Multivariable Calculus

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Introduction

This course may be like no other course in mathematics you have ever taken. We'll discuss in class some of the key differences, and eventually this section will contain a complete description of how this course works. For now, it's just a skeleton.

I received the following email about 6 months after a student took the course:

> Hey Brother Woodruff,
>
> I was reading *Knowledge of Spiritual Things* by Elder Scott. I thought the following quote would be awesome to share with your students, especially those in Math 215 :)
>
> > Profound [spiritual] truth cannot simply be poured from one mind and heart to another. It takes faith and diligent effort. Precious truth comes a small piece at a time through faith, with great exertion, and at times wrenching struggles.

Elder Scott's words perfectly describe how we acquire mathematical truth, as well as spiritual truth.

Contents

Chapter 1

Review

After completing this chapter, you should be able to do the following:

1. Give a summary of the ideas you learned in 112, including graphing, derivatives, integration, and finding area.
2. Approximate functions using differentials. Extend this idea to approximate functions using parabolas, cubics, and polynomials of any degree (in other words, create Taylor polynomials).
3. Explain how to perform matrix multiplication and compute determinants of square matrices.
4. Solve systems of linear equations, and express a solution parametrically when there are infinitely solutions.

1.1 Preparation and Suggested Homework

Most days of class we will begin by presenting in groups the material you have prepared. Typically there will be 4 problems for each day. Each member of the group should prepare one of these problems and come ready to teach the group what is needed to complete this problem. In future units you will occasionally select a problem which is entirely new to you. When this occurs, you should look for examples similar to this problem in the text, and follow those examples to learn how to do this problem. You will be exercising your faith to then go and teach the class something you have never before seen modeled, and your confidence will grow. These problems will normally be the 4th one listed on the preparation problems, so I suggest that as a group you alternate who takes this problem so that you all get a chance to grow.

Preparation Problems

Day 1	1e, 3a, 2g, 2c
Day 2	1f, 3b, 3d, 4a
Day 3	4e, 5h, 6b, 6e

This is a review chapter, and the content inside is designed to be a review. However, the amount and type of reviewing that each of us needs will be unique. At the end of this chapter, I have provided some problems you can print out and practice. Most chapters will be connected to the textbook (Thomas's Calculus [?]). I will point you to problems in the textbook that you can use to help you master the new material. However, in this unit the material comes from previous courses.

I suggest that you get your old calculus textbook (or head to the library to borrow one) and open up to the derivatives chapter (sections 3.2, 3.4, 3.5, and 3.8). You should be able to compute derivatives without a calculator. Practice differentiation and make sure you have mastered all the applicable derivative rules (power, quotient, product, chain, trig, implicit). Then try using the first and second derivative tests to optimize a function (4.5), and use differential (3.10) to approximate a function. Then head to the integration section and practice integrating, in particular try some problems which involve u-substitution (5.5 and 8.1) and integration by parts (8.2). Make sure you understand how to use an integral to find area. After you have done this, find a College Algebra textbook for practicing matrix multiplication, computing determinants, and solving systems of equations.

During this week, please download Mathematica and start trying problems with the computer (just learning to evaluate a derivative or integral, and graphing a function is sufficient). You can alternately use the open source project SAGE which you can use for free at sagemath.org.

Click on this link to download an extensive Mathematica technology introduction tailored to this text.

1.2 Review of First Semester Calculus

1.2.1 Graphing

We need to become comfortable graphing basic functions by hand. If you have not spent much time graphing functions by hand before this class, then please spend some time graphing the following functions:

$$x^2, x^3, x^4, \frac{1}{x}, \sin x, \cos x, \tan x, \arctan x, \ln x, e^x, \sinh x, \cosh x$$

When we start graphing functions of several variables, knowing how to graph basic functions like the ones above will allow us to rapidly visual 3D graphs.

1.2.2 Derivatives

We need to know and use the derivative rules for basic functions. Most calculus textbooks have a list of all these rules on the end pages. I'll provide you a copy of the end page of Thomas's calculus on any quiz or exam. Many of the rules we'll memorize through sheer use. You should know and be able to quickly apply the following differentiation rules:

- Power rule $(x^n)' = nx^{n-1}$
- Sum and difference rule $(f \pm g)' = f' \pm g'$
- Product and quotient rule $(fg)' = fg' + f'g$
- Chain rule $(f \circ g)' = f'(g(x)) \cdot g'(x)$

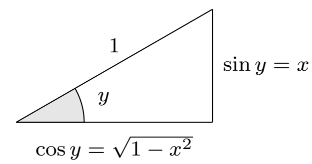
In addition to practicing the basic rules above, make sure you can use the chain rule to do implicit differentiation.

Example 1.1: Implicit Differentiation. Let's use implicit differentiation to find the derivative of $y = \arcsin(x)$. We first rewrite the expression as $x = \sin y$. We then differentiate both sides implicitly with respect to x which gives

$$\frac{d}{dx}x = \frac{d}{dx}\sin y.$$

We simplify this to $1 = \cos(y)y'$ where we applied the chain rule to $\frac{d}{dx}\sin y$ to obtain $\cos y \frac{dy}{dx}$. Solving for y' gives us

$$y' = \frac{1}{\cos(y)}.$$



The expression $x = \sin y$ means that y is the central angle of a triangle where 1 is the length of the hypotenuse and $|x|$ is the length of the opposite edge. The adjacent edge then has the length $\sqrt{1-x^2}$, which means we know $\cos y = \pm\sqrt{1-x^2}/1$. We need to determine if we should use the plus sign or minus sign. Since the range of $y = \arcsin x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we know that $\cos y$ will always be nonnegative and we can erase the negative sign. We finish by writing

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}.$$

1.2.3 Integrals

We need to be able to compute definite and indefinite integrals for any of the basic functions mentioned above. The two integration techniques we will focus on are integration by substitution (undoing the chain rule) and integration by parts (undoing the product rule). While there are many other techniques for doing integration, the two above are the crucial ones we will focus on. We will use software (Mathematica, Sage, Maple) for performing more complicated integrals. Let's look at an example to review each technique.

Example 1.2: Integration by substitution. To compute $\int e^{3x}dx$, first notice that we know $\int e^u du = e^u + C$. We let $u = 3x$, which means $du = 3dx$ or solving for dx gives $dx = \frac{du}{3}$. We now substitute, replacing $3x$ with u and dx with $du/3$, which yields

$$\int e^{3x}dx = \int e^u \frac{du}{3} = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x} + .C$$

The key is to pick the right u, solve for dx, and then compute the simpler integral (which should be on the end cover of your calculus text).

Recall the integration by parts formula is $\int udv = uv - \int vdu$. We obtain this formula by simply integrating both sides of the product rule $d(uv) = vdu + udv$ and then solving for $\int udv$. The key is often to pick u so that the derivative u' is simpler and $\int dv$ does not become more complicated.

Example 1.3: Integration by parts. To compute $\int x \sin 2x dx$, we can pick $u = 2x$ and $dv = \sin 2x$. Note that $u' = 2$ is simpler than $u = 2x$, and computing $\int \sin 2x dx = -\frac{1}{2}\cos 2x$ does not add any extra complication. We now have $du = 1dx$ and $v = -\frac{\cos 2x}{2}$. Integration by parts gives

$$\int x\sin(2x)dx = -x\frac{\cos 2x}{2} - \int -\frac{\cos 2x}{2}dx = -x\frac{\cos 2x}{2} + \frac{\sin 2x}{4}.$$

The work above can be organized into a table to simplify the work.

Tabular Integration By Parts - An organizational tool for integration by parts

The tabular method organizes and simplifies all integration by parts problems. This method sorts the information from multiple integration by parts into one simple table. Let's illustrate this method with the same example as above, where $f(x) = x\sin(2x)$. We'll start by creating the table.

1. Start by creating a table with two columns. I like to put a D above the first column and an I above the second column (for reasons you'll see in moment).

2. In the first column we place a factor which will get simpler with differentiation (what we called u above). Here we place x on the left.

D	I
x	$\sin(2x)$

3. In the second column we place the rest of the integrand, which needs to be something we can integrate and does not make the problem more complex. Here we place $\sin(2x)$ on the right.

4. We now differentiate the function in the first column and and place the derivative below the function. We can repeat this step several times, adding a new row each time, stopping when further differentiation will no longer simplify the problem. Here we differentiated twice (a total of three rows) because we obtained 0 for the second derivative.

D	I
x	
1	
0	

5. Now we integrate the function in the second column, placing the integral on the next row, and repeat this the same number of times we differentiated. Here we integrated twice so that all three rows are complete.

D	I
x	$\sin(2x)$
1	$-\cos(2x)/2$
0	$-\sin(2x)/4$

6. The integrating by parts formula requires that we compute uv *minus* $\int vdu$. To account for the minus sign, to the left of each row we alternately write $+$ or $-$. The positive in the third row is really two negative signs. Each additional application of integration by parts adds another negative sign, which is why the signs alternate.

	D	I
$+$	x	$\sin(2x)$
$-$	1	$-\cos(2x)/2$
$+$	0	$-\sin(2x)/4$

Now that we have our table, we can rapidly combine the information to obtain the integral. When we put u and dv in the table, we obtain the table to the right. Notice that $uv - \int vdu$ is the same as multiplying u by the entry in the right column that is one row lower (giving uv) and then adding the integral of the product of the terms on the bottom row. The tabular method uses this exact same pattern. For each entry in the first column except the last, multiply it by the entry right 1 and down 1. Sum these products and the add the integral of the product of the entries on the bottom row. When the bottom left entry is zero, this integral is zero and can be ignored.

	D	I
$+$	u	dv
$-$	du	v

For the example above, we multiply each entry in the first column (except the last) by the entry which is over and down one. This gives us the sum

$$(+x)(-\cos(2x)/2) + (-1)(-\sin(2x)).$$

The integral of the product of the bottom row entries is zero, so we can ignore it at this stage. , In our case, the product of the bottom row is zero, so our solution is simply

$$\begin{aligned}\int x\sin(2x) &= (+x)(-\cos(2x)/2) + (-1)(-\sin(2x)/4) + \int (+0)(-\sin(2x)/4)dx \\ &= -x\frac{\cos(2x)}{2} + \frac{\sin(2x)}{4} + C.\end{aligned}$$

Example 1.4. Let's compute $\displaystyle\int \ln x dx$. Since we don't know how to integrate $\ln x$, but its derivative is $1/x$ which is simpler, we'll place $\ln x$ on the left side. Since there is nothing left in the integrand and $\ln x = \ln x \cdot 1$, we place a 1 on the right side. The derivative of $\ln x$ is $1/x$. Further differentiation will not simplify the problem, so we stop. We only have one integration to do, where the integral of 1 is x. We alternately place $+$ and $-$ signs in the first column,

	D	I
$+$	$\ln x$	1
$-$	$1/x$	x

giving us the table to the right. The diagonal product is $x \ln x$, and the product of the bottom row entries is $-\frac{1}{x}x = -1$. The solution is

$$\int \ln x dx = x \ln x + \int -1 dx = x \ln x - x + C.$$

With practice, this method is extremely fast, and can be used in both practical and theoretical settings. Many textbooks suggest to use this method only when repeatedly differentiating the first column will eventually give a zero. If you remember to always add the integral of the product of the bottom terms, then you can use this method in any setting to simplify your work. This extends what you find in most undergraduate textbooks. There are several theorems you may encounter in future courses which are extremely simple to prove if you just apply tabular integration by parts (see [**?**]).

1.2.4 Differentials and tangent lines

Recall that the derivative of a function at a point gives us the slope of the tangent line to the function at that point. We can use this tangent line to make estimates how much the output (y-values) will change if we change the inputs (x-values). This is extremely useful when we don't know the function but instead we understand how the function changes. As one example, if we know how quickly something is moving, then we can approximate the position function because we understand the velocity.

For the function $y = f(x)$, recall that we define the differential to be $df = f'(x)dx$. We can use this to approximate changes in f based on changes in x. We can quickly obtain this formula by writing the derivative in two ways, namely $\frac{dy}{dx} = f'(x)$ and then multiplying both sides by dx to obtain $dy = f'(x)dx$.

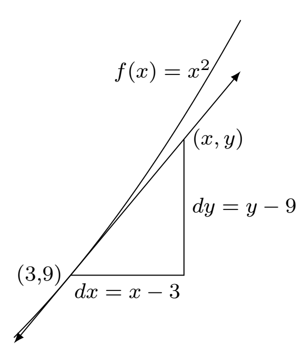
The derivative is $f'(x)$. The differential is $dy = f'(x)dx$. A differential is a derivative times a change in inputs.

Example 1.5. We know that a circle of radius r has area $A = \pi r^2$. The derivative with respect to r is $\frac{dA}{dr} = 2\pi r$, so the differential is $dA = 2\pi r dr$. We can approximate a change in the area based upon a change in the radius. Increasing the radius of a circle from 1 cm to 1.2 cm (so $r = 1, dr = .2$) will result in an approximate increase of $dA = 2\pi(1)(.2)$ square centimeters. If the radius is currently $r = 3$ cm, then a change in dr cm of radius will cause a change in area of $dA = 2\pi(3)dr$ square centimeters. When dr is small, this estimate is extremely close.

We can also view tangent lines in terms of differentials. We can use the following idea to rapidly expand first semester calculus to higher dimensions.

Example 1.6. Consider the parabola $f(x) = x^2$. To find an equation of the tangent line to $y = x^2$ at $x = 3$, recall that we need a slope and a point to get the equation. We can get the point by computing $f(3) = 3^2 = 9$, so the point $(3, 9)$ is on the tangent line. The derivative is $f'(x) = 2x$, which means the slope of the tangent line at $x = 3$ is $f'(3) = 6$. Using the point-slope form for a line, we obtain an equation for the line as $(y - 9) = 6(x - 3)$. Where are the differentials? If (x, y) is some point on the tangent line, then we can write a change in y as $dy = y - 9$ and a change in x as $dx = x - 3$. The differential formula $dy = f'dx$ then becomes

$$\underbrace{(y-9)}_{dy} = \underbrace{6}_{f'(3)} \underbrace{(x-3)}_{dx}.$$



The slope of the tangent line is

$$\frac{rise}{run} = \frac{dy}{dx} = \frac{y-9}{x-3} = f'(3) = 6.$$

An equation for the tangent line is hence $dy = f'(3)dx$ or just $y - 9 = 6(x - 3)$.

If $f(x)$ is any differentiable function, then an equation of the tangent line at $x = c$ is simply

$$\underbrace{y - f(c)}_{dy} = f'(c)\underbrace{(x - c)}_{dx}.$$

1.3 Taylor Polynomials

It's time to apply our knowledge about differentials to obtain one of the most important developments of the 18th century, Taylor polynomials. You have already seen in previous course functions such as $e^x, \sin x, \cos x, \arctan x, \ln x$, and more. If I were to ask you to compute $\sin(.2)$ or to give me $e^{.4}$, you would probably just reach for your calculator, type it in, and give me the answer. But wait, how did your calculator give you that answer? The idea behind how the calculator gives this answer is one of the key ideas leading up to most of modern mathematics. Your calculator uses differentials, and an extension of this idea, to approximate the value you want. It does not give you an exact value, rather it gives you an approximation (accurate to 15 decimal places). So how does it do this?

First, recall that if $f(x)$ is any differentiable function, then an equation of the tangent line at $x = c$ is simply $y - f(c) = f'(c)(x - c)$. Solving for y we have $y = f(c) + f'(c)(x - c)$. This tangent line is fairly close to the function at x-values near c. However, rather than use a line to approximate a function, what if instead we use a parabola. A parabola could bend more than a line, and hopefully the parabola could give a better approximation to the function. If a parabola gives a better approximation, then why stop with a parabola. We could use a cubic, a quartic, or even a polynomial of any degree we choose. As we increase the degree, we would hope that the polynomial more closely approximates the function. We call these Taylor polynomials.

Example 1.7. Let's use a parabola, say $P_2(x) = a_0 + a_1x + a_2x^2$, to approximate $f(x)$ near $x = 0$. We don't yet know the value of the coefficients a_0, a_1, and a_2, so we need some equations to solve for these constants. Let's make our parabola and function pass through the same point $(0, f(0))$, have the same slope at $x = 0$, and have the same concavity, or value of the second derivative, at $x = 0$. There three constraints give us three equations, namely

$$P_2(0) = f(0), \qquad P_2'(0) = f'(0), \qquad \text{and} \qquad P_2''(0) = f''(0).$$

We now solve this system of equations. Since $P_2(0) = f(0)$, we have $a_0 + 0 + 0 = f(0)$, which tells us that $a_0 = f(0)$ the value of the function at $x = 0$. The derivative of $P_2(x)$ is $P_2'(x) = a_1 + 2a_2x$, which at $x = 0$ needs to equal $f'(0)$. This gives us the equation $a_1 + 0 = f'(0)$, which tells us the value of a_1. The second derivative is $P_2''(x) = 2a_2$, and so we have $2a_2 = f''(0)$. From these three equations we find that the coefficients of the parabola are

$$a_0 = f(0), \quad a_1 = f'(0), \quad \text{and} \quad a_2 = \frac{f''(0)}{2}.$$

We now have an equation for the second degree Taylor polynomial, namely

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

We can repeat the process above for a cubic. We now require that the function and polynomial and their first three derivatives all have the same value at $x = 0$. This gives four equation to find the four coefficients of the polynomial. Solving this equations gives us the polynomial

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3 \cdot 2}x^3,$$

where a_0, a_1, and a_2 are the same as before with $a_3 = \frac{f'''(0)}{3 \cdot 2}$. Notice that the first three coefficients of the degree 3 polynomial match the degree 2 polynomial. To find the degree 4 polynomial, the first 4 coefficients are the same as above, and we would obtain $a_4 = \frac{f''''(0)}{4 \cdot 3 \cdot 2}$.

In general, we can find the coefficient a_n by computing n derivatives, evaluating at $x = 0$, and then dividing by the product of all the integers up to an including n. We write this in shorthand notation as $a_n = \frac{f^{(n)}(0)}{n!}$ where $f^{(n)}(0)$ is the nth derivative evaluated at $x = 0$, and $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$, which we call the factorial function. The Taylor polynomial of degree m centered at $x = 0$ is simply

The factorial of zero is $0! = 1$. Feel free to ask me in class why.

$$P_m(x) = f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f^{(3)}(0)}{3 \cdot 2 \cdot 1}x^3+\cdots+\frac{f^{(m)}(0)}{m!}x^m = \sum_{n=0}^{m} \frac{f^{(n)}(0)}{n!}x^n.$$

We say the polynomial is centered at $x = 0$, because we could have instead required that the function and its derivatives matched the polynomial at another point $x = c$. A similar calculation shows that the Taylor polynomial of degree m centered at $x = c$ is $P_m(x) = \sum_{n=0}^{m} \frac{f^{(n)}(c)}{n!}(x - c)^m$.

Example 1.8. The function $f(x) = \frac{1}{1-x}$ has as its first three derivatives

$$f'(x) = \frac{1}{(1-x)^2}, f''(x) = \frac{2}{(1-x)^3}, f^{(3)}(x) = \frac{3 \cdot 2}{(1-x)^4}.$$

Continuing to differentiate, we see that the nth derivative is $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$. We evaluate each of these functions at $x = 0$, giving

$$f(0) = f^{(0)}(0) = 1, f'(0) = 1, f^{(2)}(0) = 2!, f^{(3)}(0) = 3!, \ldots, f^{(n)}(0) = n!, \ldots.$$

To obtain the coefficient a_n, we divide by $n!$ to obtain $a_n = 1$ for all n. Hence we see that the Taylor polynomials for $f(x) = \frac{1}{1-x}$ are

$$P_1(x) = 1 + x, P_2 = 1 + x + x^2, P_3 = 1 + x + x^2 + x^3, \ldots, P_n = \sum_{m=0}^{n} x^m.$$

We can organize the calculations above into the table below.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$a_n = f^{(n)}(0)/n!$	$a_n x^n$
0	$(1-x)^{-1}$	1	$1/0!$	1
1	$(-1)(1-x)^{-2}(-1)$	1	$1/1!$	x
2	$(-1)(-2)(1-x)^{-3}(-1)^2$	2	$2/2! = 1$	x^2
3	$(-1)(-2)(-3)(1-x)^{-3}(-1)^2$	$3!$	$3!/3! = 1$	x^3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Once we have created this table, we can obtain a Taylor polynomial of any degree by summing the terms in the right most column. The third degree Taylor polynomial is $P_3(x) = 1 + x + x^2 + x^3$.

Example 1.9. Let's use the table format to find the 6th degree Taylor polynomial for $f(x) = \cos x$, centered at $x = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$a_n = f^{(n)}(0)/n!$	$a_n x^n$
0	$\cos x$	1	$1/0! = 1$	1
1	$-\sin x$	0	$0/1! = 0$	0
2	$-\cos x$	-1	$-1/2!$	$-\frac{1}{2!}x^2$
3	$\sin x$	0	$0/3! = 0$	0
4	$\cos x$	1	$1/4!$	$\frac{1}{4!}x^4$
5	$-\sin x$	0	$1/5! = 0$	0
6	$-\cos x$	-1	$-1/6!$	$-\frac{1}{6!}x^6$

Summing the right column give the 6th degree Taylor polynomial centered at $x = 0$ as

$$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6.$$

Notice that $\frac{1}{6!}$ is extremely small, so the 6th order term doesn't change the value of $P_6(x)$ much for small values of x.

Example 1.10. If we want to center the Taylor polynomial at a point other than zero, then the table format below illustrates this for the 3rd degree Taylor polynomial centered at $x = \pi$ for $f(x) = \sin x$.

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$	$a_n = f^{(n)}(\pi)/n!$	$a_n(x-\pi)^n$
0	$\sin x$	0	$0/1! = 0$	0
1	$\cos x$	-1	$-1/1!$	$-\frac{1}{1!}(x-\pi)^1$
2	$-\sin x$	0	$0/2! = 0$	0
3	$-\cos x$	1	$1/3!$	$\frac{1}{3!}(x-\pi)^3$

Summing the last column gives the 3rd degree Taylor polynomial centered at $x = \pi$ as

$$P_3(x) = -\frac{1}{1!}(x-\pi)^1 + \frac{1}{3!}(x-\pi)^3.$$

The graphs of several Taylor polynomials centered at $x = 0$ for the functions e^x, $\frac{1}{1-x}$, $\cos x$, and $\sin x$ are shown below. The higher the order of the polynomial, the closer it is to the actual function.

As we increase the degree of a Taylor polynomial, we should expect to see that our accuracy of approximating a function increases. In Table 1.1 we see some examples of a function and several of its Taylor polynomials. Notice that the polynomials more closely follow the function as the degree increases. Increasing the degree of the polynomial will provide a better approximation for functions such as $\cos x$, $\sin x$, e^x, polynomials, and combination of these functions obtained through addition, subtraction, and multiplication. We can also divide any two of these function and still obtain good approximations, but vertical asymptotes pose a problem.

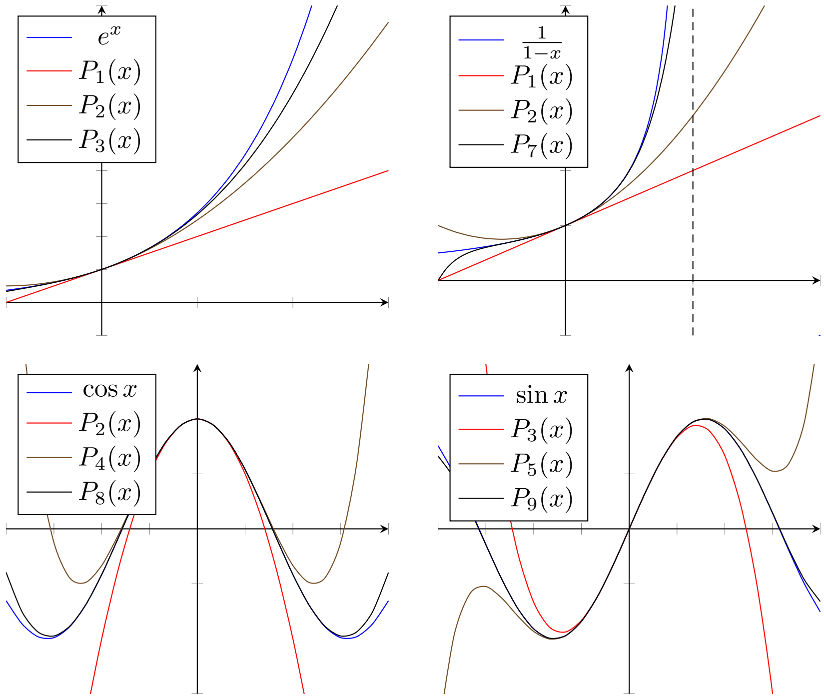


Table 1.1: Four functions are shown together with several Taylor polynomials centered at $x = 0$. Notice that as the degree of the polynomial increase, the polynomial more closely approximates the function.

The function $\frac{1}{1-x}$ has a vertical asymptote at $x = 1$. When we create Taylor polynomials centered at $x = 0$, those polynomials follow the function up the left hand side of the vertical asymptote, causing the polynomials to tend toward infinity at $x = 1$ as we increase the degree of the polynomial. However, none of these polynomials will do a good job of approximating the function to the right of $x = 1$. In addition, none of the polynomials do a good job of approximating the function to the left of $x = -1$. When approximations break down on one side, they break down on the other side as well. We can only obtain useful approximations of $1/(1-x)$ for values of x in the interval $(-1, 1)$. The center of this interval is $x = 0$, where we centered our Taylor polynomial. The distance from the center $x = 0$ to the asymptote at $x = 1$ we called the radius of convergence, and the interval of convergence $(-1, 1)$ is obtained by moving 1 unit (the radius) to the right and left of 0 (the center). We will use these ideas and explore them more in depth when we study differential equations.

1.3.1 A Geometric View of Approximation

The differential notation $dy = f'dx$ reminds us that we can approximate the change dy in a function by using the derivative. Taylor polynomials help us create higher order approximations to functions. Here is how that idea is developed. The first order Taylor polynomial centered at $x = a$ is $P_1(x) = f(a)+f'(a)(x-a)$. This means that we can approximate f using $f(x) \approx f(a) + f'(a)(x - a)$ or by subtracting $f(a)$ from both sides we obtain an approximation to the change in the y values as

$$\Delta y = f(x) - f(a) \approx f'(a)(x - a) = f'(a)dx = dy$$

For a change in x from a to $a + dx$, this means $\Delta y \approx f'(a)dx$ or $\Delta y \approx dy$. This first order approximation can be improved by using higher order approximations. The second degree Taylor polynomial gives us

$$\Delta y \approx f'(a)dx + \frac{1}{2!}f'(a)dx^2.$$

Similarly, a third degree approximation yields

$$\Delta y \approx f'(a)dx + \frac{1}{2!}f''(a)dx^2 + \frac{1}{3!}f'''(a)dx^3.$$

Taylor's remainder theorem (which I'll let you look up if you are interested) can then help us determine how far apart our estimated change in y is from the actual change in y.

Example 1.11. Let's look at approximating a change in area. Suppose we manufacture a 1 inch diameter washer blanks (no hole in the center). Each washer should be 1 inch in diameter, but because of imperfections in the manufacturing process, some washers are slightly larger and some are slightly smaller. By about how much would the area of the face of the washer increase if the diameter were to increase by 0.02 inches? Let's use a first, second, and third order approximation to analyze this question.

