the flux of a vector field across a surface S. In the line integral section, we defined the outward flux of a vector field F across a curve C to be the line integral $\int_C \vec{F} \cdot \vec{n} ds$, where $\vec{n}$ is a normal vector point out of region enclosed by a curve C. When we want to find the flux of a vector field across a surface, we must state in which direction we want to compute the flux. We then must make sure that normal vector $\vec{n}$ we choose to use actually points in the desired direction. The flux of a vector field $\vec{F}$ across a surface S is the surface integral

$$\text{Flux} = \Phi = \iint_S \vec{F} \cdot \vec{n} d\sigma.$$

The next problem will help us simplify the computation of $\vec{n} d\sigma$.

Problem 11.8 Consider again the surface $z = 9 - x^2 - y^2$.

1. Using the parametrization $\vec{r}(x, y) = (x, y, 9 - x^2 - y^2)$, find a unit normal vector $\vec{n}$ to the surface so that $\vec{n}$ points upwards away from the z-axis. State what $d\sigma$ equals, as well as $\vec{n} d\sigma$. Make sure you explain how you know the normal vector you give is pointing upwards away from the z axis.
2. Using the parametrization $\vec{r}(r, \theta) = (r \cos\theta, r \sin\theta, 9 - r^2)$, find a unit normal vector $\vec{n}$ to the surface so that $\vec{n}$ points downwards towards the z-axis. State what $d\sigma$ equals, as well as $\vec{n} d\sigma$. Make sure you explain how you know the normal vector you give is pointing downwards towards the z axis.

[For both parts above, the computations involved were actually done in previous problems. You just need to compile the information here.]

In the problem above, we showed that $\vec{n} d\sigma = \pm(\vec{r}_x \times \vec{r}_y) dxdy$ and that $\vec{n} d\sigma = \pm(\vec{r}_r \times \vec{r}_\theta) dr d\theta$. We no longer need to find the magnitude of the cross product, but we must determine the correct sign to put on our cross product. This shows us that we can write flux as

$$\text{Flux} = \Phi = \iint_S \vec{F} \cdot \vec{n} d\sigma = \iint_{R_{uv}} \vec{F} \cdot (\pm \vec{r}_u \times \vec{r}_v) du dv.$$

Problem 11.9 Consider the cone $z^2 = x^2 + y^2$ and vector field $\vec{F} = (2x + 3y, x - 2x, yz)$. Set up an iterated integral that would give the flux of $\vec{F}$ outwards (away from the z-axis) for the portion of the cone between $z = 1$ and $z = 3$. [Hint: Start by parameterizing the cone by using a polar parametrization

$$x = r\cos\theta, y = r\sin\theta, z = ?.$$

You should obtain bounds for r and θ that are constants. Compute the normal vector and look at the third component to determine if it points up or down. Then just plug everything into the formula.]

When the surface is flat, often you can determine the normal vector without having to perform any cross products. We'll now compute a flux of a vector field outwards across the 6 faces of a cube.

Problem 11.10 Find the flux of $\vec{F} = (x+y, y, z)$ outward across the surface of the cube in the first quadrant bounded by $x = 2, y = 3, z = 5$. The cube has 6 surfaces, so we have to compute the flux across all 6 surfaces. Fill in the table below to complete the flux across each surface, and then compute each integral to find the total flux.

Surface	$\vec{r}(u,v)$	$\vec{n}$	$\vec{F}(\vec{r}(u,v))$	$\vec{F}\cdot\vec{n}$	Flux
Back $x = 0$	$\langle 0, y, z\rangle$	$\langle -1, 0, 0\rangle$	$\vec{F}(0, y, z) = \langle y, y, z\rangle$	$-y$	$\iint_{\text{Back}} -y d\sigma = -\bar{y}\sigma = -(\frac{3}{2})(15)$
Front $x = 2$	$\langle 2, y, z\rangle$		$\vec{F}(2, y, z) = \langle 2+y, y, z\rangle$		
Left $y = 0$					0 (Why?)
Right $y = 3$	$\langle x, 3, z\rangle$	$\langle 0, 1, 0\rangle$	$\vec{F}(x, 3, z) = \langle x+3, 3, z\rangle$	3	30 (Why?)
Bottom $z = 0$					
Top $z = 3$					

You should be able to complete each integral by considering centroids and surface area of each of the 6 different flat surfaces. Show that the total flux is 90.

In the double integral chapter, we learned a way to greatly simplify flux computations when working with simple closed curves. Green's theorem stated that $\int_C \vec{F}\cdot\vec{n}\ ds = \iint_R M_x + N_y dA$. The divergence of $\vec{F}$ is the quantity $\text{div}(\vec{F}) = M_x + N_y$. This generalizes to higher dimensions, and is called the divergence theorem. The next problem illustrates how. We'll study this more in the triple integral unit.