The area function is $A = \pi r^2$ (ignore the small hole in the middle), and has derivatives $A' = 2\pi r$ and $A'' = 2\pi$ where $r = .5$ inches is the radius of the washer. To match the notation above, our function $f(x)$ is $A(r)$, we are centering our polynomials at $r = .5$ inches, and the change in radius is half the change in diameter, so $dr = 0.01$ inches.

- The first order approximation gives $\Delta y \approx dy = f'(.5)dr = 2\pi(.5)(.01) \approx .0314159$.
- The second order approximation gives $\Delta y = f'(.5)dr + \frac{1}{2}f''(.5)dr^2 = 2\pi(.5)(.01) + \frac{1}{2}2\pi(.01)^2 \approx .0317301$.
- The third order approximation gives $\Delta y = f'(.5)dr + \frac{1}{2}f''(.5)dr^2 + \frac{1}{3!}f'''(.5)dr^3 = 2\pi(.5)(.01) + \frac{1}{2}2\pi(.01)^2 + \frac{1}{6}0.01^3 \approx .0317301$.
- The actual difference is the difference in the two areas, namely $\pi(.51)^2 - \pi(.5)^2 \approx .0317301$.

The second and third order approximations are identical to the actual change in area because area is a 2nd order problem. This means that for each washer we make, we may have to account for an additional .0317 square centimeters of material. This is not a very large amount, but when you make one million washers, that could mean an additional 30 thousand square feet of metal.

Example 1.12. Let's examine what happens if we create a cube with side lengths $x = 2$ inches. Suppose the error tolerance in creating the length of a side of the cube is 0.1 inches. By about how much will the volume increase each side length increases by 0.1 inches? Again let's examine this using a first, second, and third order approximation. Since volume is a third order problem, the third order approximation should be exact. Our function is $f(x) = x^3$, we will center our Taylor polynomials at $x = 2$, and we have a change in x of $dx = 0.1$. The derivatives of f are $f' = 3x^2$, $f'' = 6x$, and $f''' = 6$.

- The first order approximation gives $\Delta y \approx dy = f'(2)dx = 3(2)^2(0.1) = 1.2$.

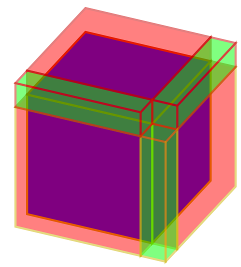
- The second order approximation gives

$$\Delta y \approx f'(2)dx + \frac{1}{2}f''(2)dx^2 = 3(2)^2(0.1) + \frac{1}{2}6(2)(0.1)^2 = 1.26$$

- The third order approximation gives

$$\Delta y \approx f'(2)dx+\frac{1}{2}f''(2)dx^2+\frac{1}{3!}f'''(2)dx^3 = 3(2)^2(0.1)+\frac{1}{2}6(2)(0.1)^2+\frac{1}{6}6(0.1)^3 = 1.261.$$

- The actual difference is the difference in the two volumes, namely $(2.1)^3 - (2)^3 = 1.261$.



The image to the right shows how each level of approximation gets closer to the actual difference. We start with a cube with edges of length 2. We attach three 2 by 2 by 0.1 inch boxes to three faces of the cube (shown in red). The volume of these 3 red boxes is precisely Δy from our first order approximation. The extra term in the second order approximation comes from attaching three 0.1 by 0.1 by 2 inch boxes to fill in the edges of the new 2.1 inch box (shown in green). The third order approximation comes by adding a single 0.1 by 0.1 by 0.1 inch box to fill in the final corner of the box.

Example 1.13. As a final example, suppose we are trying to construct a 3,4,5 triangle. We've been able to construct the side edge of 3 cm exactly, but the side with 4 cm has an error tolerance of 0.1 cm. We know the hypotenuse of the triangle has length $h = \sqrt{a^2 + b^2}$ where we have $a = 3$, $b = 4$, and $h = 5$ with $da = 0$ and $db = 0.1$. The first order approximation to the error is

$$dh = \frac{1}{2}(a^2 + b^2)^{-1/2}(2bdb) = \frac{1}{2}(3^2 + 4^2)^{-1/2}(2(4)(0.1)).$$

Similarly we could compute a second and third order approximation. However, will our estimates for Δh using Taylor polynomials ever be exact? The answer here is no. The derivative of h will never become zero, and so every time we increase the degree of our approximation, we will get better approximation. However, our first order approximation is already pretty close. I'll let you work through this example as homework.

1.4 Matrices

We will soon discover that matrices represent derivatives in high dimensions. When you use matrices to represent derivatives, the chain rule is precisely matrix multiplication. For now, we need to become comfortable with matrix multiplication.

We perform matrix multiplication "row by column". Wikipedia has an excellent visual illustration of how to do this. See http://en.wikipedia.org/wiki/Matrix_multiplication and/or http://www.texample.net/tikz/examples/matrix-multiplication/ for an explanation and visualization.

1.4.1 Determinants

Determinants measure area, volume, length, and higher dimensional versions of these ideas. Determinants will appear as we study cross products, surface integrals, and the high dimensional version of integration by substitution.

We associated with every square matrix a number, called the determinant, which relates relates to length, area, and/or volume. We can use the determinant to generalize volume to higher dimensions. We can compute the determinant of a 2 by 2 and 3 by 3 matrix using the formulas

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \text{and}$$

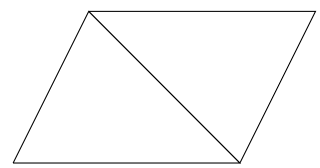
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \det \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \det \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \det \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= a(ei - hf) - b(di - gf) + c(dh - ge).$$

We use vertical bars next to a matrix to state we want the determinant. Notice the negative sign on the middle term of the 3 by 3 determinant. Also, notice that we had to compute three determinants of 2 by 2 matrices in order to find the determinant of a 3 by 3.

Example 1.14. Here's an example for a 3 by 3 matrix.

$$\det \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 2 & -3 & 1 \end{bmatrix} = 1 \det \begin{bmatrix} 3 & 4 \\ -3 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} + 0 \det \begin{bmatrix} -1 & 3 \\ 2 & -3 \end{bmatrix}$$
$$= 1(3 + 12) - 2(-1 - 8) + 0(3 - 6).$$

Determinants were discovered more than 2000 years ago, [1] though the name determinant did not show up until the 1800s. The same expression kept showing up in different places, and eventually that expression was given a name. One application of determinants is connected to finding the area of a triangle, or parallelogram, and the volume of a tetrahedron, or a three dimensional version of a parallelogram (called a parallelepiped). Let's look at this example.

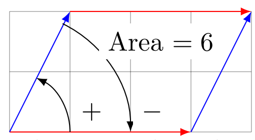


Think of each column of a 2 by 2 matrix as a point in the plane. These two points, together with the origin, give the vertices of a triangle or three of the four vertices of a parallelogram (see the image to the right). The determinant gives the area of that parallelogram, up to a sign (sometimes the determinant is negative). The determinant not only keeps track of area, but it lets you know something about the angle between the two edges of the parallelogram which contain the origin. Take the edge between (0,0) and the first column and rotate it (keeping the origin fixed) till it lies over the second edge. This rotation can be done in no more than 180 degrees. If the rotation was counterclockwise, then the determinant is positive. If the rotation was clockwise, then the determinant is negative. This is often called the right hand rule. You can achieve the same thing by placing the index finger of your right hand on the edge given by the first column, and the middle finger of your right hand on the the second edge, both fingers pointing away from (0,0). One of two things will happen. Either your thumb will point up out of the page (in which case the determinant is positive and a counterclockwise turn gets you from the first edge to the second) or your thumb will point into the page (in which case the determinant is negative). It follows that if you interchange two columns of a matrix, then the determinant will change sign but not magnitude. This idea generalizes to higher dimensions, in particular the determinant gives volume in 3D (up to a sign). Mathematicians have used the determinant to measure size in higher dimensions.

Example 1.15. Consider the 2 by 2 matrix $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ whose determinant is $3 \cdot 2 - 0 \cdot 1 = 6$. Draw the column vectors $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with their base at

[1] See http://www-history.mcs.st-and.ac.uk/history/HistTopics/Matrices_and_determinants.html for an interesting history.

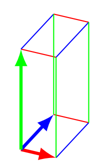
the origin. These two vectors give the edges of a parallelogram whose area is the determinant 6. If I swap the order of the two vectors in the matrix, then the determinant of $\begin{bmatrix}1 & 3\\ 2 & 0\end{bmatrix}$ is -6. The reason for the difference is that the determinant not only keeps track of area, but also order. Starting at the first vector, if you can turn counterclockwise through an angle smaller than 180° to obtain the second vector, then the determinant is positive. If you have to turn clockwise instead, then the determinant is negative. This is often termed "the right-hand rule," as rotating the fingers of your right hand from the first vector to the second vector will cause your thumb to point up precisely when the determinant is positive.



$$\begin{vmatrix}3 & 1\\ 0 & 2\end{vmatrix} = 6 \text{ and } \begin{vmatrix}1 & 3\\ 2 & 0\end{vmatrix} = -6$$

Figure 1.1: The determinant gives both area and direction. A counter clockwise rotation from column 1 to column 2 gives a positive determinant.

For a 3 by 3 matrix, the columns give the edges of a three dimensional parallelepiped and the determinant produces the volume of this object. The sign of the determinant is related to orientation. If you can use your right hand and place your index finger on the first edge, middle finger on the second edge, and thumb on the third edge, then the determinant is positive. For example, consider the matrix $A = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$. Starting from the origin, each column represents an edge of the rectangular box $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$ with volume (and determinant) $V = lwh = (1)(2)(3) = 6$. The sign of the determinant is positive because if you place your index finger pointing in the direction (1,0,0) and your middle finger in the direction (0,2,0), then your thumb points upwards in the direction (0,0,3). If you interchange two of the columns, for example $B = \begin{bmatrix}0 & 1 & 0\\ 2 & 0 & 0\\ 0 & 0 & 3\end{bmatrix}$, then the volume doesn't change since the shape is still the same. However, the sign of the determinant is negative because if you point your index finger in the direction (0,2,0) and your middle finger in the direction (1,0,0), then your thumb points down in the direction (0,0,-3). If you repeat this with your left hand instead of right hand, then your thumb points up.



1.5 Solving Systems of equations

We'll need to solve systems of 2 or more equations. In particular, please practice solving linear systems. It wouldn't hurt to learn Gaussian elimination, but it won't be needed for this course. What we will need is solving linear and quadratic systems of equations. The most common techniques involve substitution and the addition (elimination) method.

Many problems involve finding an intersection of two objects. If each object is given by an equation, then finding the point of intersection is the same as solving a system of equations. When we study Lagrange multipliers, we will need to be able to solve complex systems of nonlinear equations.

Sometimes the solution to a system of equations is not unique. For example the system $x+y=2, 2x+2y=4$ has infinitely many solutions, as both equations represent the same line. We can give a solution to such systems by introducing a new variable, our parameter. For example, we could say that $y=t$ (the variable t is a parameter we are free to choose) and then solve for x to obtain $x=2-t$. This would give a solution $(x,y)=(2-t,t)$ for any t. If there are more variables than equations, then you will often get solutions of this type.

1.6 Review Problems for 215

1. (Derivatives) Differentiate the following functions. If they are difficult, please head to your book first.
   (a) $\cos(3x)$
   (b) $x\sin x$
   (c) $\frac{\arctan x}{\ln x}$
   (d) $\tan(\sec(x^2+1))$
   (e) $f(x) = \dfrac{e^{2x}\cos(x^2+1)}{\ln(\tan^{-1} x)}$
   (f) $\arcsin x$ using implicit differentiation
   (g) $\arctan x$ using implicit differentiation
   (h) Sec. 3.2 (product and quotient rule)
   (i) Sec. 3.5 (chain rule ** practice here)
   (j) Sec. 3.8 (Inverse trig functions)

2. (Integrals) Then find the following
   (a) $\int e^{2x}dx$ (u-sub)
   (b) $\int \frac{x}{x^2+1}dx$ (u-sub)
   (c) $\int x\sin x dx$ (parts)
   (d) $\int \ln x dx$ (parts)
   (e) $\int x^2e^{3x}dx$
   (f) $\int e^x\cos x dx$
   (g) Find the area under the function e^{3x} for $0 \le x \le 3$. (u-sub)
   (h) Sec. 5.5 and 8.1 (u-substitution)
   (i) Sec. 8.2 (integration by parts)

3. (Differentials and tangent lines)
   (a) Find the differential df of the function $f(x) = x^2 + 4\arctan x$.
   (b) The volume of a sphere of radius r is $V = (4/3)\pi r^3$. A tennis ball has a radius of about 3 cm from the center of the ball to the beginning of the rubber exterior. The rubber exterior is about .2 cm thick. Use differentials to estimate the volume of the rubber exterior. Compare this to the actual difference between the volume of a sphere of radius 3 and 3.2 cm.
   (c) A manufacturer creates a cylindrical can of height 10 cm and radius 4 cm. The manufacturing process results in cans with radii between 3.9 and 4.1 cm, but the height stays at just about 10 cm always. About how much change in volume should you expect if the can's radius is .1 cm different than 4 cm? Use differentials to make your estimate.
   (d) Find an equation of the tangent line to $y = x^3$ at $x = 2$.
   (e) More differential problems are in Section 3.10:19-50 (11th edition) and Section 3.11:19-50 (12th edition).

4. (Taylor Polynomials)
   (a) Find the degree 1, 2, 3, 4, and 5 Taylor polynomials to $f(x) = \ln(x+1)$ centered at $x = 0$.
   (b) More problems are in section 11.8:1-14 (11th edition) and 10.8:1-22 (12th edition).
   (c) In constructing a square of side lengths 3 inches, use a first, second, and third order approximation to estimate the change in area of the square if the side lengths increase from 3 to 3.2 inches. Compare this to the actual change in area.
   (d) In constructing a circle of radius 4 inches, use a first, second, and third order approximation to estimate the change in area of the circle if the radius increases from 4 to 4.1 inches. Compare this to the actual change in area.
   (e) In constructing a right triangle with sides lengths 3 and 4 inches, use a first, second, and third order approximation to estimate the change in the length of the hypotenuse if one side increases from 3 to 3.1 inches. Compare this to the actual change in the length.
   (f) In constructing a ball of radius 2 inches, use a first, second, and third order approximation to estimate the change in volume ($V = \frac{4}{3}\pi r^3$) of the ball if the radius increases from 2 to 2.3 inches. Compare this to the actual change in volume.

5. (Matrices) Compute the following.
   (a) $\begin{bmatrix} 5 & -7 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 3 & 1 \end{bmatrix}$
   (b) $\begin{bmatrix} 3 & 2 \\ 9 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 & -2 \\ 5 & 2 & 6 \end{bmatrix}$
   (c) $\begin{bmatrix} 6 & 1 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 9 \\ -2 \end{bmatrix}$
   (d) $\begin{bmatrix} 2 & 8 & 5 \\ 3 & 6 & -1 \end{bmatrix}\begin{bmatrix} 7 & -1 \\ 0 & 4 \\ 2 & 3 \end{bmatrix}$
   (e) $\det\begin{bmatrix} 5 & 1 & -1 \\ 2 & 7 & 3 \\ 4 & 0 & 2 \end{bmatrix}$
   (f) $\det\begin{bmatrix} 3 & 2 & 7 \\ 2 & 0 & 1 \\ 4 & 1 & 5 \end{bmatrix}$

(g) Let $A = \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 0 \end{bmatrix}$. Find AB, BA and $\det(A)$

(h) Let $C = \begin{bmatrix} 0 & 3 & -1 \\ 2 & 1 & 1 \\ -2 & 5 & 1 \end{bmatrix}, D = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find CD and $\det(C)$.

6. (Systems) Solve the following systems of equations. If there are multiple solutions, give your answer in terms of a parameter t.

(a) $\begin{cases} x + y &= 3 \\ 2x - y &= 4 \end{cases}$

(b) $\begin{cases} x + y + z &= 3 \\ 2x - y &= 4 \end{cases}$

(c) $\begin{cases} -x + 4y &= 8 \\ 3x - 12y &= 2 \end{cases}$

(d) $\begin{cases} 6x + 2y &= 2 \\ 3x - 4y &= 5 \end{cases}$

(e) $\begin{cases} 6x + y &= 4 \\ -x - \frac{1}{6}y &= -\frac{2}{3} \end{cases}$

(f) $\begin{cases} 3x + 4y + z &= 1 \\ x - 2y + 3z &= 3 \end{cases}$

Here are some solutions.

5. (a) $\begin{bmatrix} -11 & 23 \\ 15 & 37 \end{bmatrix}$

(b) $\begin{bmatrix} 13 & 16 & 6 \\ 9 & 36 & 18 \end{bmatrix}$

(c) $\begin{bmatrix} 13 \end{bmatrix}$

(d) $\begin{bmatrix} 24 & 45 \\ 19 & 18 \end{bmatrix}$

(e) 106

(f) -1

6. (a) $x = 7/3, y = 2/3$

(b) infinitely many solutions $x = t, y = 2t - 4, z = 7 - 3t$

(c) inconsistent

(d) $x = \dfrac{3}{5}, y = -\dfrac{4}{5}$

(e) infinitely many solutions $x = t, y = 4 - 6t$

(f) infinitely many $x = t, y = -\dfrac{4}{7}t, z = 1 - \dfrac{5}{7}t$

Chapter 2

Vectors

After completing this chapter, you should be able to do the following:

1. Define, draw, and explain what a vector is in 2 and 3 dimensions.
2. Add, subtract, and multiply vectors (scalar product, dot product, and cross product). Illustrate each operation geometrically.
3. Use vector products to find angles, length, area, projections, and work.
4. Use vectors to give equations of lines and planes, and draw these objects in 3D.

2.1 Preparation and Homework Suggestions

You can find the following problems in Thomas's Calculus[**?**].

	Preparation Problems (11th ed)				Preparation Problems (12th ed)			
Day 1	12.1:50	12.2:23	12.2:26	12.3:4	12.1:56	12.2:23	12.2:26	12.3:4
Day 2	12.4:7	12.4:17	12.5:18	12.5:23	12.4:7	12.4:17	12.5:18	12.5:23
Day 3	12.3:19	12.3:45	12.5:33	12.5:59	12.3:30	12.3:43	12.5:33	12.5:59

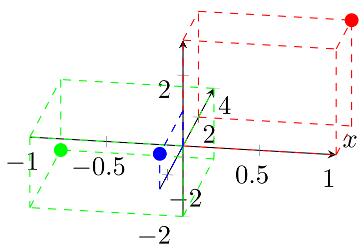
Your homework is to do at least 7 problems for each day of class, at least of 2 of these should be done with the computer. The following homework problems line up with the topics we will discuss in class. The basic practice problems should be quick problems to help you master the ideas, and the good problems will require a little more work. The theory and application problems are ones that will challenge you more if you want to fully master the material.

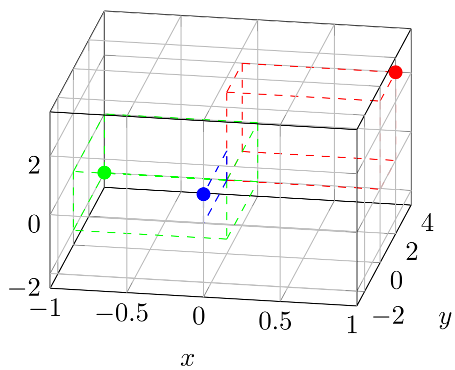
(11th ed) Topic	Sec	Basic Practice	Good Problems	Thy/App
Multiple Dimensions	12.1	1-24, 35-48	25-34, 49-52, 53-56	
Vectors	12.2	1-24	25-34, 35-40, 43-47, 54	41, 42, 48, 49, 50, 51, 53,
Dot Product	12.3	1-14, 17-19,	15, 20, 27, 28, 30, 33-42, 43-46, 47-52	16, 21, 22-26, 29, 31, 32, 53-56
Cross Product	12.4	1-18, 35-42	19-22, 23, 24, 25-26, 27-31, 43	32-34, 44
Lines and Planes	12.5	1-12, 21-28, 33-46,	13-20, 29-32, 47-48, 53-62, 65.67.70-72	63, 64, 66, 68, 69, 73, 74

(12th ed) Topic	Sec	Basic Practice	Good Problems	Thy/App
Multiple Dimensions	12.1	1-30, 41-54	31-40, 55-58, 59-66	
Vectors	12.2	1-24	25-34, 35-40, 43-49, 54	41, 42, 50-55
Dot Product	12.3	1-14,	15, 23, 24, 26, 30 31-40, 41-44, 45-50	16, 17-22, 25, 27, 28,
Cross Product	12.4	1-18, 35-47	19-22, 23, 24, 25-26, 27-31	32-34, 48-50
Lines and Planes	12.5	1-12, 21-28, 33-46,	13-20, 29-32, 47-48, 53-62, 65.67.70-72	63, 64, 66, 68, 69, 73, 74

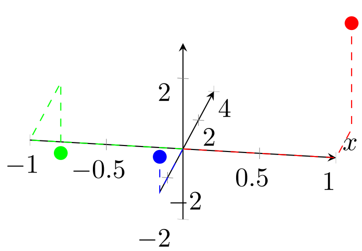
2.2 Graphing and distance in three dimensions

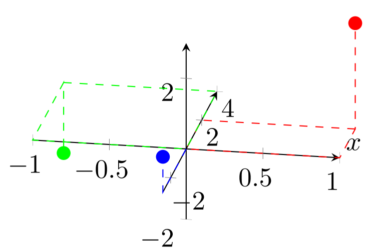
To plot points in 3 dimensions, we have to introduce a way of graphing things three dimensionally. The most common method of graphing is to use a right-hand system. Your pointer finger represents the positive x-axis, your middle finger represents the positive y-axis, and your thumb represents the positive z-axis. On the printed page, we'll often have the x-axis point right, the y-axis point back, and the z-axis point up. For one way to plot a point in 3D, we start by making a rectangular prism with one corner at the origin, and the opposing corner at the point of interest, as shown in the two pictures below which plot the three points $(1, 2, 3)$, $(-1, 4, -2)$, and $(0, -3, 1)$.





Drawing the full rectangular prism above can be time consuming, so sometimes we'll just draw the parallelogram in the xy-plane with a line up or down getting us to the point. The quickest option is to just draw half the parallelogram with a line up or down. The two pictures below show examples of each of these options. Note that every time we draw an edge, it is parallel to one of the axes.





2.2.1 Distance Between Two Points

To find the distance between the origin and a point (x, y, z), the Pythagorean theorem gives the distance from $(0, 0, 0)$ to $(x, y, 0)$ as $\sqrt{x^2 + y^2}$. The length of the hypotenuse of the triangle with vertices $(0, 0, 0)$, $(x, y, 0)$, and (x, y, z) is hence $\sqrt{(\sqrt{x^2+y^2})^2+z^2} = \sqrt{x^2+y^2+z^2}$. So the distance from the origin to $(3,5,-2)$ is $\sqrt{(3)^2+(5)^2+(-2)^2} = \sqrt{9+25+4} = \sqrt{38}$. We can generalize this formula to show that the distance between two points is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

This formula gives us the distance between $P_1 = (1,0,2)$ and $P_2 = (3,1,0)$ to be $\sqrt{(3-1)^2+(1-0)^2+(0-2)^2} = 3$.

2.2.2 Equations of Spheres

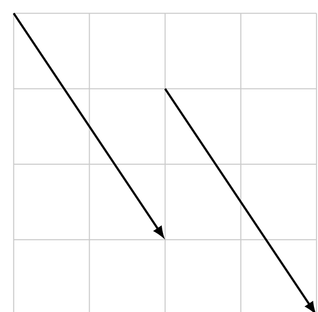
Since a sphere of radius a centered at (x_0, y_0, z_0) is all points (x, y, z) which are distance a from the center, we get (by squaring both sides of the distance formula) that the equation of a sphere is

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = a^2.$$

We can rewrite the equation $x^2+y^2+z^2+2x-4y = 0$, using completing the square, in the form $x^2+2x+1+y^2-4y+4+z^2 = 1+4$ or $(x+1)^2+(y-2)^2+z^2 = 5$. Once we've done this, we know that this is an equation of a sphere of radius $\sqrt{5}$ centered at $(-1,2,0)$.

We will focus most of our time this semester in 2 and 3 dimensions. However, many problems in the real world require much higher dimensions. When you hear the word dimension, it does not always represent a physical dimension. If a quantity depends on 30 different measurements, then immediately the problem involves 30 dimensions. Dimension is often a word used to refer to the number of variables in a problem. As a quick illustration, the formula for distance between 2 points in space depends on 6 numbers, so distance is really a 6 dimensional problem.

2.3 Vectors



A vector (written $\vec{v}$, or in bold face $\mathbf{v}$) is a magnitude in a certain direction. We can use vectors to represent forces, velocity, acceleration, and many other quantities. One way to think of a vector is to imagine an arrow, starting at the tail and ending at the head where we draw an arrow on the head. We know two vectors are equal if they both represent the same magnitude in the same direction, regardless of where the vectors are drawn. The two vectors on the right both represent the vector $\langle 2, -3\rangle$.

The vector which points one unit in the x direction we can write in many ways, such as $\mathbf{i} = \vec{i} = \langle 1,0,0\rangle$. Similarly we define $\mathbf{j} = \vec{j} = \langle 0,1,0\rangle$ to be the vector that points one unit in the y direction, and $\mathbf{k} = \vec{k} = \langle 0,0,1\rangle$ to be the vector that points one unit in the z direction.

We'll often vectors with their tail at the origin and their head at (v_1, v_2) for 2D vectors, or at (v_1, v_2, v_3) for 3D vectors. The component form of a vector $\vec{v}$ centered at the origin with head at (v_1, v_2, v_3) can be written $\langle v_1, v_2, v_3\rangle$ or $v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$. Many textbooks use bold face font to represent vectors. However, since it is rather difficult to write in bold face font on paper and chalkboards, I will write vectors with an arrow above them in the text, as this is how vectors are commonly expressed in writing other than texts. In addition, this book uses the form $\langle v_1, v_2, v_3\rangle$ much more often than $v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$.

2.4 Vector Arithmetic

We add and subtract vectors component wise, so $\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle$. I find that writing vectors vertically as column vectors make much of the arithmetic simpler, as then we can write

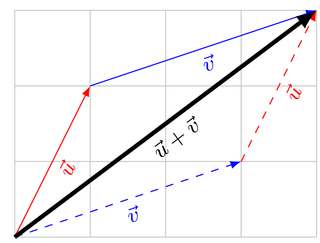
$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}.$$

Multiplying a vector by a scalar requires that we multiply each component by the scalar. As an example, we can combine addition, subtraction, and scalar multiplication to compute

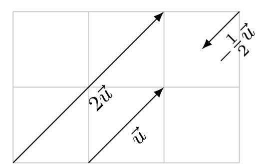
$$\langle 1, 3\rangle - 2\langle -1, 2\rangle + \langle 4, 0\rangle = \langle 1 - 2(-1) + 4, 3 - 2(2) + 0\rangle = \langle 7, -1\rangle .$$

If we write the vectors using column notation instead, we obtain

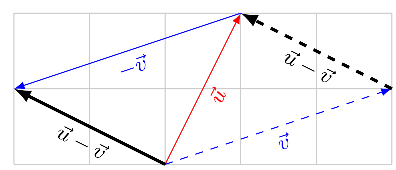
$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-2(-1)+4 \\ 3-2(2)+0 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}.$$



We can also perform vector addition geometrically. We draw the first vector anywhere we want. Then we draw the second vector with its tail located at the head of the first. The sum of these two vectors, which we call the resultant vector, is the vector which starts at the tail of the first and ends at the head of the second. We call this the parallelogram law of addition, as repeating the process with the order of the vectors switched gives the other half of the parallelogram, as seen in the picture on the right.