Problem 11.11 Consider the exact same vector field and box as the previous problem. So we have the vector field $\vec{F} = (x+y, y, z)$ and S is the surface of the cube in the first quadrant bounded by $x = 2, y = 3, z = 5$.

1. Compute the divergence of $\vec{F}$, which is $\text{div}(\vec{F}) = M_x + N_y + P_z$.
2. The divergence theorem states that if S is a closed surface (has an inside and an outside), and the inside of the surface is the solid domain D, then the flux of $\vec{F}$ outward across S equals the triple integral

$$\iint_S \vec{F}\cdot\vec{n}\ d\sigma = \iint\int_D \text{div}(\vec{F})dV.$$

Use the divergence theorem to compute the flux of $\vec{F}$ across S. [Hint: Just as the area is found by adding up little bits of area, which is what we mean by $A = \iint_R dA$, the volume is found by adding up little bits of volume.]

Problem 11.12 In problem 11.6, we found

$$\vec{n}d\sigma = \vec{r}_\phi \times \vec{r}_\theta d\phi d\theta = a^2 \sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)d\phi d\theta$$

for a sphere of radius a. Use this to compute the outward flux of

$$\vec{F} = \frac{\langle -x, -y, -z\rangle}{(x^2+y^2+z^2)^{-3/2}}$$

across a sphere of radius a. You should get a negative number since the vector field has all arrows pointing in. [Hint: Remember that for a sphere of radius a we have $a^2 = x^2+y^2+z^2$. When you perform the dot product of $\vec{F}$ and $\vec{n}$, you'll save yourself a lot of time if you remember that $\vec{u}\cdot\vec{u} = |\vec{u}|^2$; the dot product of a vector with itself is the length squared.]

Problem 11.13 Repeat the previous problem, but this time don't use the formula from problem 11.6. In fact, you don't even need to parametrize the surface. Instead, if you are at the point (x, y, z) on a sphere of radius a, give a formula for the outward pointing unit normal vector $\vec{n}$. Give this formula by only using a geometric argument. Then find the outward flux of $\vec{F} = \frac{\langle -x, -y, -z\rangle}{(x^2+y^2+z^2)^{-3/2}}$ across a sphere of radius a. You should find that $\vec{F}\cdot\vec{n}$ simplifies to a constant, so that you never actually have to compute $d\sigma$. Then you can use known facts about the surface area of a sphere.

11.2 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to Brainhoney and download the quiz. Once you have taken the quiz, you can upload your work back to brainhoney and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 12

Triple Integrals

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Explain how to setup and compute triple integrals, as well as how to interchange the bounds of integration. Use these ideas to find area and volume.
2. Explain how to change coordinate systems in integration, with an emphasis on cylindrical, and spherical coordinates. Explain what the Jacobian of a transformation is, and how to use it.
3. Use triple integrals to find physical quantities such as center of mass, radii of gyration, etc. for solid regions.
4. Explain how to use the Divergence theorem to compute the flux of a vector fields out of a closed surface.

You'll have a chance to teach your examples to your peers prior to the exam.

12.1 Triple Integral Definition and Applications

Problem 12.1 Consider the iterated integral

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{0}^{9-x^2-y^2} dzdxdy.$$

This is an integral of the form $\iiint_D dV$, which means along some solid region D in the plane, we are adding up little bits of volume. This integral should give the volume of some solid region in space. Sketch the region D in space. Compute the inside integral, and compare this to the first problem in the double integral unit. Then evaluate the remaining integrals (though you might want to change coordinate systems before doing so).

When working with double integrals, there were two different ways to set up the bounds for our integrals, as $dA = dxdy = dydx$. When working with triple integrals, there are six different ways to set up the bounds for our integrals, as

$$dV = dxdydz = dxdzdy = dydxdz = dydzdx = dzdxdy = dzdydx.$$

Problem 12.2 Consider again the iterated integral

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{0}^{9-x^2-y^2} dzdxdy.$$

There are 5 other iterated integrals that are equal to this integral, by switching the order of the bounds. One of the integrals is $\int_0^9\int_0^{\sqrt{9-z}}\int_{-\sqrt{9-x^2-z}}^{\sqrt{9-x^2-z}} dydxdz$. Set up the equivalent integrals using the bound $dydzdx$ and $dxdzdy$. We'll look at the remaining 2 in class (though you're welcome to finish them and present them with your work).

Problem 12.3 Consider the iterated integral

$$\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{y} dzdydx.$$

The bounds for this integral describe a region in space which satisfies the 3 inequalities $-1 \leq x \leq 1$, $0 \leq y \leq 1-x^2$, and $0 \leq z \leq y$.