We can visualize scalar multiplication as equivalent to stretching a vector by the scalar. If the scalar is negative then the vector turns around to point in the opposite direction. The diagram to the right shows a vector as well as what happens if we times the vector by 2 or by $-\frac{1}{2}$.



We can combine addition and scalar multiplication to obtain a geometric way to subtract vector. Since we have $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$, then we start by drawing $\vec{u}$ and then draw $-\vec{v}$ with its tail is at the head of $\vec{u}$. Connecting from the tail of $\vec{u}$ to the head of $-\vec{v}$ gives the difference, as seen on the right. Notice also that if we drew both $\vec{u}$ and $\vec{v}$ with the same tail, then their difference $\vec{u} - \vec{v}$ connects the heads, pointing from the head of $\vec{v}$ to the head of $\vec{u}$. This gives us a simple way to connect a vector between two points.

2.4.1 Magnitude

We can find the magnitude (or length, or norm, or absolute value) of a vector by using the distance formula. We'll use either $||\vec{u}||$ or $|\vec{u}|$ to denote the magnitude. The distance formula gives the magnitude as

$$||\vec{u}|| = |\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

Example 2.1. The length of $\langle 1, 2, 4\rangle$ is $|\langle 1, 2, 4\rangle| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}$.

We call a unit vector a vector with magnitude 1. We can normalize a vector by dividing the vector by its magnitude, which makes the new vector a unit vector. We can write any vector as the product of its magnitude and a unit vector in the direction of the vector.

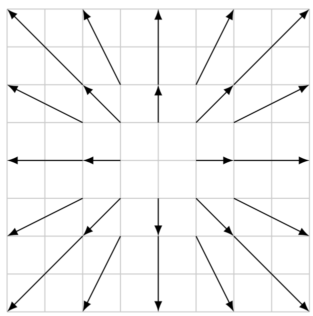
Example 2.2. We can write $\langle 1, 2, 4\rangle$ as the product of its magnitude and a unit vector by writing (recall the magnitude is $\sqrt{21}$)

$$\langle 1, 2, 4\rangle = \left(\sqrt{21}\right)\left(\frac{\langle 1, 2, 4\rangle}{\sqrt{21}}\right) = \underbrace{\left(\sqrt{21}\right)}_{\text{magnitude}} \underbrace{\left\langle \frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{4}{\sqrt{21}} \right\rangle}_{\text{unit vector}}.$$

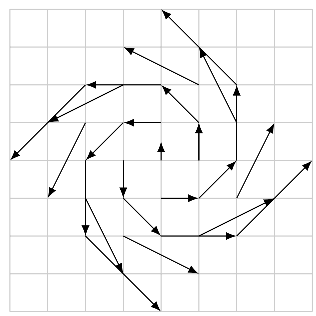
Example 2.3. Let's obtain a vector of length 5 which points in the same direction as $\langle -2, 3\rangle$. The magnitude of $\langle -2, 3\rangle$ is $\sqrt{(-2)^2 + 3^2} = \sqrt{13}$, which means a unit vector in the same direction is $\left\langle \frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$. A vector of length 5 in this direction is then $(5)\left(\left\langle \frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle\right) = \left\langle \frac{-10}{\sqrt{13}}, \frac{15}{\sqrt{13}} \right\rangle$.

2.4.2 Vector Fields

One of the most important uses of vectors is the idea of a vector field. At every point (x, y) in the plane, or (x, y, z) in space, we place a vector $\vec{F}(x, y)$, or $\vec{F}(x, y, z)$. Two types of vector fields which occur in nature are radial vector fields and spin fields, shown below. We'll study vector fields more as the semester progresses.



$\vec{F}(x, y) = \langle x, y\rangle$ (radial vector field)

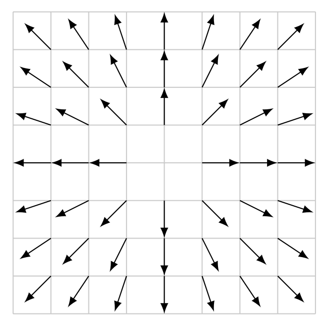


$\vec{F}(x, y) = \langle -y, x\rangle$ (spin field)

Example 2.4. Let's obtain a formula for a vector field such that at each point (x, y) other than the origin we obtain a vector of length 1 that points directly away from the origin. Note first of all that the vector $\langle x, y\rangle$ points from the origin to (x, y), so is precisely the direction we need. A unit vector in this direction is simply $\frac{\langle x, y\rangle}{\sqrt{x^2 + y^2}}$. A formula for our vector field is hence

$$\vec{F}(x, y) = \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right\rangle.$$

A graph of this vector field is shown to the right.



A vector field where every vector points away from the origin with magnitude one.

2.5 The Dot Product

We define the dot product of two vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ to be the scalar quantity

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i.$$

We define the dot product in other dimensions similarly, by just multiplying together corresponding components and then summing the products.

2.5.1 The dot product gives magnitudes and angles

The dot product helps find magnitude. Note that if $\vec{u} = \langle u_1, u_2, u_3\rangle$, then $\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + u_3^2$ which is the square of the magnitude, or symbolically we have

$$\vec{u} \cdot \vec{u} = |\vec{u}|^2.$$

In dimensions higher than three, we can use this relationship between the dot product and magnitude to define distance.

Using the law of cosines and vector subtraction, we can show that if θ is the angle between $\vec{u}$ and $\vec{v}$, then $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$ (I'll leave this as an exercise). We can then quickly compute the angle between both $\vec{u}$ and $\vec{v}$. We can rearrange the formula to compute the dot product using

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta.$$

This shows that the dot product is zero if and only if the angle between $\vec{u}$ and $\vec{v}$ is 90 degrees ($\pi/2$) radians or one of the vectors is the zero vector.

Definition 2.5: Orthogonal. We say that two vectors are *orthogonal* if their dot product is zero.

Note also that since $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$, then if $\theta < \pi/2$, we know the dot product is positive and if $\theta > \pi/2$ then the dot product is negative.

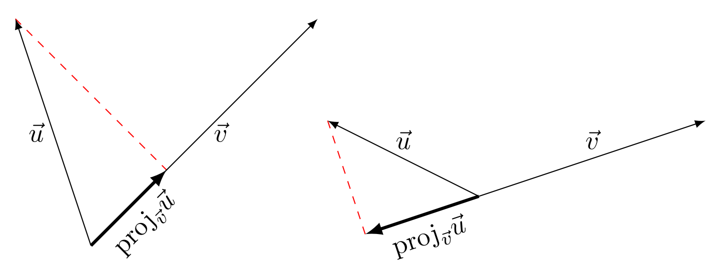
Example 2.6. The vectors $\langle 1,3,-2\rangle$ and $\langle 4,0,2\rangle$ are orthogonal because their dot product $1(4)+3(0)-2(2)=0$ is zero. The vectors $\langle 1,3,-2\rangle$ and $\langle 4,1,2\rangle$ are not orthogonal because their dot product $1(4)+3(1)-2(2)=3$ is not zero. In particular, since $|\langle 1,3,-2\rangle| = \sqrt{14}$ and $|\langle 4,1,2\rangle| = \sqrt{21}$, the angle θ between these two vectors satisfies

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{3}{\sqrt{14}\sqrt{21}}.$$

We can then find the angle between these two vectors using the inverse cosine function, namely $\theta = \cos^{-1}\left(\frac{3}{\sqrt{14}\sqrt{21}}\right)$.

2.5.2 Projections

The projection of $\vec{u}$ onto the vector $\vec{v}$, written $\text{proj}_{\vec{v}}\vec{u}$, is a vector parallel to $\vec{v}$ whose magnitude is the component of $\vec{u}$ in the direction of $\vec{v}$. We start by drawing both vectors with their base at the origin. We then create a right triangle with $\vec{u}$ as the hypotenuse and the adjacent edge on the line containing $\vec{v}$. We then have $\text{proj}_{\vec{v}}\vec{u}$ as the vector which starts at the origin and ends at the right angle, as shown below.



If the angle between the two vectors is less than 90 degrees, then the magnitude of the projection is

$$|\vec{u}|\cos\theta = |\vec{u}|\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}.$$

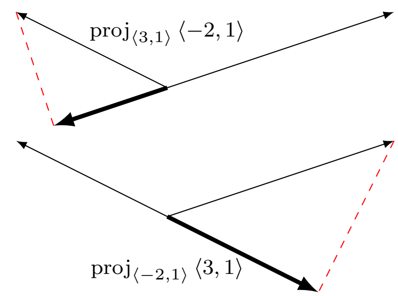
When the angle between the two vectors is greater than 90 degrees, the magnitude is opposite the quantity above. We call the quantity above the *scalar component* *of $\vec{u}$ in the direction of $\vec{v}$*. This scalar component keeps track of both magnitude and whether the projection points in the same direction as $\vec{v}$ or opposite $\vec{v}$. Since a unit vector in the direction of $\vec{v}$ is $\dfrac{\vec{v}}{|\vec{v}|}$, multiplying this vector by the scalar component of $\vec{u}$ in the direction of $\vec{v}$ gives us the projection formula

$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|}.$$

Since $|\vec{v}|^2 = \vec{v}\cdot\vec{v}$, we can rewrite the formula above as $\text{proj}_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$.

Example 2.7. The projection of $\langle -2, 1\rangle$ onto $\langle 3, 1\rangle$ is

$$\text{proj}_{\langle 3,1\rangle}\langle -2,1\rangle = \left(\frac{\langle -2,1\rangle\cdot\langle 3,1\rangle}{\langle 3,1\rangle\cdot\langle 3,1\rangle}\right)\langle 3,1\rangle = \frac{-5}{10}\langle 3,1\rangle = \left\langle \frac{-3}{2}, \frac{-1}{2}\right\rangle.$$



Reversing the order, the projection of $\langle 3, 1\rangle$ onto $\langle -2, 1\rangle$ is

$$\text{proj}_{\langle -2,1\rangle}\langle 3,1\rangle = \left(\frac{\langle 3,1\rangle\cdot\langle -2,1\rangle}{\langle -2,1\rangle\cdot\langle -2,1\rangle}\right)\langle -2,1\rangle = \frac{-5}{5}\langle -2,1\rangle = \langle 2,-1\rangle.$$

2.5.3 Work

When a force $\vec{F}$ and a displacement $\vec{d}$ are in the same direction, we define the work done by $\vec{F}$ acting through a displacement $\vec{d}$ to be $W = |\vec{F}||\vec{d}|$ (force times displacement). However if the force and displacement are in different directions, then we find the component of the force parallel to the displacement (the component of $\vec{F}$ in the direction of $\vec{d}$) and we use that quantity to compute the work. This gives us the more general work formula as $W = |\vec{F}|\cos\theta|\vec{d}|$, where θ is the angle between $\vec{F}$ and $\vec{d}$. Since we know $\cos\theta = \frac{\vec{F}\cdot\vec{d}}{|\vec{F}||\vec{d}|}$, we can simply the formula for work to just

$$W = |\vec{F}|\cos\theta|\vec{d}| = |\vec{F}|\frac{\vec{F}\cdot\vec{d}}{|\vec{F}||\vec{d}|}|\vec{d}| = \vec{F}\cdot\vec{d}.$$

Work is precisely the dot product of the force and the displace.

Example 2.8. Suppose a box moves down a ramp from the point $(0, 3)$m to the point $(6, 0)$m. Along this path, a constant force from gravity of $\vec{F} = \langle 0, -200\rangle$ Newtons acts on the box. There are other forces at play in this problem, but we want to focus on just the work done by gravity. What is the work done by the force $\vec{F}$ as the box moves down the ramp from $(0, 3)$ to $(0, 6)$?

We need a vector for the displacement. Recall that vector subtraction $\vec{u} - \vec{v}$ gets us a vector with tail at $\vec{v}$ and head at $\vec{u}$. We need a vector from $(0, 3)$ to $(6, 0)$, so the desired vector is

$$\vec{d} = \langle 6, 0\rangle - \langle 0, 3\rangle = \langle 6, -3\rangle.$$

Alternately, we could have just drawn a picture and realized that the box moved right 6 and down 3. Either way, the work (in Newton meters) done by $\vec{F}$ acting through the displacement $\vec{d}$ is then

$$W = \langle 0, -200\rangle\cdot\langle 6, -3\rangle = 600.$$

2.5.4 Parallel and orthogonal components

Notice that in finding work, we used the portion of $\vec{F}$ parallel to $\vec{d}$ to compute work. The portion of $\vec{F}$ orthogonal to $\vec{d}$ contributes nothing to the work done. We'll occasionally need to decompose a vector $\vec{F}$ as the sum of a vector parallel to $\vec{d}$ (written $\vec{F}_{\|\vec{d}}$) and a vector orthogonal to $\vec{d}$ (written $\vec{F}_{\perp\vec{d}}$). Since we know the vector parallel to $\vec{d}$ is the projection of $\vec{F}$ onto d, we have the formula

$$\vec{F} = \vec{F}_{\|\vec{d}} + \vec{F}_{\perp\vec{d}} = \text{proj}_{\vec{d}}\vec{F} + \vec{F}_{\perp\vec{d}}.$$

Solving for $\vec{F}_{\perp\vec{d}}$ gives the component of $\vec{F}$ orthogonal to $\vec{d}$ as

$$\vec{F}_{\perp\vec{d}} = \vec{F} - \text{proj}_{\vec{d}}\vec{F}).$$

Example 2.9. Let's return to the previous example where $\vec{F} = \langle 0, -200 \rangle$ and $\vec{d} = \langle 6, -3 \rangle$. We can compute the component of $\vec{F}$ parallel to $\vec{d}$ as

$$\vec{F}_{\|\vec{d}} = \text{proj}_{\langle 6,-3 \rangle} \langle 0, -200 \rangle = \frac{\langle 0, -200 \rangle \cdot \langle 6, -3 \rangle}{|\langle 6, -3 \rangle|} \frac{\langle 6, -3 \rangle}{|\langle 6, -3 \rangle|} = \langle 80, -40 \rangle .$$

We can then decompose the force as the sum

$$\vec{F} = \langle 0, -200 \rangle = \langle 80, -40 \rangle + (\langle 0, -200 \rangle - \langle 80, -40 \rangle) = \langle 80, -40 \rangle + \langle -80, -160 \rangle .$$

Remember that the only portion of the force that contributes to the work is the parallel component. We could have computed the work by first computing the projection, and then using the formula

$$W = |F_{\|\vec{d}}||\vec{d}| = |\langle 80, -40 \rangle||\langle 6, -3 \rangle| = \sqrt{6400 + 1600}\sqrt{36 + 9} = 600.$$

We obtained the exact same answer, but it took a whole lot more effort.

2.6 The Cross Product

The cross product of $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is a new vector

$$\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

Note that $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$. Because of this, we say that the cross product is anti-commutative. Be careful not to switch the order on the cross product.

2.6.1 The cross product is orthogonal to both vectors

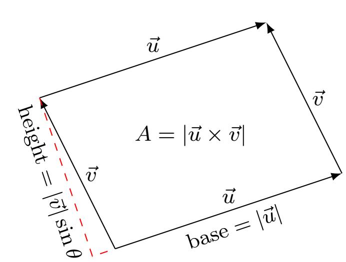
The cross product of two vectors always is orthogonal to both $\vec{u}$ and $\vec{v}$. We could verify this by performing the dot products $\vec{u} \cdot (\vec{u} \times \vec{v})$ and $\vec{v} \cdot (\vec{u} \times \vec{v})$ and showing that in each case we get zero.

2.6.2 The magnitude of the cross product gives area

We could also show that the magnitude of the cross product is

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin \theta.$$

This formula is very similar to the dot product formula, but involves $\sin \theta$ instead of $\cos \theta$. The area of the parallelogram formed using the two vectors $\vec{u}$ and $\vec{v}$ has the same area, as the base length is $|\vec{u}|$ and the height is $|\vec{v}| \sin \theta$, as shown in the picture on the right. The key to remember is that we can use the cross product to find the area of a parallelogram.



2.6.3 Direction comes from the right hand rule

Two vectors $\vec{u}$ and $\vec{v}$ generally determine a plane in space. Since the cross product is orthogonal to both $\vec{u}$ and $\vec{v}$, it must point away from this plane in one of two directions. We can visually obtain the direction of the cross product using the right hand rule. Place the base of your right hand on the first vector with the inside of your hand facing the second vector. As you curl your right hand from the first vector to the second vector, your thumb will point in the direction of the cross product. Alternately, you can place your right index finger pointing in the direction of the first vector and your middle finger pointing in the direction of $\vec{v}$. The direction of $\vec{u} \times \vec{v}$ is in the direction of your thumb.

2.6.4 Torque, or moment of force

There are many other applications of the cross product. Most engineers will compute torque (or moment, or moment of force) using the formula $\vec{\tau} = \vec{r} \times \vec{F}$ to find the tendency of the force to rotate an object about an axis.

Example 2.10. A vector which is orthogonal to both $\langle 1, -2, 3\rangle$ and $\langle 2, 0, -1\rangle$ is the cross product

$$\begin{aligned}\langle 1, -2, 3\rangle \times \langle 2, 0, -1\rangle &= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 2 & 0 & -1 \end{bmatrix} \\ &= \langle (-2)(-1) - (0)(3), -[(1)(-1) - (2)(3)], (1)(0) - (2)(-2)\rangle \\ &= \langle 2, 7, 4\rangle .\end{aligned}$$

Notice that if we reverse the order, then we obtain $\langle 2, 0, -1\rangle \times \langle 1, -2, 3\rangle = -\langle 2, 7, 4\rangle$ which is also orthogonal to both $\langle 1, -2, 3\rangle$ and $\langle 2, 0, -1\rangle$ but points in the opposite direction.

In addition, the area of the parallelogram formed by the vectors $\langle 1, -2, 3\rangle$ and $\langle 2, 0, -1\rangle$ is $|\langle 2, 7, 4\rangle| = \sqrt{4 + 49 + 16} = \sqrt{69}$. If we wanted the area of the triangle with vertices $(0, 0, 0)$, $(1, -2, 3)$, and $(2, 0, -1)$, then we would just half this amount.

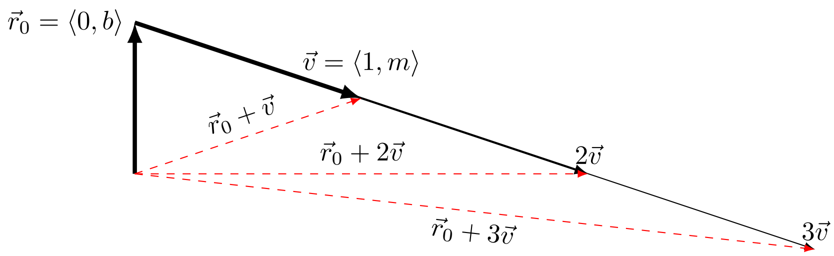
2.7 Lines and planes

2.7.1 Lines

Back in college algebra and high school, we wrote the equation of a line as $y = mx + b$. The slope m tells us a direction of over 1 and up m. This is a vector, which we can write in the component form $\vec{v} = \langle 1, m\rangle$ (a one unit increase in x results in an increase of m units in the y direction). The y-intercept b gives us a starting point $(0, b)$ which we can also write in the vector form $\vec{r}_0 = \langle 0, b\rangle$. This is the point from which we begin our graph. In vector form, we can write an equation of the line $y = mx + b$ as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix} x + \begin{pmatrix} 0 \\ b \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \vec{v}x + \vec{r}_0.$$

Look back at the equation above. Do you notice that we are scaling the vector $\langle 1, m\rangle$ by some amount (x) and then adding that to the vector $\langle 0, b\rangle$. We can think of this geometrically by drawing the starting point $\langle 0, b\rangle$ and placing the scaled vector $\langle 1, m\rangle x$ with its tail on the head of the starting point, as seen below.



Rather than writing $\langle x, y\rangle$ (or $\langle x, y, z\rangle$), we'll use the vector $\vec{r}$ to stand for the radial vector which starts at $(0,0)$ (or $(0,0,0)$) and points radially outwards to the point (x, y) (or (x, y, z)). We can then write the equation above as $\vec{r} = \vec{v}x + \vec{r}_0$ or in function notation as $\vec{r}(x) = \vec{v}x + \vec{r}_0$ to emphasize that the variable x is the parameter we are free to choose. We'll often use the variable t instead of x as our parameter, which means we would write an equation of a line in the form

$$\vec{r}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \vec{v}t + \vec{r}_0.$$

The benefit of writing an equation of a line in this vector form is that now given any starting point $\vec{r}_0$ and direction vector $\vec{v}$, a vector equation of the line is simply $\vec{r}(t) = \vec{v}t + \vec{r}_0$. The tips of the vectors $\vec{r}(t)$ whose tails are always at the origin will trace out the line.

Example 2.11. Let's give an equation of the the line which passes through the points $P(1, 2, 3)$ and $Q(0, -1, 4)$. To get the direction vector, we think of each point as a vector and then subtract, giving us

$$\vec{PQ} = \vec{Q} - \vec{P} = \langle 0-1, -1-2, 4-3\rangle = \langle -1, -3, 1\rangle .$$

We can now use either point as our start point $\vec{r}_0$ to obtain two different vector equations of the line, namely

$$\vec{r}_1(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -t+1 \\ -3t+2 \\ t+3 \end{pmatrix} \quad \text{and}$$
$$\vec{r}_2(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -t \\ -3t-1 \\ t+4 \end{pmatrix}.$$

To find a vector equation of the line parallel to the line $\vec{r}(t) = \langle 3t, -5t+2, 8t-7\rangle$ which passes through the point $(2, -8, 1)$, we need a direction vector and a point. The direction vector is parallel to the direction vector of the given line, so we use $\vec{v} = \langle 3, -5, 8\rangle$. The point was given to us as $(2, -8, 1)$, so an equation of the line is $\vec{r}(t) = \langle 3t+2, -5t-8, 8t+1\rangle$.

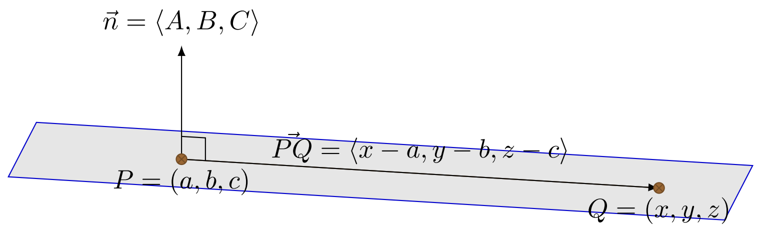
In the example above we obtained two different vector equations for the same line. In higher dimensions, we'll find that there are often many ways of expressing an equation for the same object. How can we tell that two vector equations represent the same line? The next example addresses this.

Example 2.12. Consider the line given by $\vec{r}(t) = \langle 1, 2\rangle\, t + \langle 3, 4\rangle$. A direction vector for this line is $\vec{v} = \langle 1, 2\rangle$, however any scalar multiple of this vector will be parallel to the line as well. Notice also that if we let $t = 2$, then the line passes through the point $(5, 8)$. Using $\vec{v} = \langle -3, -6\rangle$ and $\vec{r}_0 = \langle 5, 8\rangle$, another vector equation for the same line is $\vec{r}(t) = \langle -3, -6\rangle\, t + \langle 5, 8\rangle$.

How can we tell that these two vector equations represent the same line? One way is to remove the parameter t. From the second equation we know that $x = -3t + 5$ and $y = -6t + 8$. Solving both of these equations for t and setting them equal gives $\frac{x-5}{-3} = \frac{y-8}{-6}$ or solving for y gives $y = 2(x-5) + 8 = 2x - 2$. Repeating this with the first equation gives $\frac{x-3}{1} = \frac{y-4}{2}$ or solving for y gives $y = 2(x-3) + 4 = 2x - 2$, the same the second line.

2.7.2 Planes

Let's conclude this chapter by exploring how to obtain an equation of a plane in space. We say a vector is normal to a plane if the vector is orthogonal to every vector which lies in the plane. A normal vector "sticks out" of a plane at a right angle. If we have a point $P = (a, b, c)$ on a plane, and a normal vector $\vec{n} = \langle A, B, C\rangle$ to the plane, then we can quickly obtain the equation of the plane. For any point $Q = (x, y, z)$ in the plane, the vector $\vec{PQ} = \langle x - a, y - b, z - c\rangle$ is a vector in the plane and hence must be orthogonal to $\vec{n}$, as shown in the picture below.



Since orthogonal vectors have a dot product of zero, this gives an equation of the plane as

$$\vec{n} \cdot \vec{PQ} = 0.$$

We can rewrite this equation in any of the equivalent forms below:

$$\begin{aligned}
\langle A, B, C\rangle \cdot \langle x - a, y - b, z - c\rangle &= 0\\
A(x - a) + B(y - b) + C(z - c) &= 0\\
Ax + By + Cz &= D,
\end{aligned}$$

where D is the constant $D = Aa + Bb + Cc$. The key to obtaining an equation of a plane is to find a point on the plane and a normal vector. Any equation of the form $Ax + By + Cz = D$ is an equation of a plane, where the normal vector is $\vec{n} = \langle A, B, C\rangle$ and we can find a point on the plane by guessing any triple (x, y, z) that satisfies the equation.

Example 2.13. We can find a normal vector for the plane which passes through the points $P = (1, 0, 0)$, $Q = (2, 0, -1)$, and $R = (0, 1, 3)$ by using the cross product to obtain

$$\vec{n} = \vec{PQ} \times \vec{PR} = \langle 1, 0, -1\rangle \times \langle -1, 1, 3\rangle = \langle 1, -2, 1\rangle .$$

We can pick any of the three points to obtain an equation of the plane, so choosing P gives us the equation

$$1(x - 1) - 2(y - 0) + 1(z - 0) = 0 \quad \text{or} \quad x - 2y + z = 1.$$

If we instead use the point Q, then we obtain

$$1(x - 2) - 2(y - 0) + 1(z + 1) = 0 \quad \text{or again} \quad x - 2y + z = 1.$$

It does'nt matter which point you use, you will obtain the same equation.

Example 2.14. To find a normal vector for the plane containing the two intersecting lines $\vec{r}_1(t) = \langle 1+t, 3t, 2\rangle$ and $\vec{r}_2(t) = \langle 2+2t, 3, 2-t\rangle$, we just compute the cross product of the direction vectors for each line, namely

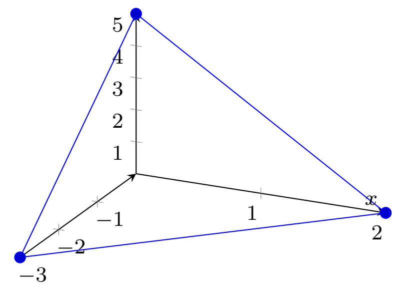
$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle 1, 3, 0\rangle \times \langle 2, 0, -1\rangle = \langle -3, 1, -6\rangle .$$

Since the plane contains both lines, we can use any point on either line as a point on the plane. Let's use $\vec{r}_1(0) = \langle 1, 0, 2\rangle$ which gives an equation of the plane as $-3(x-1) + 1(y-0) - 6(z-2) = 0$.

When two planes intersect, then they will generally intersect in a line. We need to find an equation of this line. We can find a direction vector for the line of intersection by computing the cross product of normal vectors to each plane. If that cross product is zero, then the two planes are parallel. Otherwise, we can use the dot product of the two normal vectors to find the angle of intersection of the planes.

To sketch a plane, we have to plot 3 non collinear points. The easiest way to plot a plane is to write the equation of the plane in the form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$



We can graph the plane $\frac{x}{2} + \frac{y}{-3} + \frac{z}{5} = 1$ by connecting the points $(2,0,0)$, $(0,-3,0)$, and $(0,0,5)$.

When written in this form, the plane passes through the coordinate axes at the points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$. As an exercise, take a moment to sketch the planes $2x + 3y + z = 6$, $x - 4y = 8$, and $\frac{x}{2} + \frac{y}{3} + z = 1$.

Using the dot product, cross product, and projections, we can derive the following formulas for distances between points, lines, and planes.

- The distance from a point Q to a plane (with normal vector $\vec{n}$ and a point P) is given by $|\text{proj}_{\vec{n}} \vec{PQ}|$.
- The distance from a point Q to a line (with direction vector $\vec{v}$ passing through P) is $|\vec{PQ} - \text{proj}_{\vec{v}} \vec{PQ}|$.
- The distance from a line (with direction vector $\vec{v}_1$ passing through P_1) to a line (with direction vector $\vec{v}_2$ passing through P_2) is $|\text{proj}_{\vec{v}_1 \times \vec{v}_2} \vec{P_1P_2}|$.

You can obtain all these formulas by drawing an appropriate diagram and then use some facts from earlier in the chapter. Make sure you can explain why each of these formulas is correct.

Chapter 3

Curves

After completing this chapter, you should be able to do the following:

1. Describe, graph, give equations of, and find foci for conic section such as parabolas, ellipses, and hyperbolas.
2. Model motion in the plane using parametric equations. In particular, describe conic sections using parametric equations.
3. Find derivatives and tangent lines for parametric equations.
4. Use integrals to find the lengths of parametric curves and the surface area of surfaces of revolution.

3.1 Preparation and Homework Suggestions

	Preparation Problems (11th ed)				Preparation Problems (12th ed)			
Day 1	10.1:17	10.1:33	10.1:55	10.1:59	11.6:17	11.6:33	11.6:55	11.6:59
Day 2	3.5:85	10.4:3	10.4:7	3.5:95	11.1.1	11.1:7	11.1:16	11.1:21
Day 3	3.5:101	10.4:12	6.3:3	6.5:33	11.2:1	11.1:18	11.2:26	11.2:31

The following homework problems line up with the topics we will discuss in class. Draw from these to do your homework. Remember that if you work through the details from examples in the book, you can count that as homework. You will want to make sure you can do the basic practice problems to pass the class, and be able to do the good problems to get an A.

Topic (11th ed)	Sec	Basic Practice	Good Problems	Thy/App	Examples
Conics	10.1	1-38	39-74, 83, 86, 88	75, 76, 81, 89-94	E1-5
Parametric Equations	3.5	81-94	95-108		E10-15
Parametric Equations	10.4	1-12,19-26	13-18		E1-2
Arc length	6.3	1-8	19-26		E1
Surface Area	6.5	33-38			E3-4

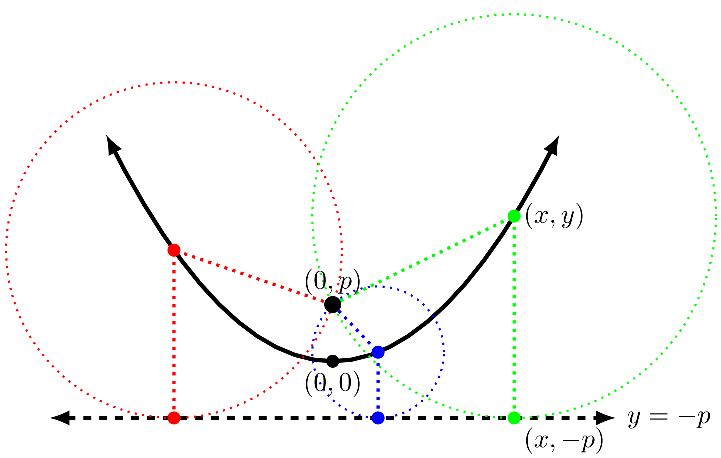
Topic (12th ed)	Sec	Basic Practice	Good Problems	Thy/App
Conics	11.6	1-38	39-68	69-80
Arc length	6.3	1-8	19-26	
Surface Area	6.5	33-38		
Parametric Equations	11.1	1-18, 19-24,	25-33	34-40
Calc. with Par. Eqns.	11.2	1-14, 25- 34	15-24, 35-36	41-48

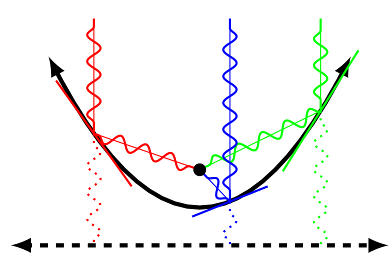
As you do homework, start with the basic practice problems. Work your way down the basic problems list and then go back to the good problems. The repetition will help you learn the vocabulary.

3.2 Conic Sections

3.2.1 Parabola

When I hear parabola, I often think $y = x^2$. However, there's much more to a parabola. Given a point (called the focus) and a line (called the directrix) which does not pass through that point, a parabola is the set of all points in the plane that are equidistant from the point and the line. The vertex is the point on the parabola closest to the directrix. A parabola will open away from the directrix. If the directrix is given by the equation $y = -p$ and the focus is the point $(0, p)$, then using the distance formula it can be shown that $x^2 = 4py$, and the parabola opens symmetrically upwards along the y-axis. Interchanging x and y shows that if the directrix is a vertical line $x = -p$ and the focus is $(p, 0)$, we get the equation $y^2 = 4px$.



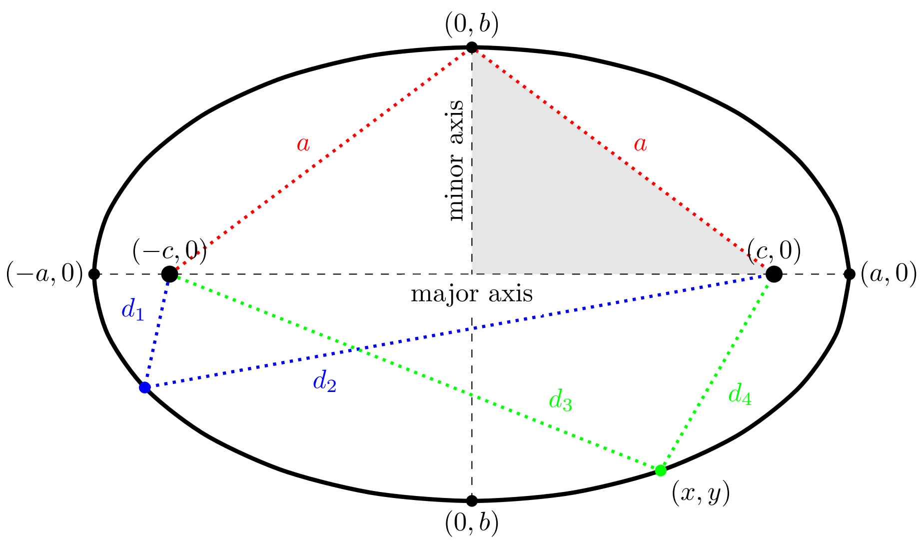


Parabolas have a very useful reflective property. If a ray of light approaches the parabola at a right angle with the directrix, then when it reflects off the parabola it will reflect toward the focus. A large parabola can gather electromagnetic waves which are aimed at the directrix, and combine all those waves so that all pass through the focus. Satellite dishes and long range telescopes utilize this property of parabolas.

The equation of a parabola with focus $(0, 3)$ and directrix $y = -3$ is $x^2 = 4{\cdot}3{\cdot}y$. If the focus is $(0, 3)$ and the directrix is $y = 1$, then notice that the vertex is at $(0, 2)$. The distance from the vertex to focus is $p = 1$, so using properties of shifting graphs we have $(x - 0)^2 = 4(1)(y - 2)$. If the focus is $(2, 5)$ and the directrix is $y = -3$, then the vertex is at $(2, 1)$, and the distance from the focus to the vertex is $p = 4$. Hence we have $(x-2)^2 = 4(4)(y-1)$ as the equation. If the parabola opens down instead of up, just multiply p by a negative number. Similarly, if the parabola opens left, with directrix $x = 0$ and focus $(-6, 1)$, then the vertex is at $(-3, 1)$, the distance from the focus to the vertex is $p = 3$, and the equation is $(y-1)^2 = 4(-3)(x+3)$.

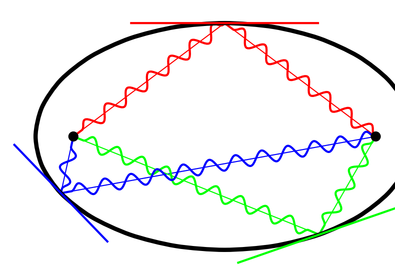
3.2.2 Ellipse

Given two points (called foci) in the plane and a fixed number, an ellipse is the set of points in the plane where the sum of the distances between the point and the two foci is the fixed number. The major axis is the segment inside the ellipse containing the two foci (each endpoint of this segment is called a vertex), the minor axis is the largest segment inside the ellipse which meets the major axis perpendicularly.



If the foci are located at $(\pm c, 0)$ and the vertices are at $(\pm a, 0)$, then using the distance formula you can show that an equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$. The distance from the center of the ellipse to the end of a minor axis we will call b, and it turns out that $b^2 = a^2 - c^2$. Hence the equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Of course we can shift the ellipse to have any center (h, k) and then the equation is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. If the foci are on the y-axis instead, then you just interchange x and y.

Ellipses have an interesting reflective property. A wave which starts at one foci will reflect off the ellipse and pass through the other foci. The Tabernacle at Temple Square has a roof with elliptical parts. One foci is at the pulpit, and the other is located in a roped off section of the audience. You can drop a tiny pin at the pulpit and hear it from the other foci. It sounds like a bag of nails was dropped. The reason why is that in every direction sound waves are created, and all of those sound waves converge at the other foci.



Let's consider an ellipse with foci $(1, \pm 2)$ and vertices $(1, \pm 3)$. The center of the ellipse is in the middle of the foci, located at $(1, 0)$. The ellipse has major axis parallel to the y-axis. If it were centered at the origin, we would have the equation $\frac{y^2}{a^2}+\frac{x^2}{a^2-c^2}=1$, or $\frac{y^2}{3^2}+\frac{y^2}{3^2-2^2}=1$. Since its center is at $(1,0)$, we write $\frac{(x-1)^2}{5}+\frac{(y-0)^2}{9}=1$. You can tell immediately that the ellipse has its larger side in the y direction because the number under y is larger.

The graph of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ is an ellipse centered at $(0,0)$ where the vertices are at $(\pm 5,0)$ and the ends of the minor axis are at $(0,\pm 4)$ ($a^2=25, b^2=16$). The computation $c^2=a^2-b^2=9$ shows that $c=3$, so the foci are at $(\pm 3,0)$.

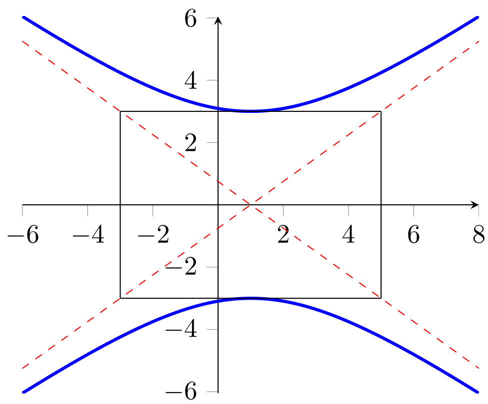
3.2.3 Hyperbola

Given two points (called foci) in the plane and a fixed number, a hyperbola is the set of points in the plane where the difference of the distances between the point and the two foci is the fixed number. The focal axis is the line containing the two foci. The vertices are the points of the hyperbola on the focal axis. Hyperbolas have an interesting reflective property. A wave which is headed toward one focus will reflect off the hyperbola and start heading toward the other focus. Hyperbolas are used in long range telescopes, as well as in triangulation (finding the location of somethings based on three measurements).

If the foci are located at $(\pm c,0)$ and the vertices are at $(\pm a,0)$, then using the distance formula you can show that an equation of a hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1$. If we define b so that $b^2=c^2-a^2$, then we can write the equation as $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Of course we can shift the hyperbola to have any center (h,k) and then the equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$. If the foci are on the y-axis instead, then you just interchange x and y (so that the negative sign is in front of the x terms instead of the y term).

To graph a hyperbola of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we first notice that hyperbolas follow an asymptote. Solving for y we obtain $y^2=b^2\left(\frac{x^2}{a^2}-1\right)=\frac{b^2}{a^2}x^2\left(1-\frac{a^2}{x^2}\right)$. This means that $y=\pm\frac{b}{a}x\sqrt{1-\frac{a^2}{x^2}}$. As x grows, $\sqrt{1-\frac{a^2}{x^2}}$ approaches one, and so we see that the hyperbola follows the two lines $y=\pm\frac{b}{a}x$ as $x\to\infty$. To quickly make these asymptotes, draw a rectangle which extends a to the left and right of the center, and b up and down from the center of the hyperbola. The asymptotes are formed by extending the diagonals of this rectangle. Then the hyperbola is drawn by plotting the vertices on the focal axis, and extending the edges to follow the asymptotes.

Let's consider a hyperbola with foci $(1,\pm 5)$ and vertices $(1,\pm 3)$. The center of the hyperbola is in the middle of the foci, located at $(1,0)$. The focal axis is parallel to the y-axis. If it were centered at the origin, we would have the equation $\frac{y^2}{a^2}-\frac{x^2}{c^2-a^2}=1$, or $\frac{y^2}{3^2}-\frac{y^2}{5^2-3^2}=1$. Since its center is at $(1,0)$, we write $-\frac{(x-1)^2}{16}+\frac{(y-0)^2}{9}=1$. The graph is formed by making a rectangle with edges $x=1\pm 4, y=0\pm 3$, then plotting the vertices at 1 ± 3, and extending the hyperbola to follow the asymptotes, as shown in the figure below.



3.2.4 Reflective Properties

We can use the the conic sections above to reflect light, sound, and radio waves and obtain some useful tools. Satellite dishes use the reflective property of a parabola to focus several signals at one point. If we build an elliptical wall, then sound waves that emanate from one focus will bounce of the wall at every point and travel to the other focus. A ray headed towards one focus of a hyperbola can bounce off the hyperbola and head towards the other focus. The Figure ... gives visual description of each property.u Let's look at each conic section one at a time.

3.2.5 Completing the square

We say that a quadratic polynomial $y = ax^2 + bx + c$ is a perfect square if we can factor it as $y = (dx + e)^2$ for some d and e. For example $x^2 + 2x + 1 = (x + 1)^2$ and $(2x - 3)^2 = 4x^2 - 12x + 9$ are perfect squares. If a quadratic is not a perfect square, we can complete the square as follows. The polynomial $x^2 + 2x$ is not a perfect square. However $x^2 + 2x = x^2 + 2x + 1 - 1 = (x + 1)^2 - 1$. We complete the square by adding and subtracting a constant term which will allow us to factor the quadratic as a square, with a constant left over. We can use completing the square to find the centers of ellipses and hyperbolas. Recall that in general if we have $x^2 + bx$, we can complete the square by writing $x^2 + bx + (b/2)^2 - (b/2)^2 = (x + b/2)^2 - (b/2)^2$.

The equation $x^2 + 2x + y^2 - 6y + 6 = 0$ is actually the equation of an ellipse. We can complete the square by looking at the x and y terms separately. We write $x^2 + 2x = x^2 + 2x + 1 - 1 = (x + 1)^2 - 1$, and for the y terms we write $y^2 - 6y = y^2 - 6y + 9 - 9 = (y - 3)^2 - 9$. Then we put these into the original equation to obtain $((x+1)^2 - 1) + ((y-3)^2 - 9) + 6 = 0$, or $(x+1)^2 + (y-3)^2 = 4$ which is a circle of radius 2 center at $(-1, 3)$ (recall that circles are ellipses). Similarly, we can show $9x^2 - 6y^2 + 36y = 0$ is a hyperbola. We write

$$-6y^2+36y = -6(y^2-6y) = -6(y^2-6y+9-9) = -6((y-3)^2-9) = -6(y-3)^2+54.$$

Hence the equation $9x^2 - 6y^2 + 36y = 0$ becomes $9x^2 - 6(y - 3)^2 + 72 = 0$, which we can write as

$$-\frac{x^2}{8} + \frac{(y-3)^2}{12} = 1$$

and recognize as a hyperbola which opens up and down, centered at $(0, 3)$.

3.3 Parametric Equations

How can we describe motion in the plane? One idea is to let t represent time, and then write two equations $x = f(t), y = g(t)$ to describe the (x, y) location of the point at any time t. This gives us our first example of a mathematical function where we put in a number t and get out a vector $\langle x, y\rangle$. This approach gives us a valuable way to study motion in the plane. The equations $x(t), y(t)$ are called parametric equations of the curve. Written in terms of vectors, we write $\vec{r}(t) = \langle x(t), y(t)\rangle$, where the vector $\vec{r}$ is commonly used to represent position. Throughout the semester we will be extending the language of functions to allow use to put in vectors and get out vectors. Here are some examples of parametric curves

1. (Circles) The parametric equations $x = \cos t, y = \sin t$ for $0 \leq t \leq 2\pi$ describe motion along a circle of radius 1 in the plane. In vector form we write $\vec{r}(t) = \langle \cos t, \sin t\rangle$. You can convince yourself of this by plotting the points for various values of t (make a t, x, y table and start plotting points). Alternatively, if you recall that $\cos^2 t + \sin^2 t = 1$, then you can replace $\cos t$ with x and $\sin t$ with y to obtain $x^2 + y^2 = 1$, a Cartesian equation for a circle of radius 1.

2. (Ellipes) Similarly you can show that $x = a\cos t, y = b\sin t$ ($\vec{r}(t) = \langle a\cos t, b\sin t\rangle$) for $0 \leq t \leq 2\pi$ gives equations for an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

3. (Parabolas) The graph of a function of the form $y = f(x)$ can always be written parametrically as $x = t, y = f(t)$, and then any equation we studied in first semester calculus can be analyzed using parametric equations. The parabola $y = x^2$ can be parametrized using $x = t, y = t^2$ or as a vector equation $\vec{r}(t) = \left\langle t, t^2\right\rangle$.

4. (Hyperbolas) Using the identity $\cosh^2 x - \sinh^2 x = 1$, the parametric equations $x = \cosh t, y = \sinh t$ give parametric equations for a hyperbola $x^2 - y^2 = 1$, where $-\infty < t < \infty$. Because $\cosh x$ and $\sinh x$ trace out a hyperbola as parametric equations, they are called the hyperbolic trigonometric functions.

5. (Lines) To parametrize the line segment from $(1, 2)$ to $(3, 3)$, think of a particle starting at $(1, 2)$ and moving straight towards $(3, 3)$ so that it takes 1 second to get there. The x value, which started at 1, increased 2 units per second, so we write $x = 1 + 2t$. The y value, which started at 2, increased 1 unit per second, so we write $y = 2 + t$. The equations $x = 1+2t, y = 2+t$ for $0 \leq t \leq 1$ are now parametric equations for the line. In terms of vectors, recall the vector equation $\vec{r}(t) = \vec{m}t + \vec{b}$ where $\vec{m}$ is a direction vector for the line and $\vec{b}$ is a point on the line. To get the direction vector, we use "head minus tail" to obtain $\vec{m} = \langle 3, 3\rangle - \langle 1, 2\rangle = \langle 2, 1\rangle$. Since we are starting at $(1, 2)$, we'll use the point $\vec{b} = \langle 1, 2\rangle$. We now have the vector equation $\vec{r}(t) = \langle 2, 1\rangle t + \langle 1, 2\rangle = \langle 2t + 1, t + 2\rangle$, which matches our parametric equations. I remember this as "head minus tail times t plus the tail."

We need to learn how to write parametric equations given a curve, and we need to learn how to find a Cartesian equation (remove t from the equations) when we are given parametric equations. We will practice this idea in class all semester long. Obtaining a skill in parameterizing curves requires lots of practice.

3.3.1 Derivatives and Tangent lines

Let's now look at how to find the slope $\frac{dy}{dx}$ of a parametric curve? The chain rule gives us a quick answer, we just divide the top and bottom by dt which gives

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Often times we'll use dots to denote the derivative with respect to t and primes to denote the derivative with respect to x. Using this convention, we can write the first derivative as $dy/dx = \dot{y}/\dot{x}$. We can use this formula to find tangent lines and more for parametric curves. To find the second derivative $\frac{d^2y}{dt^2}$, we first find $y' = \frac{dy}{dx}$ and then repeat the process to compute $\frac{d(y')}{dx} = \frac{d(y')/dt}{dx/dt}$ (again dividing the top and bottom by dt).

We don't really divide by dt, rather we think of y as a function of x, use the chain rule to compute $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, and then solve for dy/dx.

Example 3.1. Consider the circle given by the parametric equations $x = \cos t$ and $y = \sin t$. The derivatives with respect to t are

$$\dot{x} = dx/dt = -\sin t \quad \text{and} \quad \dot{y} = dy/dt = \cos t.$$

The derivative of y with respect to x is then

$$dy/dx = \dot{y}/\dot{x} = \frac{\cos t}{-\sin t} = -\cot t.$$

The derivative of y' with respect to t is $(-cott)' = csc^2 t$. Dividing this by $\dot{x} = -\sin t$ gives $\frac{d^2y}{dx^2} = \csc^3 t$. To find the tangent line to the curve at $t = \pi$, we simply compute $x = \sqrt{2}/2$, $y = \sqrt{2}/2$, and $dy/dx = -\cot(\pi/4) = -1$. Using the point-slope form of a line, an equation of the tangent line is $(y - \sqrt{2}/2) = -1(x - \sqrt{2}2)$.

Now let's look at finding derivatives using vectors. We can find a tangent vector to the curve, and then use that to give equations for the tangent line. Given the vector equation $\vec{r}(t) = \langle x(t), y(t)\rangle$, we know the derivative is simply $\vec{r}'(t) = \langle x'(t), y'(t)\rangle$. Hence the rise is $y'(t)$ when the run is $x'(t)$, which gives us the slope as $y'(t)/x'(t)$. This vector $\vec{r}'(t)$ gives us a velocity vector for an object moving through space. The length of this vector is the speed of the object and the second derivative is the acceleration. Most of the ideas learned from first semester calculus transfer directly over to vector form. The vector form is often easier to remember, as you just take the derivative of each component.

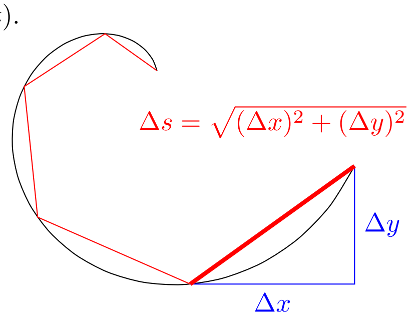
Example 3.2. Let's revisit the example above, but this time using vectors. The derivative of $\vec{r}(t) = \langle \cos t, \sin t\rangle$ is $\vec{r}'(t) = \langle -\sin t, \cos t\rangle$. At $t = \pi/4$ the position is $\vec{b} = \vec{r}(\pi/4) = \left\langle \sqrt{2}/2, \sqrt{2}/2 \right\rangle$ and a direction vector is $\vec{r}'(\pi/4) = \left\langle -\sqrt{2}/2, \sqrt{2}/2 \right\rangle$. This means that a vector equation of the tangent line is $\vec{r}(t) = \left\langle -\sqrt{2}/2, \sqrt{2}/2 \right\rangle t + \left\langle \sqrt{2}/2, \sqrt{2}/2 \right\rangle$ or just $\vec{r}(t) = \left\langle -\sqrt{2}/2t + \sqrt{2}/2, \sqrt{2}/2 + \sqrt{2}/2 \right\rangle$. The line has parametric equations $x = -\sqrt{2}/2t + \sqrt{2}/2$ and $y = \sqrt{2}/2t + \sqrt{2}/2$. We can convert these to a Cartesian equation by solving the first equation for t to obtain $t = (x - \sqrt{2}/2)/(-\sqrt{2}/2)$ and then replacing t in the second equation to obtain $y = -(x - \sqrt{2}/2) + \sqrt{2}/2$ (the same equation as before).

3.3.2 Arc Length

Recall that to find the length (Δs) along a straight line, we just need the change in x (Δx) and the change in y (Δy). The length is the hypotenuse of a right triangle with edge lengths Δx and Δy which means $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$. When a curve is not straight, we start by breaking the curve up into a bunch of small pieces of length Δs_i. Along each small piece, the curve is approximately straight, so we approximate each Δs_i with $\sqrt{\Delta x_i^2 + \Delta y_i^2}$. In differential notation this becomes $ds = \sqrt{dx^2 + dy^2}$. To find arc length s, we add up $(\sum ds)$ the little pieces of arc length and take a limit to get $s = \int_C ds = \int_C \sqrt{dx^2 + dy^2}$. For parametric equations $x(t), y(t), a \leq t \leq b$ we multiply by $\frac{dt}{dt}$ to obtain $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$. If we use vector form $\vec{r}(t) = \langle x, y \rangle$, then the derivative $\vec{r}' = \langle x', y' \rangle$ has length $\sqrt{(x')^2 + (y')^2}$ which equals the integrand in the parametric form, which means we can write

$$s = \int_C ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_C |\vec{r}'| dt.$$

I like to remember $ds = |\vec{r}'|dt$ where distance (ds) equals speed $(|r'|)$ times time (dt).



Example 3.3. Let's find the length of the curve $x = t, y = t^2$ for $-2 \leq t \leq 3$. In vector form we have $\vec{r}(t) = \langle t, t^2 \rangle$. The velocity vector is $\vec{r}'(t) = \langle 1, 2t \rangle$, which means the speed is $|\vec{r}'(t)| = \sqrt{1 + 4t^2}$. The arc length is then

$$\int ds = \int |\vec{r}'| dt = \int_{-2}^{3} \sqrt{1 + 4t^2} dt.$$

Remember, you just have to integrate the speed.

3.3.3 Surface Area

When you revolve a curve about a line, the radius of rotation is the distance to the line. We will now develop formulas for find the surface area of a surface of revolution given by rotating about a line.