1. Draw the solid domain D in space described by the bounds of the iterated integral.
2. There are 5 other iterated integrals equivalent to this one. Set up the integrals that use the bounds $dydxdz$ and $dxdzdy$. We'll create the other 3 in class (though you are welcome to include them as part of your presentation).

Problem 12.4 In each problem below, you'll be given enough information to determine a solid domain D in space. Draw the solid D and then set up an iterated integral (pick any order you want) that would give the volume of D. You don't need to evaluate the integral, rather you just need to set them up.

1. The region D under the surface $z = y^2$, above the xy-plane, and bounded by the planes $y = -1$, $y = 1$, $x = 0$, and $x = 4$.
2. The region D in the first octant that is bounded by the coordinate planes, the plane $y + z = 2$, and the surface $x = 4 - y^2$.
3. The pyramid D in the first octant that is below the planes $\frac{x}{3} + \frac{z}{2} = 1$ and $\frac{y}{5} + \frac{z}{2} = 1$. [Hint, don't let z be the inside bound.]
4. The region D that is inside both right circular cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$.

We can find average value, centroids, centers of mass, moments of inertia, and radii of gyration exactly as before, We just now need to integrate using three integrals, and replace ds, dA or $d\sigma$, with dV.

Problem 12.5 Consider the triangular wedge D that is in the first octant, bounded by the planes $\frac{y}{7} + \frac{z}{5} = 1$ and $x = 12$. In the yz plane, the wedge forms a triangle that passes through the points $(0, 0, 0)$, $(0, 7, 0)$, and $(0, 0, 5)$. Set up integral formulas that would give the centroid $(\bar{x}, \bar{y}, \bar{z})$ of D. Actually compute the integrals for $\bar{y}$. Then state $\bar{x}$ and $\bar{z}$ by using symmetry arguments.

Problem 12.6 Consider the tetrahedron D in the first octant that is underneath the plane that intersects the coordinate axes in the three point $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, where you can assume that $a, b, c > 0$.

1. An equation of an ellipse that passes through $(a, 0)$ and $(0, b)$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$. An equation of a line through these same two points is $\frac{x}{a} + \frac{y}{b} = 1$. An equation of an ellipsoid through the three points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Guess an equation of the plane through these same three points, and then verify that your guess is correct by plugging the 3 points into your equation. This will provide you with an extremely fast way to get an equation of a plane.
2. Set up an iterated integral that would give the volume of D.
3. If the density is $\delta(x, y, z) = 3x + 2yz$, set up iterated integrals that would give the mass m and moment of inertia I_y about the y-axis.

12.2 Changing Coordinate Systems: The Jacobian

Just as we did with polar coordinates in two dimensions, we can compute a Jacobian for any change of coordinates in three dimensions. We will focus on cylindrical and spherical coordinate systems. Remember that the Jacobian of a transformation is found by first taking the derivative of the transformation, then finding the determinant, and finally computing the absolute value.

Problem 12.7 The cylindrical change of coordinates is

$$x = r\cos\theta, y = r\sin\theta, z = z, \text{ or in vector form } \vec{C}(r, \theta, z) = (r\cos\theta, r\sin\theta, z).$$

The spherical change of coordinates is

$$x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi, \quad \text{or in vector form}$$
$$\vec{S}(\rho, \phi, \theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi).$$

1. Verify that the Jacobian of the cylindrical transformation is $\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = |r|$. If you want to make sure you don't have to use absolute values, what must you require?
2. The Jacobian of the spherical transformation is $\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = |\rho^2 \sin\phi|$. If you want to make sure you don't have to use absolute values, what must you require?

The previous problem shows us that we can write

$$dV = dxdydz = rdrd\theta dz = \rho^2 \sin\phi d\rho d\phi d\theta,$$

provided we require $r \geq 0$ and $0 \leq \phi \leq \pi$. Cylindrical coordinates are extremely useful for problems which involve cylinders, paraboloids, and cones. Problems which involve cones and spheres often have simple integrals in spherical coordinates.

Problem 12.8 The double cone $z^2 = x^2 + y^2$ has two halves. Each half is called a nappe. Set up an integral in the coordinate system of your choice that would give the volume of the region that is between the xy plane and the upper nappe of the double cone $z^2 = x^2 + y^2$, and between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$. Then evaluate the integral.

See Sage.

Problem 12.9 Set up an integral in the coordinate system of your choice that would give the volume of the solid ball that is inside the sphere $a^2 = x^2 + y^2 + z^2$. Compute the integral to give a formula for the volume of a sphere of radius a. Then set up (don't evaluate) an iterated integral that would give the moment of inertia I_x about the x-axis, if the density is a constant, so $\delta = c$.