Start by breaking the curve up into small pieces (as done before). The length of each piece is approximately given by the arc length approximate Δs_i, which is a straight line segment from one end of the small portion of the curve to the other. We assume that the radius is constant, namely the distance $radius_i$ to the line. If we rotate about the x-axis, then $radius_i = y_i$. If we rotate about the y-axis, then $radius_i = x_i$. The surface area of a frustrum of a cone is $\Delta \sigma_i = 2\pi radius_i \Delta s_i$ (we use σ to designate surface area). Adding each small pieces of surface area up $\sigma = \sum \Delta \sigma_i$, we get the integration formulas

$$\sigma = \int d\sigma = \int 2\pi \; radius \; ds = \int_a^b 2\pi \; radius \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Note that if you can remember $d\sigma = 2\pi \; radius \; ds$, then you just have to know what the radius is, and what to use for ds. If you are revolving about the

x-axis, then the radius of rotation is r$adius = y$. Similarly, the radius is x when revolving about the y-axis.

Let's rotate the curve $x = t, y = t^2$ for $0 \leq t \leq 3$ about both the x and y-axis. From the example in the arc length section, we have $\vec{r}(t) = \left\langle t, t^2 \right\rangle$, $\vec{r}'(t) = \langle 1, 2t \rangle$, and $|\vec{r}'(t)| = \sqrt{1+4t^2}$. If we rotate about the x-axis, then the radius of rotation is $y = t^2$, so we have

$$\int 2\pi r \ ds = \int 2\pi y |\vec{r}'| dt = \int_{-2}^{3} 2\pi (t^2) \sqrt{1+4t^2} dt.$$

If we rotate about the y-axis, then the radius of rotation is $x = t$, so we have

$$\int 2\pi r \ ds = \int 2\pi x |\vec{r}'| dt = \int_{-2}^{3} 2\pi (t) \sqrt{1+4t^2} dt.$$

Chapter 4

New Coordinates

After completing this chapter, you should be able to do the following:

1. Be able to convert between rectangular and polar coordinates in 2D. Convert between rectangular and cylindrical or spherical in 3D.
2. Graph polar functions in the plane. Find intersections of polar equations, and illustrate that not every intersection can be obtained algebraically (you may have to graph the curves).
3. Find derivatives and tangent lines, area, arc length, and surface area using polar equations.

4.1 Preparation and Homework Suggestions

	Preparation Problems (11th ed)				Preparation Problems (12th ed)			
Day 1	10.5:3	10.5:19	10.5:33	10.5:59	11.3:3	11.3:23	11.3:37	11.3:63
Day 2	10.6:5	10.6:19	10.6:33	10.7:9	11.4:5	11.4:19	11.4:??	11.5:11
Day 3	10.6:35	10.7:21	10.7:29	10.7:17	11.4:??	11.5:23	11.5:??	11.5:19

I'll be creating HW problems for the missing Day 2 and Day 3 problems.

The following homework problems line up with the topics we will discuss in class.

Topic (11th ed)	Sec	Basic Practice	Good Problems	Thy/App	Comp	Examples
Polar Coordinates	10.5	1-62		63, 64		E1-6
Graphing Polar Coordinates	10.6	1-24, 29-30	25-28, 31-38	49, 50, 51	39-48	E1-5
Area and Length	10.7	1-16, 19-27, 29-32	17, 18, 28, 34	33, 35, 36		E1-5
Cylindrical and Spherical	15.6	Derive Eqns				

Topic (12th ed)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Polar Coordinates	11.3	1-66		67, 68	
Graphing Polar Coordinates	11.4	1-24, see handout	25-28		29-34
Area and Length	11.5	1-18, 21-28, see handout	29	30-32	
Cylindrical and Spherical	15.6	Derive Eqns			

Don't worry about trying to solve by hand all the integrals in 10.7 or 11.5. If you can set them up, and solve the simpler ones, you are doing great.

4.2 Polar Coordinates

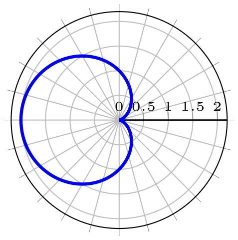
We now introduce a new coordinate system, the polar coordinate system. This is just another way of referencing points in the plane. We use r and θ to reference points. The distance from the origin to the point P is called r, and the angle made by the positive x-axis and a ray from the origin through P is called θ. The key equations $x = r\cos\theta, y = r\sin\theta, x^2 + y^2 = r^2$ come directly from a right triangle with one vertex at the origin, another at the point P, and another directly above or below P on the x-axis.

The point $(r, \theta) = (2, \pi/2)$ represents the point $(x, y) = (0, 2)$. The point $(r, \theta) = (3, \pi/4)$ represents the point $(x, y) = (3\frac{\sqrt{2}}{2}, 3\frac{\sqrt{2}}{2})$. Notice that the point $(1, \pi)$ in polar coordinates is the same as the point $(1, 3\pi)$ or $(-1, -\pi)$ in polar coordinates. The same (x, y) point in the plane can be represented by infinitely many different polar coordinate pairs.

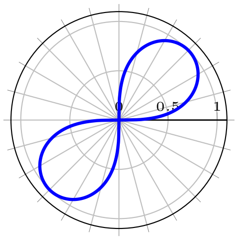
4.2.1 Converting between polar and Cartesian coordinates and graphing

The graph of $r = 2$ is a circle of radius 2 in the plane. The graph of $\theta = \pi/3$ is a straight line through the origin with angle of inclination $\pi/3$. The line $x = 4$ can be rewritten in polar coordinates as $r\cos\theta = 4$, or $r = 4\sec\theta$. The polar coordinate equation $r = 2\sin\theta$ can be converted to rectangular coordinates by multiplying both sides by r, giving $r^2 = 2r\sin\theta$, and then using the key equations to obtain $x^2 + y^2 = 2y$, which is the equation of a circle $x^2 + y^2 - 2y + 1 = 1$, or $x^2 + (y - 1)^2 = 1$ centered at $(0, 1)$ of radius 1 (I completed the square to complete this example). The key equations will allow you to convert from one system to another. You may need to multiply both sides of an equation by r to do a conversion (as in the last example).

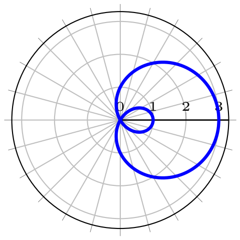
To plot a polar equation, we just make an r, θ, x, y table and start plotting points. You will discover after some practice that there are symmetries that you can use to help you learn how to plot polar curves. To plot the curves by hand, try putting in values of θ which are on the unit circle, in particular use $0, \pi/2, \pi, 3\pi/2$. When you connect dots on your graphs, realize that you wrap around counter-clockwise as theta increases, and the variable r that is changing is the distance from the origin. Some common curves are shown below:



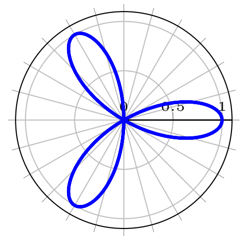
Cardioid
$r = 1 - \cos\theta$



Lemniscate
$r^2 = \sin 2\theta$



Limacon
$r = 2\cos\theta + 1$



Rose
$r = \cos 3\theta$

4.2.2 Finding Intersections

One problem with finding the intersection of two polar graphs is that there are many ways to represent the same point in polar coordinates. Hence, when you solve for the intersection of two polar graphs, you may not find all the intersection points algebraically. Often you must graph the polar curves in addition to finding the points of intersection. For example, the two curves $r = 1 - \cos\theta$ (a cardioid) and $r = \cos\theta$ (a circle of radius 1/2 centered at (1/2,0)) intersect in three points. Solving for θ we have $\cos\theta = 1 - \cos\theta$, or $2\cos\theta = 1$, or $\cos\theta = \frac{1}{2}$. This occurs when $\theta = \pm\frac{\pi}{3}$. Hence the two points of intersection are $(x, y) = (r\cos\theta, r\sin\theta) = (1/4, \pm\sqrt{3}/4)$. The algebraic solution mises completely the fact that $(x, y) = (0, 0)$ is an intersection point of the two graphs. It is best to graph any polar curves when you wish to find their intersection.

4.3 Calculus with new coordinates

4.3.1 Slope

If a curve is given parametrically as $x = f(t), y = g(t)$, then we can find $\frac{dy}{dx}$ using the chain rule. Symbolically we just divide both dx and dy by dt, and obtain $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. In vector form, we can simply compute the derivative $\vec{r}'(t) = \langle dx/dt, dy/dt\rangle$ as the direction vector, and then the slope is $(dy/dt)/(dx/dt)$ (the change in the y component over the change in the x component). For example, if $x = 3t$ and $y = t^2 - t$, then $\vec{r}'(t) = \langle 3t, 2t - 1\rangle$ which means $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t-1}{3t}$. This is how we find the slope of graphs of parametric curves.

If the curve is given by a polar coordinate equation $r(\theta) = f(\theta)$, then we use the same principle. Recall $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$. This means we have a parametric curve $\vec{r}(\theta) = \langle f(\theta)\cos\theta, f(\theta)\sin\theta\rangle$. The derivative of this vector valued function is

$$\begin{aligned}\vec{r}'(\theta) &= \left\langle \frac{d}{d\theta} f(\theta)\cos\theta, \frac{d}{d\theta} f(\theta)\sin\theta \right\rangle \\ &= \langle f'(\theta)\cos\theta - +f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta\rangle\end{aligned}$$

which gives us $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - +f(\theta)\sin\theta}$. Don't worry about memorizing this formula, rather realize that it is just $\frac{dy/d\theta}{dx/d\theta}$.

4.3.2 Area

To find the area swept out by a segment from the origin to a polar curve, we first need to recall that the area of a sector of a circle is $\frac{1}{2}r^2\theta$. To derive this, just recall the area of a circle is πr^2. So half a circle has area $\frac{\pi}{2}r^2$. If you sweep out θ radians, then you have covered $\frac{\theta}{2\pi}$ percent of the circle. So the area covered is $\frac{\theta}{2\pi} \cdot \pi r^2$.

Now we take the simple formula, $\frac{1}{2}r^2\theta$ and use it to derive the integral formula $\int_\alpha^\beta \frac{1}{2}r^2 d\theta$. Consider the region swept out by a segment from the origin to the polar curve $r = f(\theta)$ as θ ranges from α to β. Break up the region into small sectors, each having interior angle $\Delta\theta$. By making $\Delta\theta$ small enough, we can approximate the area ΔA_i of each sector by assuming the radius is constant, $f(\theta_i)$. This gives $\Delta A_i \approx \frac{1}{2}f(\theta_i)^2\Delta\theta$. To find the total area, we add up all the little areas and take a limit as $\Delta\theta \to 0$, as follows:

$$A = \lim_{\Delta\theta\to 0} \sum \Delta A_i = \lim_{\Delta\theta\to 0} \sum \frac{1}{2} f(\theta_i)\Delta\theta = \int_\alpha^\beta \frac{1}{2} f(\theta)^2 d\theta = \int_\alpha^\beta \frac{1}{2} r^2 d\theta.$$

We use the differential notation $dA = \frac{1}{2}r^2 d\theta$ to remember the integration formula. If you can remember the area of a sector of a circle, then you can remember the integration formula. You probably recall already the differential notation $dA = f(x)dx$ To find area, all you have to do is remember that area is the integral of the area differential dA, so $A = \int dA = \int_a^b f(x)dx = \int_\alpha^\beta \frac{1}{2}r^2 d\theta$. To find area between two polar curves, just find the area inside the outer curve, and subtract the area inside the inner curve. The area inside the cardioid $r = 1 - \cos\theta$ is given by $\int_0^{2\pi} \frac{1}{2}(1-\cos\theta)^2 d\theta$, which we would let a computer calculate for us.

4.3.3 Arc Length

Finding length (Δs) along a straight line is done by finding the change in x (called Δx) and the change in y (Δy), and then using the Pythagorean identity to get $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$. If a curve is not straight, then start by breaking the curve up into a bunch of small pieces of length Δs_i. Along each small piece, the curve is approximately straight, so we approximate each Δs_i with $\sqrt{\Delta x_i^2 + \Delta y_i^2}$. In differential notation this becomes $ds = \sqrt{dx^2 + dy^2}$. To find arc length s, we add up ($\sum ds$) the little pieces of arc length and take a limit to get $s = \int_C ds = \int_C \sqrt{dx^2 + dy^2}$. For a function $y = f(x), a \le x \le b$ whose independent variable is x, you can multiply ds by $1 = \frac{dx}{dx}$ to obtain $ds = \sqrt{dx^2 + dy^2}\frac{dx}{dx} = \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}dt = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dt$ which means arc length is $s = \int_C ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$. Similarly, for $x = g(y), c \le y \le d$ we multiply by $\frac{dy}{dy}$ to obtain $\int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}dy$. For parametric equations $x(t), y(t), a \le t \le b$ we multiply by $\frac{dt}{dt}$ to obtain $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$. The polar coordinate version $\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}d\theta$ comes from the nontrivial simplification $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$, where $x = r\cos\theta, y = r\sin\theta$. Hence, to find the arc length for parametric or polar curves, we just integrate the arc length differential.

4.3.4 Surface Area of a surface of revolution

When you revolve a curve about a line, the radius of rotation is the distance to the line. We will now develop formulas for find the surface area of a surface of revolution given by rotating about a line.

Start by breaking the curve up into small pieces (as done before). The length of each piece is approximately given by the arc length approximate Δs_i, which is a straight line segment from one end of the small portion of the curve to the other. We assume that the radius is constant, namely the distance $radius_i$ to the line. If we rotate about the x-axis, then $radius_i = y_i$. If we rotate about the y-axis, then $radius_i = x_i$. The surface area of a frustrum of a cone is $\Delta\sigma_i = 2\pi radius_i \Delta s_i$ (we use σ to designate surface area). Adding each small pieces of surface area up $\sigma = \sum \Delta\sigma_i$, we get the integration formulas $\sigma = \int d\sigma = \int 2\pi\ radius\ ds = \int_a^b 2\pi\ radius\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_\alpha^\beta 2\pi\ radius\ \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$. Note that if you can remember $d\sigma = 2\pi\ radius\ ds$, then you just have to know what the radius is, and what to use for ds.

The surface area of a surface of revolution formed by revolving a polar curve about the x-axis is given by the formula $\int_\alpha^\beta 2\pi|y|ds = \int_\alpha^\beta 2\pi r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$, provided $y \geq 0$. When we revolve about the y-axis we obtain instead the formula $\int_\alpha^\beta 2\pi|x|ds = \int_\alpha^\beta 2\pi r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$, provided $x \geq 0$.

If we rotate about a line such as $x = 3$, then the distance to the line $x = 3$ is given by $|x - 3|$, so our formula is $\int_\alpha^\beta 2\pi|x-3|ds = \int_\alpha^\beta 2\pi(r\cos\theta - 3)\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$, provided $x \geq 3$ (otherwise we just leave the absolute values in the problem).

4.4 Cylindrical and Spherical Coordinates

Cylindrical coordinates is an extension of polar coordinates to three dimensions. The transformation $T(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$, or in parametric form

$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z,$$

gives us a new way of viewing points in 3D. The variable r represents the distance to the z axis, where θ is the angle from the positive x-axis, and z is the distance above the xy plane. The point $(r,\theta,z) = (3,\pi/2,4)$ in cylindrical coordinates is the same as the rectangular point $(x,y,z) = (0,3,4)$. If we let r be a constant (like $r = 1$) then we get a sphere. If we let θ be constant, we get a vertical plane through the origin. If we let z be constant, we get a horizontal plane.

Spherical coordinates (ρ,θ,ϕ) are defined as follows. The distance from the origin to the point (x,y,z) is called ρ. The angle θ is the same as in cylindrical coordinates. The angle ϕ is the angle between the positive z-axis and a ray from the origin to (x,y,z). Using these definitions, we obtain the following equations by considering the two right triangles with edges x,y,r and r,z,ρ:

$$\begin{aligned} x &= r\cos\theta & \tan\theta &= y/x & x &= \rho\sin\phi\cos\theta \\ y &= r\sin\theta & r &= \rho\sin\phi & y &= \rho\sin\phi\sin\theta \\ r^2 &= x^2+y^2 & \rho^2 &= x^2+y^2+z^2 & z &= \rho\cos\phi \end{aligned}$$

Part of your homework will be to derive these equations yourself. We can describe the spherical coordinate transformation as a function

$$T(r, \theta, \phi) = \langle \rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi \rangle .$$

The point $(\rho, \theta, \phi) = (4, \pi, \pi/4)$ is the same as the point $(x, y, z) = (4/\sqrt{2}, 0, 4/\sqrt{2})$. If we let ρ be a constant (such as $\rho = 1$), then we obtain all points that are the same distance from the origin, or a sphere. If we let θ be constant, we get a vertical plane through the origin. If we let ϕ be constant, we get all points with the same angle down from the z axis, which creates a cone. Spherical coordinates is a great way to describe sphere's and cones.

Chapter 5

Functions

After completing this chapter, you should be able to do the following:

1. Be able to describe cylinders and quadric surfaces in space.
2. Describe uses for function with varying input and output dimensions. Be able to draw appropriate representations when the input and output dimensions are 3 or less. Recognize by name and graph the different types of functions, in particular parametric equations, space curves, functions of several variables, vector fields, transformations, and parametric surfaces.
3. Find derivatives of space curves, and use the derivative to find tangent lines to space curves.

5.1 Preparation and Homework Suggestions

Here are the problems to prepare for class in this module.

	Preparation Problems (11th ed)				Preparation Problems (12th ed)			
Day 1	12.6:1-12(match)	12.6:53	13.1:8	14.1:31-36(match)	12.6:1-12(match)	12.6:37	13.1:8	14.1:13-18(mat
Day 2	13.1:34	14.1:25	16.2:31	16.6:1	13.1:20	14.1:43	16.2:39	16.5:1
Day 3	14.1:38	16.2:35	16.6:5	16.6:15	14.1:58	16.2:43	16.5:5	16.5:15

Here is the homework that matches up with the material we are learning.

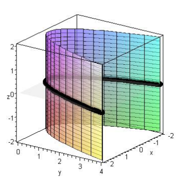
Topic (11th ed)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Cylinders and quadric surfaces	12.6	1-44	45-76	77-84	85-94
Space Curves	13.1	1-18, 21-26,	38-42, 45	27-32, 37, 44, [48-55], 56-57	42, 43, 58-63
Space Curves	16.1	1-8			
Functions of Multiple variables	14.1	1-18(do 13-18), 29-32, 41-44	19-28, 33-40, 45-46	47-48	49-60
Cylindrical and Spherical	15.6	Derive Eqns			
Vector Fields	16.2	31-36 (by hand and maple)			31-36
Parametric Surfaces	14.1	57-60 (by hand and maple)			57-60
Parametric Surfaces	16.6	1-16	53a, 54a, 55, 56, 58		

Topic (12th ed)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Cylinders and quadric surfaces	12.6	1-32	33-44	45-48	49-58
Space Curves	13.1	1-22	23-26	27-34	35-40
Space Curves	16.1	1-8			
Functions of Multiple variables	14.1	1-16, 31-48, 53-60	17-30, 49-52, 61-64	65-68	69-80
Cylindrical and Spherical	15.7	Derive Eqns			
Vector Fields	16.2	39-44 (by hand and maple)			39-44
Parametric Surfaces	14.1	77-80 (by hand and maple)			77-80
Parametric Surfaces	16.5	1-16	31a, 32a, 33, 34, 36		

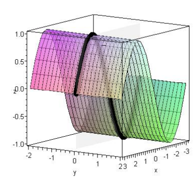
5.2 Cylinders and Quadric Surfaces

5.2.1 Cylinders

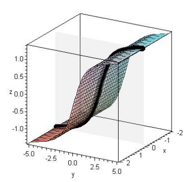
A right circular cylinder is formed by taking a circle in a plane (hence the word "circular"), and then extending through each point of the circle a straight line with direction vector orthogonal (hence the word "right") to the plane. In general, a cylinder is any surface which is created by extending through each point of a curve a straight line in a fixed direction. The curve through which the lines are drawn is called a generating curve for the cylinder. The intersection of a cylinder with a coordinate plane is called a cross-section or a trace. Some examples of cylinders are below, where the generating curve is in bold.



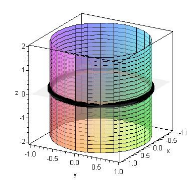
$y = x^2$



$z = \sin(x)$



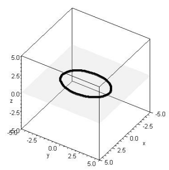
$y = \tan(z)$



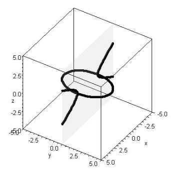
$x^2 + y^2 = 1$

5.2.2 Quadric Surfaces

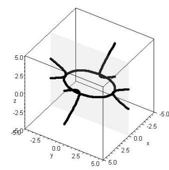
A quadric surface is a generalization of conic sections to three dimensions. In general, it is the graph in 3D of any expression involving at most second degree terms in x, y, and/or z. To graph a quadric surface, hold one variable constant and then graph the resulting conic section in the plane which represents the variable you held constant. Repeat this for a few different variables and constants until you can piece together the surface. An illustration of this process for $\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$ follows on the next page, as well as some typical quadric surfaces.



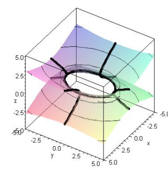
Let $z = 0$



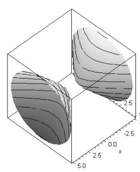
Let $y = 0$



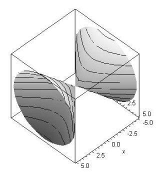
Let $x = 0$



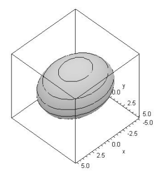
$\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$
Hyperboloid of one sheet



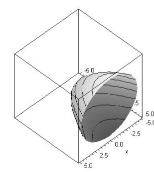
$x^2 - y^2 - z^2 = 1$
Hyperboloid of 2 sheets



$-x^2 + y^2 + z^2 = 0$
Cone



$\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1$
Ellipsoid



$x^2 - 4y + z^2 = 1$
Paraboloid

5.3 Functions in General

A function is a set of instructions (a relation) involving two sets (called the domain and range). A function assigns to each element of the domain D exactly one element in the range R. It is customary to write $f : D \to R$ when we want to specify the domain and range. In this class, we will study what happens when the domain and range are subsets of $\mathbb{R}^n$ (Euclidean n-space). In particular we will study functions of the form $f : \mathbb{R}^n \to \mathbb{R}^m$.

5.3.1 Functions of one variable: $f : \mathbb{R}^1 \to \mathbb{R}^1$

In the first year of calculus, the domain and range are always subsets of the real numbers $\mathbb{R}$. Many ideas from first semester calculus generalize to all dimensions, but some do not. A typical example is $f(x) = x^2$.

5.3.2 Parametric curves: $f : \mathbb{R} \to \mathbb{R}^2$

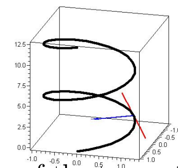
Parametric curves represent motion in the plane. A typical example is given by $x = 2\cos t, y = 3\sin t$, which traces out an ellipse. We will also write these functions as $\vec{r}(t) = \langle 2\cos t, 3\sin t\rangle$.

5.3.3 Spacecurves: $\vec{r} : \mathbb{R} \to \mathbb{R}^3$

Space curves generalize parametric curves, but the output dimension is in 3D. So this is a curve in space. The notation $\vec{r} : \mathbb{R} \to \mathbb{R}^3$ suggests that we are graphing somting one dimensional in three dimensions. The graph of a space curves can be thought of as a bent wires in space. A space curve traces out a path in space. For each t, the position vector $\vec{r}(t)$ gives the position of a particle whose motion is described by the space curve. Since the output is a vector, we often call space curves vector valued functions. A graph of a space curve is made by picking values for t and plotting the corresponding points.

The derivative of a space curve is found by differentiating each component of the space curve, which follows immediately by looking at the limit $\dfrac{d\vec{r}}{dt} = \lim\limits_{h\to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$. An equation of the tangent line to a space curve at $t = c$ has direction vector equal to $\frac{d\vec{r}}{dt}$ and passes through the point $\vec{r}(c)$, so an equation is $\vec{l}(t) = \vec{r}'(c)t + \vec{r}(c)$. If a space curve is used to describe motion, then velocity is $\vec{v}(t) = \frac{d\vec{r}}{dt}$, speed is the magnitude of velocity $|\vec{v}|$, and acceleration is $\vec{a}(t) = \frac{d\vec{v}}{dt}$, just as was taught in first semester calculus.

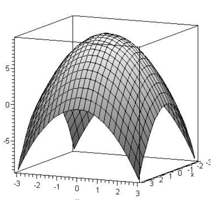
The space curve $\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$ is a helix which wrapps around the z-axis. Its graph is shown on the right, where $0 \leq t \leq 4\pi$, as well as the tangent line and acceleration vector. The velocity and acceleration at any time t are $\vec{v}(t) = \langle -\sin(t), \cos(t), 1\rangle$ and $\vec{a}(t) = \langle -\cos(t), -\sin(t), 0\rangle$. When $t = 2\pi/3$, we have $\vec{r}(2\pi/3) = \left\langle 1/2, \sqrt{3}/2, 2\pi/3\right\rangle$, $\vec{v}(2\pi/3) = \left\langle -\sqrt{3}/2, -1/2, 1\right\rangle$, and $\vec{a}(2\pi/3) = \left\langle 1/2, -\sqrt{3}/2, 0\right\rangle$. An equation of the tangent line is $\vec{l}(t) = \vec{v}(2\pi/3)t + \vec{r}(2\pi/3) = \left\langle -\sqrt{3}/2, -1/2, 1\right\rangle t + \left\langle 1/2, \sqrt{3}/2, 2\pi/3\right\rangle$.



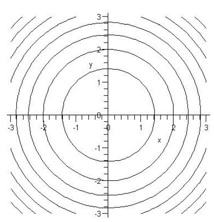
5.3.4 Functions of several variables: $f : \mathbb{R}^2 \to \mathbb{R}$, $f : \mathbb{R}^3 \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$

With functions of this type, the output dimension is always 1, while the input dimension may be as large as needed. This type of function is used to measure a quantity at each point in the plane ($f : \mathbb{R}^2 \to \mathbb{R}$), at each point in space ($f : \mathbb{R}^3 \to \mathbb{R}$), or for every combination of n inputs ($f : \mathbb{R}^n \to \mathbb{R}$). The temperature at each point in the plane would be modeled by a function of the form $T(x, y)$. The wind speed at each point in space could be modeled by a function of the form $f(x, y, z)$.

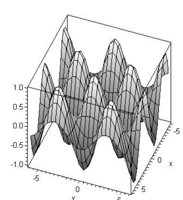
To graph functions of the form $f : \mathbb{R}^2 \to \mathbb{R}$, we typically write $z = f(x, y)$ and then plot the function in xyz coordinates. Every pair (x, y) corresponds to exactly one point $(x, y, f(x, y))$ in space. So functions of this form still pass the "vertical line test." We get 2D traces (vertical cross sections) of the function by replacing x or y with a constant, and then creating a 2D graph of the resulting function. A level curve is a graph in the plane of the equation $c = f(x, y)$ for some constant c (essentially a level curve is a horizontal cross section drawn in the xy-plane). A few graphs and several level curves follow.



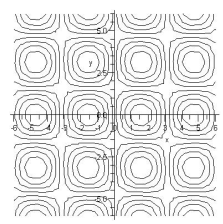
$f(x, y) = 9 - x^2 - y^2$



level curves of f



$g(x, y) = \sin x \cos y$



level curves of g

Functions with three inputs $f : \mathbb{R}^3 \to \mathbb{R}$ would require 4 dimensions to graph. Rather than graph functions in 4D, we instead look at level surfaces. For the function $w = f(x, y, z)$, we pick a constant $w = c$ and graph the surface $c = f(x, y, z)$. The level surface $w = 1$ for the function $f(x, y, z) = x^2 + y^2 + z^2$ is a sphere of radius 1. Quadric surfaces will show up often when you graph level surfaces of functions with three inputs.

5.3.5 Transformations (changing coordinates): $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f : \mathbb{R}^3 \to \mathbb{R}^3$

The polar coordinate equations $x = r\cos\theta, y = r\sin\theta$ require 2 inputs (r, θ) and give two outputs (x, y). This can be thought of as a function $T(r, \theta) = \langle r\cos\theta, r\sin\theta\rangle$. The book will never recognize this "transformation" as a function, because it does not define functions in general. However, every time we change coordinate system, it will be valuable to give that tranformation a name and recognize that it is indeed a function.

5.3.6 Cylindrical and Spherical Coordinates

Cylindrical coordinates is an extension of polar coordinates to three dimensions. The transformation $T(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$, or in parametric form

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

gives us a new way of viewing points in 3D. The variable r represents the distance to the z axis, where θ is the angle from the positive x-axis, and z is the distance above the xy plane. The point $(r,\theta,z) = (3,\pi/2,4)$ in cylindrical coordinates is the same as the rectangular point $(x,y,z) = (0,3,4)$. If we let r be a constant (like $r = 1$) then we get a sphere. If we let θ be constant, we get a vertical plane through the origin. If we let z be constant, we get a horizontal plane.

Spherical coordinates (ρ,θ,ϕ) are defined as follows. The distance from the origin to the point (x,y,z) is called ρ. The angle θ is the same as in cylindrical coordinates. The angle ϕ is the angle between the positive z-axis and a ray from the origin to (x,y,z). Using these definitions, we obtain the following equations by considering the two right triangles with edges x,y,r and r,z,ρ:

$$\begin{aligned} x &= r\cos\theta & \tan\theta &= y/x & x &= \rho\sin\phi\cos\theta \\ y &= r\sin\theta & r &= \rho\sin\phi & y &= \rho\sin\phi\sin\theta \\ r^2 &= x^2+y^2 & \rho^2 &= x^2+y^2+z^2 & z &= \rho\cos\phi \end{aligned}$$

Part of your homework will be to derive these equations yourself. We can describe the spherical coordinate transformation as a function

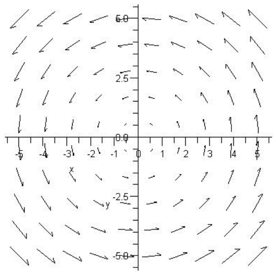
$$T(r,\theta,\phi) = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle .$$

The point $(\rho,\theta,\phi) = (4,\pi,\pi/4)$ is the same as the point $(x,y,z) = (4/\sqrt{2}, 0, 4/\sqrt{2})$. If we let ρ be a constant (such as $\rho = 1$), then we obtain all points that are the same distance from the origin, or a sphere. If we let θ be constant, we get a vertical plane through the origin. If we let ϕ be constant, we get all points with the same angle down from the z axis, which creates a cone. Spherical coordinates is a great way to describe sphere's and cones.

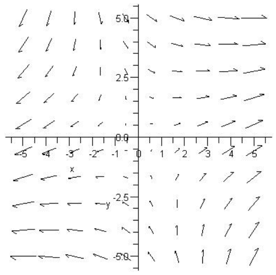
5.3.7 Vector Fields - $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$

A vector field $\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ (or $\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$) is a function which assigns to each point in the plane (or space) a vector. Vector fields are used to model gravity, forces, velocity, wind, acceleration, electric fields, and many other things in nature. Wind depends on location, so it is important to be able to have a way of assigning a different vector to every point in the domain. Vector fields may be one of the most useful tools we have. You should practice creating vectors given a description. For example, a vector field which points to the origin in the plane, and has magnitude equal to the square of its distance from the origin is $\vec{F}(x,y) = (x^2+y^2)\frac{\langle -x,-y\rangle}{\sqrt{x^2+y^2}}$. Remember (magnitude) times (direction).

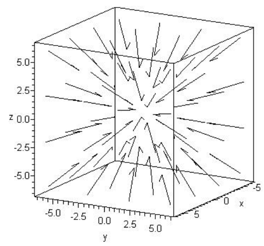
To graph a vector field $\vec{F}(x,y) = \langle M, N\rangle$ in the plane, at each point (x,y) we draw the vector $\vec{F}(x,y)$ with it's base at (x,y). Since the vectors may be rather large, computers will proportionally rescale all vectors so that the vectors fit in the graph. A few examples follow.



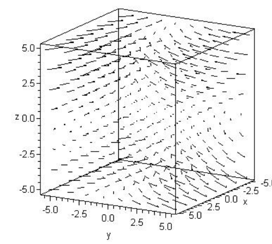
$F(x,y) = \langle -y, x \rangle$



$F(x,y) = \langle 2x+y, x-y \rangle$



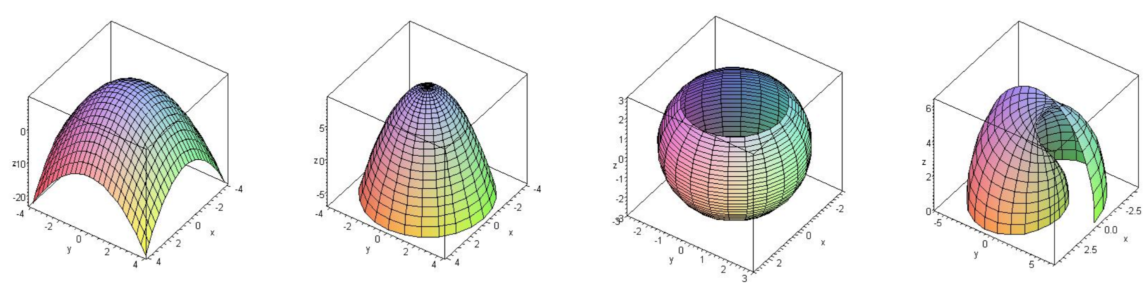
$F(x,y,z) = \frac{\langle -x,-y,-z \rangle}{\sqrt{x^2+y^2+z^2}}$



$F = \langle -x+y, -yz+1, z \rangle$

5.3.8 Parametric Surfaces - $f : \mathbb{R}^2 \to \mathbb{R}^3$

Just as parametric and space curves describe 1 dimensional objects, we use parametric surfaces to describe 2 dimensional objects in 3D. Notice that we are mapping 2 dimensions into 3, so think of a parametric surface as a set of instructions of how to place the 2D plane in space (where you can twist the plane and stretch it based on the set of instructions). If you hold one variable constant, then the graph of the resulting function is a space curve. To graph a parametric surface, hold one variable constant and draw the resulting space curve. Do this for a few values of each variable, and you will have created a net of overlapping space curves from which you can piece together the surface. Any surface of the form $z = f(x,y)$ can be made a parametric surface by writing $\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$, which just says for each (x,y) to plot the point $(x, y, f(x,y))$. We often use u, v as variables for a parametric surface if those variables do not represent some other standard quantity. Cylindrical and spherical coordinates may be very helpful as you learn to create parametric surfaces. Here are some examples:



The functions are, from left to right,

$\vec{r}(x,y) = \left\langle x, y, 9 - x^2 - y^2 \right\rangle$ for $-4 \le x \le 4, -4 \le y \le 4$

$\vec{r}(u,v) = \left\langle u\cos v, u \sin v, 9 - u^2 \right\rangle$ for $0 \le u \le 3, 0 \le v \le 2\pi$ (cylindrical coordinates)

$\vec{r}(\theta,\phi) = \langle 3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi \rangle$ for $0 \le \theta \le 2\pi, \frac{\pi}{4} \le \phi \le \frac{3\pi}{4}$ (spherical coordinates)

$\vec{r}(u,v) = \langle u \sin u \cos v, u\cos u \cos v, u \sin v \rangle$ for $0 \le u \le 2\pi, 0 \le v \le \pi$.

Chapter 6

Derivatives

After completing this chapter, you should be able to do the following:

1. Find limits, and determine where functions of several variables are continuous.
2. Compute partial derivatives. Use them to find tangent lines and tangent planes.
3. Be able to find the derivative of a function (as a matrix), and use it to find tangent planes.
4. Find derivatives of composite functions, using the chain rule (matrix multiplication). In addition, find derivatives when constraints are in a problem.

Following you will find some suggested homework, and then my best attempt at condensing the information we are learning into a set of concise lecture notes. These are not intended to be a complete resource. As you read my lecture notes, please try an example like each example you see. Read the material in the text as well, and expand your knowledge of the material you see here. I am trying to have my lecture notes follow the model found in "Preach My Gospel," where the gospel is taught in a short complete concise manner, with additional study suggestions given.

6.1 Preparation and Homework Suggestions

	Preparation Problems (11th ed)				Preparation Problems (12th ed)			
Day 1	14.2:37	14.3:39	14.3:43	14.6:11	14.2:43	14.3:39	14.3:43	14.6:11
Day 2	Handout #4	14.4:9	14.4:15	16.6:49	Handout #4	14.4:9	14.4:15	16.5:27
Day 3	14.9:2	14.4:47	14.3:51-52	14.4:42a	14.10:2	14.4:49	14.3:55-56	14.4:44a

Here is the homework that matches up with the material we are learning. Remember to work down the first column ASAP in the next day or two. Just seeing the problems and attempting them for 2 minutes will prepare you for what we are learning.

Topic (11th ed)	Sec	Basic Practice	Good Problems	Thy/App
Limits and Continuity	14.2	1-34,35-42	45-48, 49-50	51-68
Partial Derivatives	14.3	1-46,	47-52, 63-75	53-56, 57-62, 76-77
Total Derivative	Class	handout has problems	do them all	check your solutions
Chain Rule	14.4	1-12, 13-24 (use matrices, not the tree)	25-40,	41-50
Tangent Planes	14.6	9-12		
Tangent Planes	16.6	49-52		
Differentials	14.6	19-30	47-58	
Partial Der. w/ Constraints	14.9	1-8	9-12	

Topic (12th ed)	Sec	Basic Practice	Good Problems	Thy/App
Limits and Continuity	14.2	1-40, 41-48	49-60	61-80
Partial Derivatives	14.3	1-50,	51-56, 61-62, 73-87	57-60, 63-70, 88-89
Total Derivative	Class	handout has problems	do them all	check your solutions
Chain Rule	14.4	1-12, 13-24 (use matrices, not the tree)	25-42,	43-50
Tangent Planes	14.6	9-12		
Tangent Planes	16.5	27-30		
Differentials	14.6	19-30	49-62	31,32
Partial Der. w/ Constraints	14.10	1-8	9-12	

6.2 Limits and Continuity

We start with some notation. We will denote functions using $\vec{f}:\mathbb{R}^n\to\mathbb{R}^m$ where n is the number of inputs, and m is the number of outputs. We will use $\vec{x}$ to represent an input, and $\vec{y}$ to represent an output. For example, if $z=f(x,y)$, then in vector notation we would write $\vec{y}=\vec{f}(\vec{x})$, where $\vec{y}=\langle z\rangle$ and $\vec{x}=\langle x,y\rangle$. Using this notation allows us to restate most of the theorems of multivariable calculus using the exact same notation as was learned in single variable calculus. The parametric surface $\vec{r}(u,v)=\langle x,y,z\rangle$ would be written as $\vec{f}(\vec{x})=\vec{y}$ where $\vec{x}=\langle u,v\rangle$ and $\vec{y}=\langle x,y,z\rangle$.

Recall that in first semester calculus we defined the limit of a function as follows. We say that a function $y=f(x)$ has limit L at $x=c$ if for every $\epsilon>0$ there exists a $\delta>0$ such that if $0<|x-c|<\delta$ then $|f(x)-L|<\epsilon$. The idea is that you can make the output values y as close to L as you want (within ϵ of L) by requiring the input values x to be very close to c (within δ of c). This generalizes to all dimensions by placing vector symbols above everything. We say that a function $\vec{y}=\vec{f}(\vec{x})$ has limit $\vec{L}$ at $\vec{x}=\vec{c}$ if for every $\epsilon>0$ there exists a $\delta>0$ such that if $0<|\vec{x}-\vec{c}|<\delta$ then $|\vec{f}(\vec{x})-\vec{L}|<\epsilon$. We interpret absolute values as distance (the square root of the dot product). Learning to use the formal limit definition to prove theorems about limits and derivatives will be deferred to a course in real analysis, where an appropriate amount of time can be spent on the topic. For now, the main idea is that a function has a limit of L at $\vec{c}$ if the $\vec{y}$ values are close to L for all $\vec{x}$ values close enough to $\vec{c}$. We say that a function is continuous at $\vec{x}=\vec{c}$ if the limit of the function L equals $\vec{f}(\vec{c})$.

The main difference between single variable limits and multivariate limits is the number of ways to approach a point. With one input variable, we can only study limits from the left and right. With more input variables, there are infinitely many ways to approach the point of interest. A limit exists at $\vec{c}$ if and only if the limit exists along every approach to $\vec{c}$, however there are infinitely many different ways to approach this point.

Example 6.1. The function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ has a limit at every point in the plane except at the origin. As long as $(a,b) \neq (0,0)$, we can write

$$\lim_{(x,y)\to(a,b)} \frac{x^2-y^2}{x^2+y^2} = \frac{a^2-b^2}{a^2+b^2},$$

because rational functions are continuous as long as the denominator is nonzero. If $(a,b) = (0,0)$, then let's consider three different ways of approaching the origin.

- If we look at (x,y) values on the x-axis, then we compute

$$\lim_{\substack{(x,\,y)\,\to\,(0,\,0)\\ y\,=\,0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1.$$

- If we look at (x,y) values on the y-axis, then we have

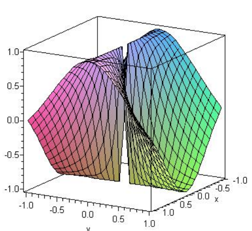
$$\lim_{\substack{(x,\,y)\,\to\,(0,\,0)\\ x\,=\,0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1.$$

- If we look at (x,y) values on the line $y = mx$, then we have

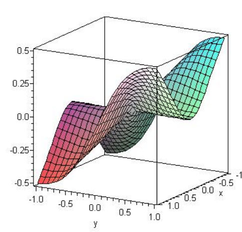
$$\lim_{\substack{(x,\,y)\,\to\,(0,\,0)\\ y\,=\,mx}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-(mx)^2}{x^2+(mx)^2} = \lim_{x\to 0} \frac{x^2(1-m^2)}{x^2(1+m^2)} = \frac{1-m^2}{1+m^2}.$$

From our work above, we see that the limit is not the same along different approaches and we say that the function f has no limit at the origin.

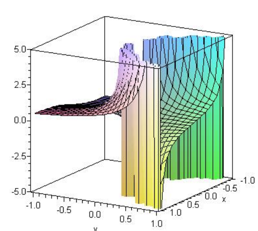
When a function has a limit, that limit will be the same regardless of the approach we use. In the functions below, the first does not have a limit at the origin, the second does, and the third is discontinuous along an entire curve $y = x^2$.



$f(x,y) = \frac{x^2-y^2}{x^2+y^2}$



$f(x,y) = \frac{x^2 y}{x^2+y^2}$



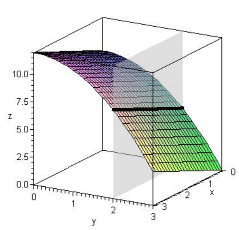
$f(x,y) = \frac{x}{x^2-y}$

6.3 Partial Derivatives

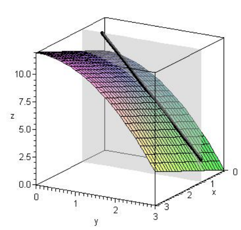
If a function f has only one input x, then the derivative is defined to be $\dfrac{df}{dx} = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$. The derivative gives the best possible linear approximation to changes in a function. This idea is represented by the equation $\Delta y \approx f'(x)\Delta x$, or in differential form we write $dy = f'(x)dx$. An equation of a tangent line is found by noticing that a change in y is approximately $y - f(c)$ when the change in x is $x - c$, so the differential form $dy = f' dx$ becomes $(y - f(c)) = f'(c)(x - c)$. I repeat, the derivative gives the best possible linear approximation to changes in a function.

If a function has more than one input variable, then division by the vector $\vec{h}$ is not well-defined, so we run into a problem with generalizing derivatives. Instead, we start with partial derivatives, which approximate change in the function if we hold all other variables constant and just differentiate with respect to one variable. For the function $f(x, y)$, we define the partial derivative of f with respect to x as $f_x(x, y) = f_x = \dfrac{\partial f}{\partial x} = \lim_{h \to 0} \dfrac{f(x + h, y) - f(x, y)}{h}$, and the partial derivative of f with respect to y as $f_y(x, y) = f_y = \dfrac{\partial f}{\partial y} = \lim_{k \to 0} \dfrac{f(x, y + k) - f(x, y)}{k}$. Notice that f_x computes a limit as y is held constant and we vary x. For $f(x, y) = 3x^2 + 4xy + \cos(xy) + y^3$, we obtain $f_x = 6x + 4y - y\sin(xy) + 0$ and $f_y = 0 + 4x - x\sin(xy) + 3y^2$. Partial derivatives are found by holding all other variables constant, and then differentiating with respect to the variable in question. Practice computing partial derivatives with lots of functions. The homework should go very quickly, but this skill needs plenty of practice so that partial differentiation is done immediately.

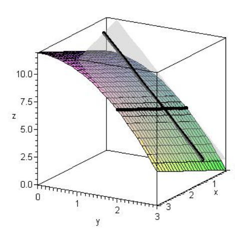
Partial derivatives approximate change in a function as you vary only one variable, hence a partial derivative gives slope in the direction which is varied. For the function $z = f(x, y)$, f_x is the slope of a line tangent to the surface $z = f(x, y)$, where the tangent line is parallel to the xz-plane. A direction vector for this line is $\langle 1, 0, f_x \rangle$ (every increase in x of 1 unit yields an increase in the output z of f_x units, and y does not change). Similarly $\langle 0, 1, f_y \rangle$ is a direction vector of a line tangent to the surface, where the line is parallel to the yz-plane. The cross product of these two vectors is $\vec{n} = \langle -f_x, -f_y, 1 \rangle$, which is a normal vector for the tangent plane to the surface. For the function $f(x, y) = 9 + x - y^2$ at $(x, y) = (1, 2)$, we have $f(x, y) = 6$, $f_x(x, y) = 1$, $f_y(x, y) = -2y$, $f_x(1, 2) = 1$, and $f_y(1, 2) = -4$. So a tangent line to the surface at $(1, 2, 6)$ in the x direction is $\vec{r}(t) = \langle 1, 0, 1 \rangle t + \langle 1, 2, 6 \rangle$, in the y direction is $\vec{r}(t) = \langle 0, 1, -4 \rangle t + \langle 1, 2, 6 \rangle$, and an equation of the tangent plane is $-1(x - 1) + 4(y - 2) + 1(z - 6) = 0$. The pictures below illustrate these lines and planes



y is constant, $\vec{v} = \langle 1, 0, f_x \rangle$



x is constant, $\vec{v} = \langle 0, 1, f_y \rangle$



Tangent plane $n = \langle -f_x, -f_y, 1 \rangle$

For the parametric surface $\vec{r}(u, v) = \left\langle u, v, 9 + u - v^2 \right\rangle$, we compute $\vec{r}_u = \langle 1, 0, 1 \rangle$ and $\vec{r}_v = \langle 0, 1, -2v \rangle$, which are direction vectors for tangent lines to the surface (notice they are the same vectors as in the previous example). At the point $(u, v) = (1, 2)$, the tangent lines and tangent plane are exactly the same as those for the function $z = f(x, y) = 9 + x - y^2$.

Since a partial derivative is a function itself, you can take derivatives of a partial derivative as well, giving what is called higher order partials. For the function $f(x, y, z) = x^2 + 4xy^2 - yz^3$, we have $f_x = 2x + 4y^2, f_y = 8xy - z^3, f_z = -3yz^2$, and so we obtain the following second order partial derivatives

$\frac{\partial}{\partial x} f_x = f_{xx} = 2$, $\frac{\partial}{\partial y} f_x = f_{xy} = 8y$, $\frac{\partial}{\partial z} f_x = f_{xz} = 0$, $\frac{\partial}{\partial x} f_y = f_{yx} = 8y$, $\frac{\partial}{\partial y} f_y = f_{yy} = 8x$, $\frac{\partial}{\partial z} f_y = f_{yz} = -3z^2$, $\frac{\partial}{\partial x} f_z = f_{zx} = 0$, $\frac{\partial}{\partial y} f_z = f_{zy} = -3z^2$, $\frac{\partial}{\partial z} f_z = f_{zz} = -6z$. Notice that $f_{xy} = f_{yx}, f_{xz} = f_{zx}$, and $f_{zy} = f_{yz}$. As long as a function is two times continuously differentiable, then second order partial derivatives using the same variables but in a different order will always be the same. This fact will get used periodically throughout the class.

6.4 The Derivative

The derivative of a function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ is an $m \times n$ matrix written $D\vec{f}(\vec{x})$, where the columns of the matrix are the partial derivatives of the function with respect to each input variable (the first column is the partial derivative with respect to the first variable, and so on). Some people call this derivative the "total" derivative instead of the derivative, to emphasize that the "total" derivative combines the "partial" derivatives into a matrix. This definition of the derivative gives the best possible linear approximation to changes in a function. A full course in linear algebra is needed to fully understand the significance of this statement, however we can use the derivative as a matrix to simplify a lot of our work in multivariable calculus.

Some examples of functions and their derivative follow. When the output dimension of a function is one, then the matrix has only one row. In this case, we often consider it as a row vector and the derivative is called the gradient of f and written ∇f.

Function	Derivative
$f(x) = x^2$	$Df(x) = [2x]$
$\vec{r}(t) = \langle 3\cos(t), 2\sin(t) \rangle$	$D\vec{r}(t) = \begin{bmatrix} -3\sin t \\ 2\cos t \end{bmatrix}$
$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$	$D\vec{r}(t) = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$
$f(x, y) = 9 - x^2 - y^2$	$Df(x, y) = \nabla f(x, y) = [-2x \quad -2y]$
$f(x, y, z) = x^2 + y + xz^2$	$Df(x, y, z) = \nabla f(x, y, z) = \begin{bmatrix} 2x + z^2 & 1 & 2xz \end{bmatrix}$
$\vec{F}(x, y) = \langle -y, x \rangle$	$D\vec{F}(x, y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$\vec{F}(r, \theta, z) = \langle r\cos\theta, r\sin\theta, z \rangle$	$D\vec{F}(r, \theta, z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\vec{r}(u, v) = \left\langle u, v, 9 - u^2 - v^2 \right\rangle$	$D\vec{r}(u, v) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2u & -2v \end{bmatrix}$

To emphasize that the derivative is the best possible linear approximation to changes in a function, replace f' in the single variable equation $dy = f'dx$ with the derivative $D\vec{f}(\vec{x})$ to obtain $d\vec{y} = D\vec{f}(\vec{x})d\vec{x}$ (d[outputs] $= Dfd$[inputs]). For a function $z = f(x, y)$ we obtain $dz = Df(x, y) \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = f_x dx + f_y dy$ using matrix multiplication. To get an equation of a tangent plane to a surface at $(a, b, f(a, b))$, we let $dz = z - f(a, b), dx = x - a, dy = y - b$, and then obtain $z - f(a,b) = \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} = f_x(a,b)(x-a) + f_y(a,b)(y-b)$. This is equivalent to $-f_x(a,b)(x-a) - f_y(a,b)(y-b) + z - f(a,b) = 0$, which is the equation of a plane with normal vector $\vec{n} = \langle -f_x, -f_y, 1 \rangle$, which we already obtained in the partial derivatives section. For the function $f(x,y) = 9+x-y^2$ at $(x,y) = (1,2)$, an equation of the tangent plane is simply $z-6 = \begin{bmatrix} 1 & -4 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$. This version of finding tangent planes generalizes to all dimensions and gives the "tangent space."

This optional paragraph explains why the definition of the derivative above makes sense. Since division by $\vec{h}$ is not defined if $\vec{h}$ is a vector and not a number, we modify the definition of the derivative to define the derivative of a function in general. The definition of the derivative when written using the formal definition of a limit requires that we examine the inequality $|\frac{f(x+h)-f(x)}{h} - f'(x)| < \epsilon$. Multiply both sides by $|h|$ and obtain $|f(x+h) - f(x) - f'(x)h| < \epsilon|h|$. The derivative of a function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ gives the best possible linear approximation to changes in the function. A course in linear algebra will show you that this means the derivative can be represented by a matrix, and then the equation $|\vec{f}(\vec{x}+\vec{h}) - \vec{f}(\vec{x}) - D\vec{f}(\vec{x})\vec{h}| < \epsilon|\vec{h}|$ shows that $D\vec{f}(\vec{x})$ must be able to multiply on the right by vectors of size n (hence it has n columns), and the product $D\vec{f}(\vec{x})\vec{h}$ must give a vector with m components (hence the matrix has m rows). By considering the vector $\vec{h} = \langle h_1, 0, \ldots, 0 \rangle$, it can be shown that the first column of $D\vec{f}(\vec{x})$ equals the partial derivative of $\vec{f}$ with respect to the first variable.

6.5 The Chain Rule

For multivariable functions, we can form the composition $\vec{f} \circ \vec{g}$ of two functions $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{f} : \mathbb{R}^r \to \mathbb{R}^s$ provided that output dimension of $\vec{g}$ is the same as the input dimension of $\vec{f}$. The composition $\vec{f}(\vec{g}(\vec{x}))$ essentially asks us to put into the function $\vec{g}$ a vector $\vec{x}$ of size r, which will give us a vector $\vec{g}(\vec{x})$ of size s. If $s = n$, then we can put $\vec{g}(\vec{x})$ into the function $\vec{f}$, and we get a vector $\vec{f}(\vec{g}(\vec{x}))$ of size m. For examples, if $f(x,y) = x^2 - y$ and $\vec{g}(r,s,t) = \langle 3r+4s, t^2 - r \rangle$, then $f(\vec{g}(r,s,t)) = f(3r+4s, t^2 - r) = (3r+4s)^2 - (t^2 - r)$. We now discuss how to differentiate such functions.

Recall in first semester calculus that the chain rule states $(f \circ g)'(x) = f'(g(x))g'(x)$ (the derivative of the outside function multiplied by the derivative of the inside function). The high dimensional version of the chain rule is exactly the same, namely $D(\vec{f} \circ \vec{g})(\vec{x}) = D\vec{f}(\vec{g}(\vec{x}))D\vec{g}(\vec{x})$, and a proof (which we will leave to another course) is essentially the same. The product is matrix multiplication. Most textbooks describe a "tree rule" which is just a way of organizing matrix multiplication without referring to a matrix.