Problem 12.10 Find the volume of the solid domain D in space which is above the cone $z = \sqrt{x^2 + y^2}$ and below the paraboloid $z = 6 - x^2 - y^2$. Use cylindrical coordinates to set up and then evaluate your integral. You'll need to find where the surface intersect, as their intersection will help you determine the appropriate bounds.

See Sage.

Problem 12.11 Consider the region D in space that is inside both the sphere $x^2 + y^2 + z^2 = 9$ and the cylinder $x^2 + y^2 = 4$. Start by drawing the region.

1. Set up an iterated integral in Cartesian (rectangular) coordinates that would give the volume of D.
2. Set up an iterated integral in cylindrical coordinates that would give the volume of D.

Problem 12.12 Consider the region D in space that is both inside the sphere $x^2 + y^2 + z^2 = 9$ and yet outside the cylinder $x^2 + y^2 = 4$. Start by drawing the region.

1. Set up two iterated integrals in cylindrical coordinates that would give the volume of D. For one integral use the order $dzdrd\theta$. For the other, use the order $d\theta drdz$.
2. Set up an iterated integral in spherical coordinates that would give the volume of D.

Problem 12.13 The integral $\int_0^{\pi}\int_0^1\int_{\sqrt{3}r}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$ represents the volume of solid domain D in space. Set up integrals in both rectangular coordinates and spherical coordinates that would give the volume of the exact same region.

Problem 12.14 The temperature at each point in space of a solid occupying the region D, which is the upper portion of the ball of radius 4 centered at the origin, is given by $T(x, y, z) = \sin(xy + z)$. Set up an iterated integral formula that would give the average temperature.

12.3 The Divergence Theorem

In definition 10.4 on page 104, we defined the divergence, or flux density, of a vector field $\vec{F}$ at a point P to be the flux per unit area, and then stated that $\text{div}(\vec{F}) = M_x + N_y$. We now extend this to 3D.

In 3D, the flux of $\vec{F}$ across S, $\iint_S \vec{F}\cdot\vec{n}d\sigma$, is a measure of flow across S where $\vec{n}$ is a continuous unit normal vector to S. Flux density at (x, y, z) is found by creating a sphere S_a of radius a centered at (x, y, z) with interior volume V_a and outward normal vector $\vec{n}$, and considering the quotient of flux per volume given by $\frac{1}{V_a}\iint_{S_a} \vec{F}\cdot\vec{n}d\sigma$. By computing $\displaystyle\lim_{a\to 0}\frac{1}{V_a}\iint_{S_a} \vec{F}\cdot\vec{n}d\sigma$, we obtain the divergence of $\vec{F}$ at (x, y, z), also called the flux density. In a future mathematics course, we could prove that the divergence equals

$$\begin{aligned}\text{div}\vec{F}(x,y,z) = \vec{\nabla}\cdot\vec{F} &= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(M,N,P)\\ &= \frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z} = M_x+N_y+P_z.\end{aligned}$$

Theorem 12.1 (Divergence Theorem)**.** *Let S be a closed surface whose interior is the solid domain D. Let $\vec{n}$ be an outward pointing unit normal vector to S. Suppose that $\vec{F}(x, y, z)$ is a continuously differentiable vector field on some open region that contains D. Then the outward flux of $\vec{F}$ across S can be computed by adding up, along the entire solid D, the flux per unit volume (divergence). Symbolically, the divergence theorem states*

$$\iint_S \vec{F}\cdot\vec{n}d\sigma = \iiint_D \vec{\nabla}\cdot\vec{F}dV = \iiint_D (M_x+N_y+P_z)\,dV$$

for S a closed surface with interior D and outward normal $\vec{n}$.

Problem 12.15 Let S be the surface of the wedge in the first octant bounded by the planes $x = 1$ and $\frac{y}{2}+\frac{z}{3} = 1$. Let $\vec{F}$ be the vector field $\langle x+3y^2, y^2-4x, 2z+xy\rangle$. Use the divergence theorem to compute the outward flux of $\vec{F}$ across S. Make sure you draw the wedge (you may find centroids and volume help complete this problem rapidly).

Problem 12.16 Consider the vector field $\vec{F} = \langle yz, -xz, 3xz\rangle$. Let D be the solid region in space inside the cylinder of radius 4, above the plane $z = 0$, and below the paraboloid $z = x^2 + y^2$. The surface S consists of 3 portions, so computing the flux would require a rather time consuming process of parameterizing these 3 surfaces. Instead, use the divergence theorem to compute the outward flux of $\vec{F}$ across the surface S.

12.4 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to Brainhoney and download the quiz. Once you have taken the quiz, you can upload your work back to brainhoney and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.