We now look at a few examples. Suppose the temperature at each point in the plane is given by $f(x,y) = 9 - x^2 - y^2$. A particle moving through the plane along the curve $x = t+1, y = t^2$ will have a temperature given by $f(r(t)) = f(t+1, t^2) = 9 - (t+1)^2 - (t^2)^2$. We will find the rate of change of the temperature of the particle as it moves through the plane, which we call $\frac{df}{dt}$. I like to start by giving the path $x = t+1, y = t^2$ a name, for example let $\vec{r}(t) = \langle t+1, t^2 \rangle$. Then the composition $f(\vec{r}(t))$ is the temperature of the particle at any time t. Hence, we are really trying to compute $\frac{d(f \circ r)}{dt}$, which is ugly to write so most people abbreviate it with $\frac{df}{dt}$, even though there are no t's in the definition of $f(x,y)$. The chain rule says that I can compute this value by finding $Df(x,y)$ and $D\vec r(t)$, evaluating $Df(x,y)$ at $\vec r(t)$ which gives me $Df(\vec r(t))$, and then multiplying the matrices together. The calculations are $Df(x,y) = \begin{bmatrix}-2x & -2y\end{bmatrix}$ and $D\vec r(t) = \begin{bmatrix}1\\2t\end{bmatrix}$, so $Df(\vec r(t)) = \begin{bmatrix}-2(t+1) & -2(t^2)\end{bmatrix}$. We then calculate

$$D(f\circ\vec r)(t) = Df(\vec r(t))D\vec r(t) = \begin{bmatrix}-2(t+1) & -2(t^2)\end{bmatrix}\begin{bmatrix}1\\2t\end{bmatrix} = (-2(t+1))(1)+(-2(t^2))(2t).$$

We can also do this by just replacing x and y with what they are in terms of t, and then differentiating $f(r(t)) = 9-(t+1)^2-(t^2)^2$. This second approach may at first seem easier, but the first approach becomes essential when you want to do implicit differentiation and high dimensional calculus. We just developed the formula $\frac{df}{dt} = \begin{bmatrix}f_x & f_y\end{bmatrix}\begin{bmatrix}x_t\\y_t\end{bmatrix} = f_xx_t+f_yy_t = \frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$.

For $f(x,y,z) = 3xy+z^2$ and $x = 2u+v, y = u-v, z = uv$, we can compute both $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ using the chain rule. I start by naming the function $x = 2u+v, y = u-v, z = uv$ using something like $\vec r(u,v) = \langle 2u+v, u-v, uv\rangle$. Then the derivative $D(f\circ\vec r)$ is found by multiplying

$$\begin{aligned}D(f\circ\vec r)(u,v) = Df(x,y,z)D\vec r(u,v) &= \begin{bmatrix}f_x & f_y & f_z\end{bmatrix}\begin{bmatrix}x_u & x_v\\ y_u & y_v\\ z_u & z_v\end{bmatrix}\\ &= \begin{bmatrix}3y & 3x & 2z\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & -1\\ v & u\end{bmatrix}\\ &= \begin{bmatrix}(3y)(2)+(3x)(1)+(2z)(v) & (3y)(1)+(3x)(-1)+(2z)(u)\end{bmatrix}.\end{aligned}$$

Replacing x,y,z with what they are in terms of u and v gives

$$D(f\circ\vec r)(u,v) = \begin{bmatrix}6(u-v)+3(2u+v)+2uv^2 & 3(u-v)-3(2u+v)+2u^2v\end{bmatrix}.$$

The first column of the matrix is the partial with respect to u, so $\frac{\partial f}{\partial u} = 6(u-v)+3(2u+v)+2uv^2$ and the second column gives $\frac{\partial f}{\partial v} = 3(u-v)-3(2u+v)+2u^2v$. We just developed the general formula

$$D(f\circ\vec r)(u,v) = \begin{bmatrix}f_u & f_v\end{bmatrix} = \begin{bmatrix}f_xx_u+f_yy_u+f_zz_u & f_xx_v+f_yy_v+f_zz_v\end{bmatrix}.$$

For an equation of the form $y^2x+x-xy = 1$, we learned how to calculate $\frac{dy}{dx}$ implicitly in first semester calculus. We now learn a quick method using high dimensional calculus. Let $f(x,y) = y^2x+x-xy$ and $\vec r(x) = \langle x, y(x)\rangle$ be a parametrization of the curve. Composition shows that $f(\vec r(x)) = 1$, so we compute $D(f\circ r)(x) = Df(\vec r(x))D\vec r(x) = \begin{bmatrix}f_x & f_y\end{bmatrix}\begin{bmatrix}1\\ \frac{dy}{dx}\end{bmatrix}$ which equals 0 because the composition equals the constant 1. Matrix multiplication gives $f_x+f_y\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{y^2+1-y}{2xy-x}$. This idea can be generalized to do implicit differentiation in any setting. This illustrates an important idea. Some problem become easier to solve in higher dimensions. Sometimes the only solution to a problem is found by looking at the problem in a higher dimensional space where there are more tools available.

6.6 Partial Derivatives with constrained variables

When you have many variables in a problem, it is important to specify which are independent, and which are dependent. Your choice could have drastic consequences on the values of the partial derivatives. When there are many variables in a problem the notation $\left(\frac{\partial w}{\partial x}\right)_{y,t}$ means that x, y, t are the independent variables, and the other variables all depend on x, y, t. If the variables that you see in the problem are x, y, z, and t, and w is a function of all of these variables, then often it is helpful to create a diagram of the form $\begin{bmatrix} x \\ y \\ t \end{bmatrix} \to \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \to \begin{bmatrix} w \end{bmatrix}$ to remind yourself how the variables relate, namely that z depends on the other variables. Then you can use the chain rule to find the partials of w with respect to any variable you wish.

For example, let $w = x^2 - y^2 + z - \sin(t)$ and $x - y^2 + 3z = 5t$ (the equation $x - y^2 + 3z = 5t$ is called a constraint because it puts a limitation on the values you can pick for x, y, z, and t). Then $\left(\frac{\partial w}{\partial x}\right)_{y,z}$ is found in one of two ways. The first approach (which you can use for the homework and exams) is to replace t with what it is in terms of x, y, and z, and then take the partial with respect to x. This is by far the quickest route, if you are just after a number. In this case we get $t = (x - y^2 + 3z)/5$ and so $w = x^2 - y^2 + z - \sin((x - y^2 + 3z)/5)$. Then $\left(\frac{\partial w}{\partial x}\right)_{y,z} = 2x - 0 + 0 - \cos((x - y^2 + 3z)/5) \cdot \frac{1}{5}$.

The other option involves a theoretical understanding. Start by making the diagram $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \to \begin{bmatrix} w \end{bmatrix}$. Notice that t is the dependent variable, so we have $t = \frac{1}{5}(x - y^2 + 3z)$. This is a composite function, and the derivative is

$$Dw(x,y,z) = \begin{bmatrix} w_x & w_y & w_z & w_t \end{bmatrix} \begin{bmatrix} x_x & x_y & x_z \\ y_x & y_y & y_z \\ z_x & z_y & z_z \\ t_x & t_y & t_z \end{bmatrix} = \begin{bmatrix} 2x & -2y & 1 & -\cos t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/5 & -2y/5 & 3/5 \end{bmatrix}.$$

Since I am only after $\left(\frac{\partial w}{\partial x}\right)_{y,z}$, I want the partial with respect to x which means the first column of $Dw(x,y,z)$ or $2x - \frac{1}{5}\cos(t) = 2x - \frac{1}{5}\cos(\frac{1}{5}(x - y^2 + 3z))$. The second column would give $\left(\frac{\partial w}{\partial y}\right)_{x,z}$, and the third column $\left(\frac{\partial w}{\partial z}\right)_{x,y}$. If on the other hand I were asked to compute $\left(\frac{\partial w}{\partial z}\right)_{y,t}$, then $x = 5t - 3z + y^2$ and so

$$Dw(y,z,t) = \begin{bmatrix} w_x & w_y & w_z & w_t \end{bmatrix} \begin{bmatrix} x_y & x_z & x_t \\ y_y & y_z & y_y \\ z_y & z_z & z_y \\ t_y & t_z & t_y \end{bmatrix} = \begin{bmatrix} 2x & -2y & 1 & -\cos t \end{bmatrix} \begin{bmatrix} 2y & -3 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which means that $\left(\frac{\partial w}{\partial z}\right)_{y,t} = 2(x)(-3) + (1)(1) = -6(5t - 3z + y^2) + 1$ is the second column of $Dw(y,z,t)$.

As a last example, if $w = x^2 + y + z^2$ and $x^2 + y = z^3$, then we can compute $\left(\frac{\partial w}{\partial x}\right)_{z}$ by writing $y = z^3 - x^2$ and $w = x^2 + (z^3 - x^2) + z^2 = z^3 - z^2$. The partial with respect to x when y is dependent is hence 0. On the other hand, we can compute $\left(\frac{\partial w}{\partial x}\right)_y$ by writing $z = \sqrt[3]{x^2 + y}$ and $w = x^2 + y + (\sqrt[3]{x^2 + y})^2$. The derivative of this with respect to x is $2x + \frac{2}{3}(x^2 + y)^{-1/3}2x$, which is not zero. Specifying which variables depend on which makes a difference when taking derivatives where constraints are present.

Chapter 7

Motion

After completing this chapter, you should be able to do the following:

1. Develop formulas for the position of a particle in projectile motion if we neglect air resistance and consider only acceleration due to gravity. Develop formulas for the range, maximum height, and flight time of the projectile.

2. Develop the TNB frame for describing motion. Add to your model the concepts of curvature, osculating circle, torsion, and the tangential and normal components of acceleration. Be able to prove the relationships that you develop in the TNB frame.

In this unit, we will focus more on the relationships between the ideas we are learning, rather than just the computations.

7.1 Preparation and Homework Suggestions

	Preparation Problems (11th ed)				Preparation Problems (12th ed)			
Day 1	13.2:3	13.2:5	13.3:3	13.3:11	13.2:21	13.2:23	13.3:3	13.3:11
Day 2	13.2: 11	13.3:18	13.4:1	13.4:9	13.2: 29	13.3:18	13.4:1	13.4:9
Day 3	13.4:3	13.4:21	13.5:1	13.5:9	13.4:3	13.4:21	13.5:9	13.5:1
Day 4	13.4:7	13.5:11	13.5:15	13.5:17	13.4:7	13.5:11	13.5:15	13.5:17

Here are the suggested homework problems for this unit.

Topic (11th Ed.)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Space Curves (review)	13.1				
Projectile Motion	13.2	1-13	14-23	24-31	
Arc Length	13.3	1-8, 11-14	9-10, , 15-17	18-20	
Curvature *	13.4	1-4, 9-16, 21-22	5-7(do for sure), 17-19,	20	23-26, 27-34
Torsion *	13.5	1-16,	17-24, 27	25, 26, 28	29-32

Topic (12th Ed.)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Space Curves (review)	13.1				
Projectile Motion	13.2	1-24	25-36	37-42	
Arc Length	13.3	1-8, 11-14	9-10, , 15-17	18-20	
Curvature *	13.4	1-4, 9-16, 21-22	**5-7(do for sure), 17-19,	20	23-26, 27-34
Torsion *	13.5	1-16,	17-24, 27	25, 26, 28	29-32

* Don't do too many by hand in the curvature and torsion sections, as they can be very time consuming. Practice a few so that you can make sure you understand the computations, but then spend the rest of your time reviewing the theory and proving the relationships that exist among the vectors. Though difficult, the best problems in my opinion in 13.4 and 13.5 are problems 5-7 in 13.4.

7.2 Projectile Motion

If an object has initial velocity $\vec{v}_0$, initial position $\vec{r}_0$, and acceleration $\vec{a}(t)$, then you can find the position at any given time by integrating, as $\vec{v}(t) = \int \vec{a}(t)dt$, and $\vec{r}(t) = \int \vec{r}(t)dt$. Projectile motion describes the ideal path of motion of an object which is fired into the air at a given angle, α the firing angle, and a given speed, v_0, assuming that gravity $\vec{a}(t) = \langle 0, -g\rangle$ is the only force which acts on the particle. Notationally we let $v_{x_0} = v_0 \cos\alpha$, $v_{y_0} = v_0 \sin\alpha$, $\vec{v_0} = \langle v_{x_0}, v_{y_0}\rangle$, $\vec{r}_0 = \langle x_0, y_0\rangle$. Integration gives $\vec{v}(t) = \vec{a}t + \vec{v}_0$, and $\vec{r}(t) = \frac{1}{2}\vec{a}t^2 + \vec{v}_0 t + \vec{r}_0$. Hence we have $x(t) = v_{x_0} t + x_0 = v_0 \cos\alpha + x_0$ and $y(t) = -\frac{1}{2}gt^2 + v_{y_0} t + y_0 = -\frac{1}{2}gt^2 + v_0 \sin\alpha t + y_0$. To simplify the calculations, most of the time the coordinate axis is placed with the origin at (x_0, y_0), which simplifies the formulas to be $x(t) = v_{x_0} t$ and $y(t) = -\frac{1}{2}gt^2 + v_{y_0} t$.

Using this formula, we can compute the height, flight time, and range by using tools from first semester calculus. The max height is achieved when the velocity is 0 in the y direction. Hence we want to solve $0 = -gt + v_{y_0}$ or $t = v_{y_0}/g = v_0 \sin\alpha/g$. The max height is the y value at this value of t, so the max height is $y_{\max} = -\frac{1}{2}g(v_{y_0}/g)^2 + v_{y_0} v_{y_0}/g = \frac{1}{2}v_{y_0}^2/g$. The range is found by calculating the x value at twice this time, or $R = v_{x_0}(2v_{y_0}/g) = 2v_0^2 \sin\alpha\cos\alpha/g = \sin(2\alpha)/g$.

The main point of this section is not if you can memorize the formulas and put numbers in them, but rather that you can actually derive these formulas. The homework in the textbook provides you with a bunch of problems where you are just putting in numbers and solving. I suggest that you start each problem from scratch and derive the formulas (as this is what I will test you on).

The following example comes from the textbook. However, the approach I use to solve it here is slightly different. An archer stands 6ft above ground level and shoots an arrow at an object which is 90 feet away in the horizontal direction and 74 ft above ground. The archer needs the arrow to hit the target at the peak of its parabolic path. For the purposes of this example, let $g = 32\text{ft/s}^2$. (see the textbook, page 908) What initial velocity and firing angle are needed to achieve this result? To answer this, we first decide where to place the origin. We will place the origin at 6ft above ground, so that the max height is 68 ft and it is achieved 90 ft away horizontally. We need to solve $\vec{r}(t_m) = \langle 90, 68\rangle = \left\langle v_{x_0} t_m, -\frac{1}{2}gt_m^2 + v_{y_0} t\right\rangle$, and $\vec{v}(t_m) = \langle v_{x_0}, 0\rangle = \langle v_{x_0}, -gt_m + v_{y_0}\rangle$ for t_m, v_{x_0}, v_{y_0}, as then $v_0 = \sqrt{v_{x_0}^2 + v_{x_0}^2}$ and $\alpha = \arctan(v_{y_0}/v_{x_0})$. We have $0 = -gt_m + v_{y_0}$ or $t_m = v_{y_0}/g$. The y coordinate of the position gives $68 = -\frac{1}{2}g(v_{y_0}/g)^2 + v_{y_0} v_{y_0}/g = \frac{1}{2}v_{y_0}^2/g$. Hence $v_{y_0} = \sqrt{2 \cdot 68 \cdot g}$. The x coordinate of the position gives $90 = v_{x_0} v_{y_0}/g$, or $v_{x_0} = 90g/v_{y_0} = 90g/\sqrt{2 \cdot 68 \cdot g}$. The rest of the variables can now be solved for to find the initial firing angle.

7.3 Arc Length

Recall the formulas for arc length are $s = \int_a^b \sqrt{[\frac{dy}{dx}]^2 + 1}dx$, $s = \int_c^d \sqrt{1 + [\frac{dx}{dy}]^2}dys = \int_a^b \sqrt{[\frac{dx}{dt}]^2 + [\frac{dy}{dt}]^2}dt$. Using differential notation $ds = \sqrt{dx^2 + dy^2}$, these formulas can all be summarized by the formula $s = \int_C ds$, where C represents the curve over which we are integrating. This notation introduces the notation for line integrals. By parametrizing the curve C as $r(t) = \langle x(t), y(t)\rangle$, we see $s = \int_C ds = \int_a^b |r'(t)|dt$. In other words, a little change of arc length ds is equal to the product of speed $|\vec{r}'(t)|$ and a little change in time dt. This is the same formula learn in grade school: distance = rate × time, where now we are just adding up a bunch of distances using definite integrals.

7.3.1 Reparametrizing by arc length

From now on we will assume that curves are smooth, which means that the curve is differentiable and $\vec{r}'(t) \neq \vec{0}$ (the velocity is never zero). When we follow a space curve $\vec{r}(t)$, the speed $|\vec{r}'(t)|$ traveled depends on the parameter t. At some points along the curve the speed could be larger than at others. We can introduce a new parametrization by speeding up if the the speed is less than one and slowing down if the speed is greater than one. This new parametrization will move at constant speed 1, so that every one unit increase in time results in a one unit increase in length. For a curve $\vec{r}(t)$, $a \leq t \leq b$, let $s(t) = \int_a^t |\vec{r}'(\tau)d\tau$ (note that we use τ as a dummy variable since t is already used in the bounds of the integral). We see that $s(t)$ calculates the length traveled by the curve between a and t. The fundamental theorem of calculus shows that $\frac{ds}{dt} = |\vec{r}'(t)|$. Since the speed is never zero, we can find an inverse function $t(s)$. The inverse function theorem states that $\frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{|\vec{r}'(t)|}$. The chain rule then gives the derivative of $\vec{r}(t(s))$ with respect to the arc length parameter s as $\frac{dr}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds} = \frac{\vec{r}'}{|\vec{r}'(t)|}$, which is a unit vector. Hence if we use s as the parameter, we traverse the curve at constant speed 1. Theoretically it is possible to always reparametrize any smooth curve. However, in practice it may not always be easy to actually find the parametrization.

For the helix $\vec{r}(t) = \langle\cos(t), \sin(t), t\rangle$, we can compute the following: $r'(t) = \langle -\sin(t), \cos(t), 1\rangle$, $|r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$. The length of one coil is $s = \int_0^{2\pi} \sqrt{2}dt = 2\pi\sqrt{2}$. The arc length parameter is $s(t) = \int_0^t \sqrt{2}dt = t\sqrt{2}$. Hence $t = s/\sqrt{2}$. So if we reparametrize the curve using the composite $\vec{r}(t(s)) = \left\langle \cos(\frac{s}{\sqrt{2}}), \sin(\frac{s}{\sqrt{2}}), \frac{s}{\sqrt{2}} \right\rangle$, we find $\frac{d\vec{r}}{ds} = \left\langle -\frac{1}{\sqrt{2}}\sin(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}}\cos(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}} \right\rangle$ and $\left|\frac{d\vec{r}}{ds}\right| = 1$.

7.4 The TNB frame

For a space curve $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, the TNB frame is an orthogonal collection of unit vectors which describe the tangential, normal, and binormal (tangential cross normal) directions of motion. Such a frame is necessary for a stationary observer to understand how the world looks to a moving object. The TNB frame gives an observer a way to place points in an xyz coordinate frame. Imagine two friends, one on the ground and another on a spacecraft which moves in the tangential direction ($\vec{T}$) with the left wing always pointing in the direction $\vec{N}$ of acceleration which is orthogonal to the tangential direction of motion. Then the binormal direction ($\vec{B}$) is the direction the head of the person on a spacecraft would point. The TNB frames gives a way of allowing the observer on the ground to give directions to the person in the spacecraft. The following table summarizes the discussion which follows.

Unit Tangent Vector	$\vec{T}$	$\frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$
Curvature Vector	$\vec{\kappa}$	$\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt} = \frac{d\vec{T}/dt}{\|\vec{v}\|} = \frac{\vec{T}'(t)}{\|\vec{r}'(t)\|}$
Curvature (not a vector, but a scalar)	κ	$= \left\|\frac{d\vec{T}}{ds}\right\| = \left\|\frac{d\vec{T}/dt}{ds/dt}\right\| = \frac{\left\|d\vec{T}/dt\right\|}{\|\vec{v}\|} = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$
Principal unit normal vector	$\vec{N}$	$\frac{1}{\kappa}\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{\|d\vec{T}/dt\|} == \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$
Radius of curvature	ρ	$1/\kappa$
Center of curvature		$\vec{r}(P) + \rho(P)\vec{N}(P)$
Binormal vector	$\vec{B}$	$\vec{T} \times \vec{N}$
Torsion	τ	$\pm\left\|\frac{d\vec{B}}{ds}\right\|$ (you have to pick the sign) or $-\frac{d\vec{B}}{ds}\cdot\vec{N}$
Tangential Component of acceleration	a_T	$\vec{a}\cdot\vec{T} = \frac{d}{dt}\|\vec{v}\|$
Normal Component of acceleration	a_N	$\vec{a}\cdot\vec{N} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa\|\vec{v}\|^2$

7.4.1 The unit tangent vector $\vec{T}$

The velocity $\vec{r}'(t) = \vec{v}(t)$ gives us the tangential direction of motion. Division by the magnitude gives the unit vector (called the unit tangent vector) $\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\vec{v}}{|\vec{v}|}$ Alternatively, if the curve is parametrized by arc length, then the speed is already 1, so the unit tangent vector is also $\vec{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds} = \vec{r}'(t)\frac{1}{ds/dt} = \vec{r}'(t)\frac{1}{|\vec{r}'(t)|} = \frac{\vec{v}}{|\vec{v}|}$.

7.4.2 If a vector valued function has constant length, then its derivative is orthogonal to the function.

First note that the product rule works for vector valued functions when considering the dot or cross product of two space curves (in general mathematicians define operations as products if those operations obey the product rule for derivatives). If the length of a vector is always constant, i.e. $|\vec{r}(t)| = c$, then square both sides to obtain $|r(t)|^2 = c^2$. Recall that $|\vec{r}|^2 = \vec{r}\cdot\vec{r}$, so we can write $\vec{r}(t)\cdot\vec{r}(t) = c^2$. Taking derivatives of both sides (using the product rule) gives $\vec{r}(t)\cdot\vec{r}'(t) + \vec{r}'(t)\cdot\vec{r}(t) = 0$, or simplifying since the dot product is commutative we have $2\vec{r}(t)\cdot\vec{r}'(t) = 0$. Divide both sides by 2 to obtain $\vec{r}(t)\cdot\vec{r}'(t) = 0$, which means that $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal. So vector valued functions with constant length have an orthogonal derivative. One way of visualizing this result is to realize that if the length of the position vector is never changing, then it must trace out a path on a circle (in 2D) or a sphere (in 3D). Then notice that on circles and spheres, tangent lines and tangent planes are always normal to a vector from the center to the point of tangency.

7.4.3 Curvature κ and the Principal Unit Normal Vector $\vec{N}$

The unit tangent vector $\vec{T}$ gives us the tangential direction of motion. The derivative of the unit tangential vector tells us how the direction of motion is changing. Since the unit tangent vector always has length one, its derivative is normal to the tangential direction. Curvature is a measure of the rate of change of the unit tangent vector per unit length. The curvature vector $\vec{\kappa} = \frac{d\vec{T}}{ds}$ points in a direction normal to $\vec{T}$. The magnitude of the curvature vector is called the curvature, and written $\kappa = |\frac{d\vec{T}}{ds}|$. The unit vector in the direction of the curvature vector is called the principal unit normal vector, and written $\vec{N}$. As reparametrizing by arc length can be difficult, it is convenient to give formulas for curvature that can be computed from a given parametrization. The formulas are listed at the beginning of this section.

7.4.4 Circle of curvature (osculating circle)

The circle of curvature at a point P on a curve where the curvature is nonzero is a circle in the plane containing the unit tangent and principle unit normal vectors which is tangent to the curve at P, has the same curvature as the curve at P, and whose center lies in the direction of the principal unit normal (i.e. the center is $\vec{r} + \rho\vec{N}$). This circle is the best approximating circle to the curve at P. The radius of the circle is $\rho = \frac{1}{\kappa}$. Note that the curvature of a circle of radius $\rho = a$ is $\kappa = 1/a$. The center of the circle of curvature is called the center of curvature.

7.4.5 The Binormal vector $\vec{B}$ and Torsion τ

The binormal vector is $\vec{B} = \vec{T} \times \vec{N}$. Notice that $\vec{B}$ is already a unit vector. The binormal vector provides the z axis for describing the world from the viewpoint of an object in motion where the x and y axes are given by the $\vec{T}$ and $\vec{N}$ directions, respectively.

The derivative $\frac{d\vec{B}}{ds} = \frac{d\vec{B}/dt}{ds/dt}$ measures how quickly the binormal vector changes as you move along a curve (or how quickly the object is twisting). Since $\vec{B}$ is constant length, the derivative $\frac{d\vec{B}}{ds}$ is orthogonal to $\vec{B}$. The following computations show that $\frac{d\vec{B}}{ds}$ is also orthogonal to $\vec{T}$, which means that $\frac{d\vec{B}}{ds}$ must be parallel to $\vec{N}$. We compute (using the product rule) $\frac{d\vec{B}}{ds} = \frac{d(\vec{T}\times\vec{N})}{ds} = \vec{T} \times \frac{d\vec{N}}{ds} + \frac{d\vec{T}}{ds} \times \vec{N} = \vec{T} \times \frac{d\vec{N}}{ds} + \vec{0}$ The last $\vec{0}$ comes because $\frac{d\vec{T}}{ds}$ is parallel to $\vec{N}$, and the cross product of parallel vectors is the zero vector. We see from this computation that $\frac{d\vec{B}}{ds} = \vec{T} \times \frac{d\vec{N}}{ds}$, which means that $\frac{d\vec{B}}{ds}$ is orthogonal to $\vec{T}$. Since $\frac{d\vec{B}}{ds}$ is orthogonal to both $\vec{B}$ and $\vec{T}$, it must be a scalar multiple of $\vec{N}$. Torsion τ is the opposite of the scalar component of $\frac{d\vec{B}}{ds}$ in the direction of $\vec{N}$, i.e. $\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}$. Torsion is a measure of the rate at which acceleration is causing an object to rotate out of the plane containing $\vec{T}$ and $\vec{N}$. Objects which are spiraling clockwise (as seen from behind the object) around some axis have positive torsion. Spiraling counterclockwise results in negative torsion. If you wrap your hand around the $\vec{T}$ vector in the direction of $\frac{d\vec{B}}{ds}$, then a clockwise rotation has your thumb point in the $\vec{T}$ direction. This is the reason for the choice of sign.

7.4.6 Tangential and Normal Components of acceleration

We now decompose the acceleration into tangential and normal components. The scalar component of acceleration in the tangential direction is called the Tangential Component of Acceleration a_T, and the scalar component of acceleration in the normal direction is called the Normal Component of Acceleration a_T. These can be computed using projections, namely $a_T = \text{comp}_{\vec{T}}\vec{a} = \frac{\vec{a}\cdot\vec{T}}{|\vec{T}|} = \vec{a}\cdot\vec{T}$, and $a_N = \text{comp}_{\vec{N}}\vec{a} = \frac{\vec{a}\cdot\vec{N}}{|\vec{N}|} = \vec{a}\cdot\vec{N}$. Alternatively, we can write velocity as magnitude times direction, $\vec{v} = |\vec{v}|\vec{T}$, and then compute (using the product rule)

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(|\vec{v}|\vec{T}\right) && \vec{v} = |\vec{v}|\vec{T} \\
&= \frac{d|\vec{v}|}{dt}\vec{T} + |\vec{v}|\frac{d}{dt}\vec{T} && \text{use the product rule} \\
&= \frac{d|\vec{v}|}{dt}\vec{T} + |\vec{v}|^2\frac{\frac{d}{dt}\vec{T}}{|\vec{v}|} && \text{multiply and divide by } |\vec{v}| \\
&= \frac{d|\vec{v}|}{dt}\vec{T} + |\vec{v}|^2\vec{\kappa} && \vec{\kappa} = \frac{d}{ds}\vec{T} = \frac{\frac{d}{dt}\vec{T}}{|\vec{v}|} \\
&= \frac{d|\vec{v}|}{dt}\vec{T} + |\vec{v}|^2\kappa\vec{N} && \vec{\kappa} = \kappa\vec{N}
\end{aligned}$$

From this computation we know that acceleration is in the plane formed by the tangential and principal unit normal vectors. Since $a_T = \frac{d|\vec{v}|}{dt}$, the tangential component of acceleration is positive if you are speeding up and negative if you are slowing down. The normal component of acceleration $a_N = |\vec{v}|^2\kappa$ grows and shrinks much more quickly with changes in speed than it does with changes in the curvature. Doubling the speed will cause the normal component of acceleration to quadruple. This is the reason that when you are driving on a mountain switchback and encounter a sharp turn, you see a posted speed limit of 15 miles. By dropping your speed from 45 miles per hour to 15, you reduce the normal component of acceleration by a factor of 9 (thus helping you from falling of the roads edge).

There are many other formulas for computing a_T and a_N. You don't need to memorize them, rather you should be able to derive them. Following are useful formulas for doing basic computations. Modern technology makes doing the computations in all cases simple, regardless of the formula. You should know how to derive these formulas given the hints I provide: $a_N = \sqrt{|\vec{a}|^2 - a_T^2}$ (this is very useful for computing a_N, just draw the vectors and use the Pythagorean theorem), $a_N = \vec{a}\cdot\vec{N}$ (the hard part here is computing $\vec{N}$), $\kappa = \dfrac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}$ (write $\vec{v}$ and $\vec{a}$ in terms of $\vec{T}$ and $\vec{N}$, compute $|\vec{v}\times\vec{a}|$ by distributing the cross product, and then solve for κ).

7.4.7 An Example

We will compute the quantities above for the helix $\vec{r}(t) = \langle\cos t, \sin t, t\rangle$. $\vec{v}(t) = \vec{r}'(t) = \langle-\sin t, \cos t, 1\rangle$, and speed $= |\vec{r}'(t)| = \sqrt{2}$, $\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)| = \frac{1}{\sqrt{2}}\langle-\sin t, \cos t, 1\rangle$, $d\vec{T}/dt = \frac{1}{\sqrt{2}}\langle-\cos t, -\sin t, 0\rangle$, $\vec{\kappa} = d\vec{T}/ds = \frac{1}{2}\langle-\cos t, -\sin t, 0\rangle$, $\kappa = |\vec{\kappa}| = \frac{1}{2}$, so $\rho = 2$, $\vec{N} = \vec{\kappa}/\kappa = \langle-\cos t, -\sin t, 0\rangle$, the center of curvature is at $\vec{r} + \rho\vec{N} = \langle\cos t, \sin t, t\rangle + 2\langle-\cos t, -\sin t, 0\rangle = \langle-\cos t, -\sin t, t\rangle$, $\vec{B} = \frac{1}{\sqrt{2}}\langle\sin t, -\cos t, 1\rangle$, $d\vec{B}/dt = \frac{1}{\sqrt{2}}\langle\cos t, \sin t, 0\rangle$, $d\vec{B}/ds = \frac{1}{2}\langle\cos t, \sin t, 0\rangle$, $\tau = -\frac{1}{2}\langle\cos t, \sin t, 0\rangle\cdot\langle-\cos t, -\sin t, 0\rangle = \frac{1}{2}$, $a_T = 0$, $a_N = 1$. In general you should not expect to find that $\kappa, \rho, \tau, a_T, a_N$ are integers, but rather some complex function of t. Even for parabolas, these formulas get messy really soon. Do a few problems from the text to make sure you understand how the computations proceed, but spend a majority of your time making sure you can prove the relationships found between the vectors.

Chapter 8

Line Integrals

After completing this chapter, you should be able to do the following:

1. Describe how to integrate a function along a curve. Use line integrals to find the area of a sheet of metal with height $z = f(x, y)$ above a curve $\vec{r}(t) = \langle x, \rangle$ and the average value of a function along a curve.
2. Compute work and flux of vector fields along and across piecewise smooth curves. Note that flow and circulation along curves are equivalent to finding work along a curve.
3. Find the following geometric properties of a curve: centroid, mass, center of mass, moments of mass, moments of inertia, and radii of gyration.
4. Be able to draw vector fields, determine if a field is a gradient field (hence conservative), and use the fundamental theorem of line integrals to simplify work calculations.

8.1 Preparation and Homework Suggestions

	Preparation Problems (11th ed)				Preparation Problems (12th ed)			
Day 1	16.1:1-8 (matching)	16.1:13	16.2:7	16.2:13 Day 1	16.1:1-8 (matching)	16.1:13	16.2:7	16.2:19
Day 2	16.1:24	16.2:17	16.2:25	16.2:37 Day 2	16.1:34	16.2:23	16.2:31	16.2:47
Day 3	Handout 1	3	5	6 Day 3	Handout 1	3	5	6
Day 4	16.3:5	16.3:15	16.3:21	16.3:26 Day 4	16.3:5	16.3:15	16.3:21	16.3:26

Here is the homework which lines up with the text.

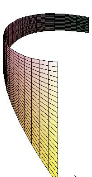
Topic (11th ed.)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Line integrals	16.1	1-8, 9-22, 23-32			33-36
Work, Flow, Circulation, Flux	16.2	7-16, 25-28, 37-40	17-24, 29-30, 41-44	45-46	47-52
Gradient Fields	16.2	1-6			
Gradient Fields	14.5	1-8			
Potentials	16.3	1-12,13-24	25-33	34-38	

Topic (12th ed.)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Line integrals	16.1	1-8, 9-26, 33-42	27-32		43-46
Work, Flow, Circulation, Flux	16.2	7-12, 19-24, 31-36, 47-50	13-18, 25-30, 37-38,	51-54	55-60
Gradient Fields	16.2	1-6			
Gradient Fields	14.5	1-10			
Potentials	16.3	1-12,13-24	25-33	34-38	

8.2 Line Integrals

The integral $\int_a^b f(x)dx$ is an integral in the plane of a function $y = f(x)$ over the interval (a, b). This integral gives the area of the region above the interval (assuming $f \geq 0$). A little piece of area dA is width dx times height f, so area can be computed using the integral $A = \int dA = \int_a^b f dx$. We now generalize this to integration along a curve.

To find the area of a metal sheet that has height $z = f(x, y)$ over a curve $\vec{r}(t) = \langle x(t), y(t)\rangle$, we approximate the area by breaking the curve up into little pieces of length ds. The height along a very small portion of the sheet can be assumed to be constant and is approximated by f. A small piece of area dA is approximately height times width, or $f\ ds$. To find the total area, we sum the approximate areas $A \approx \sum dA = \sum f ds$. As the small pieces ds of arc length approaches zero, we obtain the integral formula $A = \int dA = \int_C f ds = \int_a^b f|r'(t)|dt$, recall that $ds = |r'|dt =$ speed $\times\, d(\text{time})$. This is called a line integral. You parametrize a curve C, and then compute the integral using the formula $\int_C f ds = \int_a^b f|r'(t)|dt$. This formula extends to all dimensions. It is an extension of integrating along an interval. Line integrals are used to find work done by a non constant force to move an object along an arbitrary path. Work leads to the concepts of flow, circulation, and flux, and is central in understanding how energy is used to generate power.



Example: The area of a sheet of metal above the curve $\vec{r}(t) = \langle t, t^2\rangle$ for $-1 < t < 2$, with height given by $f = (x + 2)(y + 2)$, is $\int_{-1}^2 (t + 2)(t^2 + 2)\sqrt{(1)^2 + (2t)^2}dt = \int_{-1}^2 (t + 2)(t^2 + 2)\sqrt{1 + 4t^2}dt$. Solving this integral is rather time consuming, and technology will quickly get an answer. Please spend some time sharpening your integration skills (in particular u substitution and integration by parts), but do not spend so much time doing complex integrals that you do not get to practice the new ideas. Most of the solutions I have online just setup an integral.

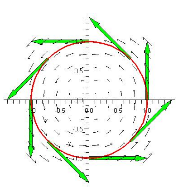
8.3 Work

If a constant force $\vec{F}$ acts on an object through a displacement in a straight line, then work is $W = \vec{F} \cdot \vec{r}$ (we use the dot product). If the displacement is not linear or the force is not constant, then this formula breaks down. To find work done by a non constant force $\vec{F}$ along an arbitrary curve $\vec{r}$, start by breaking the curve up into little pieces $d\vec{r} = \vec{T}ds$ (direction times magnitude, where $\vec{T}$ is the unit tangent vector). On each little piece of the curve, the force is approximately constant, and the displacement is approximately in a straight line, so we approximate the work done on each little piece as $dW = \vec{F} \cdot dr = \vec{F} \cdot \vec{T}ds$. Sum the little pieces of work to get an approximate total work

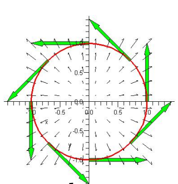
$W \approx \sum \vec{F} \cdot \vec{T} ds$. Taking limits gives the integral formula $W = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F} \cdot \frac{\vec{r}'}{|\vec{r}'|} |\vec{r}'| dt = \int_a^b \vec{F} \cdot \vec{r}' dt = \int_C \vec{F} \cdot d\vec{r}$. We use the differential notation $d\vec{r} = \vec{r}' dt$, which comes from the equation $\frac{d\vec{r}}{dt} = \vec{r}'$ (just multiply both sides by dt). If $\vec{F} = \langle M, N, P \rangle$ for some functions M, N, and P, then the work can be written $W = \int_C \langle M, N, P \rangle \cdot \langle dx, dy, dz \rangle = \int_C M dx + N dy + P dz$. These many formulas are different ways of representing the exact same quantity. It is common to represent vector fields using the notation $\vec{F} = \langle M, N, P \rangle$ or $\vec{F} = \langle P, Q, R \rangle$, depending on the book.

8.3.1 Flow (synonym for work) and Circulation (work on a closed curve)

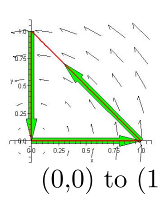
Flow along a curve C is a measure of how much fluid (with velocity field $\vec{F} = \langle M, N, P \rangle$) flows along a curve $C : \vec{r}(t), a \leq t \leq b$ per unit time. This quantity is particularly useful in the study of fluid mechanics, and studying how air flow near a wing provides lift for an airplane. The component of the velocity in the direction of the curve is $\text{comp}_{\vec{T}} \vec{F} = \vec{F} \cdot \vec{T}$. For each little piece of curve ds, the product $\vec{F} \cdot \vec{T} ds$ is approximately how much fluid will flow across this portion of the curve. Summing these small bits of flow gives the total flow by the line integral $\int_C \vec{F} \cdot T ds = \int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy + P dz$. If the curve is a simple closed curve (meaning it is piecewise smooth, starts and ends at the same point, and does not cross itself), and is oriented in the counterclockwise direction, then flow along C is called circulation around C, and we add a closed circle to the integral as in $\oint_C \vec{F} d\vec{r}$. The only difference between work, flow, and circulation is how we interpret the vector field, the computations are the same. A vector field is said to be **conservative** if circulation along every closed loop is zero, or equivalently if work calculations are independent of the path, and only depend on initial and starting point.



Example: Consider the vector field $\vec{F}(x, y) = \langle -y, x \rangle$. The circulation of $\vec{F}$ along a circle or radius a is found by first parametrizing the circle $C : r(t) = \langle a \cos t, a \sin t \rangle$. Hence $\vec{r}'(t) = \langle -a \sin t, a \cos t \rangle$ and $\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle -y, x \rangle \cdot \langle -a \sin t, a \cos t \rangle \, dt = \int_0^{2\pi} \langle -a \sin t, a \cos t \rangle \cdot \langle -a \sin t, a \cos t \rangle \, dt = \int_0^{2\pi} a^2 \sin^2 t + a^2 \cos^2 t dt = \int_0^{2\pi} a^2 dt = 2\pi a^2$. Notice that the integral of a constant along an interval is always the length of the interval multiplied by that constant. This will speed up the calculation of many integrals that we encounter throughout the semester.



Example: Now consider the vector field $\vec{F}(x, y) = \langle x, y \rangle$. To calculate the circulation of $\vec{F}$ along a circle or radius a, we use the same parametrization as last time. This time using the formula $\int_C M dx + N dy$, we have $x = a \cos t, y = a \sin t, dx = -a \sin t, dy = a \cos t, M = x = a \cos t, N = y = a \sin t$, so $\int_C M dx + N dy = \int_0^{2\pi} a \cos t(-a \sin t) + a \sin t(a \cos t) dt = \int_0^{2\pi} 0 dt = 0$. Notice that the vector field is always orthogonal to the unit tangent vector, which is why the flow along C is zero.



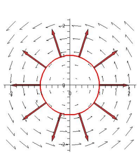
Example: Now consider the vector field $\vec{F}(x, y) = \langle -y, x \rangle$ and the curve which forms the boundary of the triangle with vertices $(0, 0), (1, 0), (0, 1)$. To find the work done by $\vec{F}$ through the displacement along C (or the flow along C, or the circulation around C), we first have to parametrize the curve. This is a piecewise smooth curve and as such is parametrized using 3 different curves. From (0,0) to (1,0), we get a direction vector for the line segment by subtracting the points, $\langle 1-0, 0-0\rangle$. We then write $C_1 : \vec r_1(t) = \langle 1,0\rangle t + \langle 0,0\rangle$ for $0 \le t \le 1$ (we know to stop at 1, because $\vec r_1(1) = \langle 1,0\rangle$). Similarly for the segment from (1,0) to (0,1), we write $C_2 : \vec r_2(t) = \langle -1,1\rangle t + \langle 1,0\rangle$ for $0 \le t \le 1$. For the segment from (0,1) to (0,0), we write $C_3 : \vec r_3(t) = \langle 0,-1\rangle t + \langle 0,1\rangle$ for $0 \le t \le 1$. We then compute

$$\begin{aligned}
\oint_C \vec F \cdot d\vec r &= \int_{C_1} \vec F \cdot d\vec r_1 && + \int_{C_2} \vec F \cdot d\vec r_2 && + \int_{C_3} \vec F \cdot d\vec r_3 \\
&= \int_0^1 \langle -y, x\rangle \cdot \langle 1,0\rangle\, dt && + \int_0^1 \langle -y, x\rangle \cdot \langle -1,1\rangle\, dt && + \int_0^1 \langle -y, x\rangle \cdot \langle 0,-1\rangle\, dt \\
&= \int_0^1 \langle -(0), (t+0)\rangle \cdot \langle 1,0\rangle\, dt && + \int_0^1 \langle -(t+0), (1-t)\rangle \cdot \langle -1,1\rangle\, dt && + \int_0^1 \langle -(1-t), 0\rangle \cdot \langle 0,-1\rangle\, dt \\
&= \int_0^1 0dt && + \int_0^1 (t+0) + (1-t)dt && + \int_0^1 0dt \\
&= 0 && +1 && +0 \\
&= 1.
\end{aligned}$$

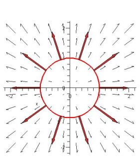
8.4 Flux across a smooth curve

Any simple closed curve (meaning the curve is piecewise smooth, doesn't cross itself, and starts and ends at the same point) divides the plane into two regions, which we will call the inside and outside of the curve. While flow is a measure of the rate of fluid flow along a curve, flux is a measure of fluid flow across a simple closed curve (a normal vector to the curve will be needed). If fluid is flowing out of region, then there is positive flux across the curve which is the boundary of the region. If you were to turn on a faucet at the origin and let water flow onto the plane, the water would flow outwards, and the flux would be positive across any curve which contained the origin in its interior. Alternatively, if you placed the drain of a sink at the origin, then water would flow into any region containing the origin and flux across a curve containing the origin in its interior would be negative.

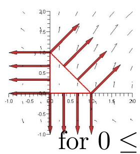
If $\vec n$ is an outward pointing unit vector, then $\vec F \cdot \vec n$ is the component of $\vec F$ in the outward normal direction. Break up the curve into small bits of curve ds, and then $d\text{Flux} = \vec F \cdot \vec n ds$ is the small amount of fluid which flows across each small bit ds of curve. If $\langle dx, dy\rangle$ represents the tangential direction, then $\langle dy, -dx\rangle$ gives the direction of the outward pointing normal (the dot product with the tangential vector is zero, and the vector points outward). A unit outward normal vector is hence $\frac{\langle dy, -dx\rangle}{\sqrt{dy^2+dx^2}}$. For flux across a curve C, we add up the small bits of flux and take limits to get the formula Flux $= \int_C dFlux = \int_C \vec F \cdot \vec n ds = \int_C \vec F \cdot \frac{\langle dy, -dx\rangle}{\sqrt{dy^2+dx^2}} \sqrt{dx^2+dy^2} = \int_C \langle M, N\rangle \cdot < dy, -dx >= \int_C Mdy - Ndx$.



Example: Consider the vector field $\vec F(x,y) = \langle -y, x\rangle$. The flux of $\vec F$ across a circle or radius a is found by first parametrizing the circle $C : r(t) = \langle a\cos t, a\sin t\rangle$. Hence $\vec r'(t) = \langle -a\sin t, a\cos t\rangle$, or $dx = -a\sin t$ and $dy = a\cos t$. Hence we have $\int_C Mdy - Ndx = \int_0^{2\pi} (-a\sin t)(a\cos t) - (a\cos t)(-a\sin t)dt = \int_0^{2\pi} 0dt = 0$. Notice that this spin field is always orthogonal to the outward normal. This should visually show that the flux is zero. It is valuable to look at a picture and try to decide if the flux is zero, positive, or negative, as this will give you some intuition about flux.



Example: Consider the vector field $\vec F(x,y) = \langle x, y\rangle$. We use the same parametrization as the previous examples, so $x = a\cos t, y = a\sin t, dx = -a\sin t, dy = a\cos t, M = x = a\cos t, N = y = a\sin t$. This gives the flux of $\vec F$ across C as $\int_C Mdy - Ndx = \int_0^{2\pi} a\cos t(a\cos t) - a\sin t(-a\sin t)dt = \int_0^{2\pi} a^2(\cos^2 t + \sin^2 tdt = 2\pi a^2$. You should notice that the vector field in this instance is always pointing out of the circle. Since the vector field and the outward normal are in the same direction, the flux is positive. Fluid is moving out of th interior of the circle.



Example: Consider the vector field $\vec{F}(x,y) = \langle xy, x+y\rangle$ and the curve C which forms the boundary of the triangle with vertices $(0,0),(1,0),(0,1)$. We first parametrize C by finding a parametrization for each curve. This already done above, and we had the three curves $C_1 : \vec{r}_1(t) = \langle 1,0\rangle t + \langle 0,0\rangle$ for $0 \le t \le 1$, $C_2 : \vec{r}_2(t) = \langle -1,1\rangle t + \langle 1,0\rangle$ for $0 \le t \le 1$, and $C_3 : \vec{r}_3(t) = \langle 0,-1\rangle t + \langle 0,1\rangle$ for $0 \le t \le 1$. We then compute the flux of $\vec{F}$ across C as

$$\begin{array}{llll}
\oint_C Mdy - Ndx & = \int_{C_1} (M)(0) - (x+y)(1)dt & + \int_{C_2} (xy)(1) - (x+y)(-1)dt & + \int_{C_3} (xy)(-1) - (N)(0)dt \\
 & = \int_0^1 -(t+0)dt & + \int_0^1 (1-t)(t) + (1-t) + (t)dt & + \int_0^1 (0)(1-t)(-1)dt \\
 & = \int_0^1 -tdt & + \int_0^1 t - t^2 + 1dt & + \int_0^1 0dt \\
 & = -\frac{1}{2} & + \frac{1}{2} - \frac{1}{3} + 1 & +0 \\
 & = \frac{2}{3} & &
\end{array}$$

8.5 Physical Applications

We now develop average value, centroid, mass, center of mass, moment of inertia, and radius of gyration. There are examples at the end of this section of how to do all of this, as well as a "Short Version" at the beginning which summarizes the ideas found herein. I have an alternate version of the material below which goes into more details about how the ideas are calculated. If you are going to graduate school, and want to spend a little more time with the theory, send me an email and I will provide you with the other handout.

8.5.1 The Short Version

Essentially in this learning module you are learning to use and derive differential formulas to find quantities such as area, centroid, mass, center of mass, moment of mass, moment of inertia, radii of gyration, work, and flux. The following formulas summarize the entire learning unit.

- Area: $dA = fdx, fds$
- Average Value: $(AV)s = \int fds$
- Mass: $dm = \delta ds$
- Centroid: $\langle \bar{x}, \bar{y}, \bar{z}\rangle s = \int \langle x,y,z\rangle ds$ (find average x, y, z values)
- Center of Mass: Replace s with m in centroid, $\langle \bar{x}, \bar{y}, \bar{z}\rangle m = \int_C \langle x,y,z\rangle dm$, where $M_{yz} = \int_C \bar{x}dm = \int xdm$ is a first moment of mass.
- $I = \int rad^2 dm$ is a moment of inertia, and $\int R^2 dm = \int rad^2 dm$ gives $R^2 m = I$ or $R = \sqrt{I/m}$ as the radius of gyration (where rad represents the generic distance from a point in space to the axis of rotation).
- Work: $d\text{Work} = \vec{F}\cdot\vec{T}ds = \vec{F}\cdot d\vec{r} = Mdx + Ndy(\text{in 2D}) = Mdx + Ndy + Pdz(\text{in 3D})$ where $\vec{T}$ is a unit tangent vector to C.
- Flux: $d\text{Flux} = \vec{F}\cdot\vec{n}ds = Mdy - Ndx$ where $\vec{n}$ is a unit normal vector to C.
- Fundamental Theorem of calculus: Change in f from a to b is $f(b) - f(a) = \int df = \int \frac{df}{dx}dx = \int_a^b f'(x)dx$.

- Fundamental Theorem of Line integrals: Change in f from A to B along C is $f(B)-f(A)=\int_C df=\int_C \frac{df}{ds}ds=\int_a^b \frac{df/dt}{ds/dt}ds=\int_a^b \vec{\nabla} f\cdot\frac{d\vec{r}}{dt}dt=\int_C \vec{F}\cdot d\vec{r}$.

I strongly suggest that you start each problem from a differential formula (one involving dx, dy, ds, or dm), and then work from there to convert it to a line integral in terms of t that you can evaluate. If you do this, you will eventually internalize each formula and know where each formula comes from.

8.5.2 A more detailed version

Average Value - how do you average infinitely many things:

Recall from first semester calculus the average value of a function formula given by $\bar{f}=\frac{1}{b-a}\int_a^b f(x)dx$, or $\bar{f}(b-a)=\int_a^b fdx$. This formula gives us a way of averaging together infinitely many values. The area underneath f from a to b is given by $\int_a^b f(x)dx$. Average value is the height $\bar{f}$ of a rectangle from a to b which has the same area as the area under f from a to b. Average value can be though of as follows: if we were to replace f by a constant $\bar{f}$, what should $\bar{f}$ equal so that $\int_a^b \bar{f}dx=\int_a^b fdx$. Since $\bar{f}$ is a constant, the left integral becomes $\bar{f}(b-a)$ and we then obtain the formula $\bar{f}=\frac{1}{b-a}\int_a^b f(x)dx$.

To get a visual picture of what average value does, imagine for a moment that you have an ant farm in front of you. The top of the sand in the ant farm will be our function f, and the left and right sides of the ant farm will be a and b. Now shake the ant farm so that the sand levels off. The height of the sand when you were done shaking would be the average value. Notice that high points are made low, and low points are made high.

Average value for curves is essentially the same idea. The average value of a function f along a curve C is the constant $\bar{f}$ so that $\int_C \bar{f}ds=\int_C fds$. The left integral becomes $\bar{f}\int_C ds=f\ s$ where s is the arc length of the curve. We divide by s to obtain the formula $\bar{f}=\frac{1}{s}\int_C f\ ds$. Average value can be used to find average temperatures, average stock prices, and in the ideas that follow average value gives us formulas for centroids, center of mass, and other geometrical values which are crucial in a study of energy.

Density and Mass: A density is the measure of a quantity per unit something. The derivative $\frac{dy}{dx}$ is a measure of change in y per unit change in x. A change dy of a function for some given change dx is computed by multiplying the density $\frac{dy}{dx}$ by the change dx, giving us the familiar equation $dy=\frac{dy}{dx}dx$. Adding up these approximate changes in y gives the integral formula $\int_a^b dy=\int_a^b f'dx$ which measures the total change in y from a to b (which is called the fundamental theorem of calculus). Hopefully this was a review. The new idea is that density is the measure of a quantity per unit something.

Mass density $\delta(x,y,y)$ in space is a measure of mass per unit volume, i.e density $=\frac{\text{mass}}{\text{unit volume}}$. The (mass) density of water is found by calculating the mass of some quantity of water, and dividing by its volume. The metric unit system was created so that 1 kg of water occupies 1 liter of space, giving a density of 1kg/L. If you mix oil and water, the density of oil is less than water which is why the oil and water separate themselves with the oil on top and the water underneath. Different substances have different densities. When an object is made out of many different materials, the density of the object may differ depending on where in the object you are. This is why we write $\delta(x,y,z)$, to reinforce the idea that density may vary based upon which portion of the object you are considering. If the density of an object is constant, to find the mass we just multiply the density by its volume.

If we consider an object that is located in space along curve $C:\vec{r}(t)=$

$\langle x, y, z\rangle)$ for $a \leq t \leq b$, then it is more convenient to think of density as mass per unit arc length. This gives us a density $\delta(x, y, z)$ at each point on the curve, and then the mass of each little portion of the curve can then be found by multiplying the density by the length of each little piece. In other words, if ds is a little length of curve with density δ, then a little piece of mass is approximately $dm = \delta ds$. Total mass of an object along this curve can be found by adding up little bits of mass, giving $m = \int dm = \int_C \delta ds$.

Centroid: The centroid $(\bar{x}, \bar{y}, \bar{z})$ of an object is the point in space whose x, y, z-values are the average x, y, z-values (where we assume the density is constant). The average value formula gives $\bar{x} = \frac{1}{s}\int_C xds, \bar{y} = \frac{1}{s}\int_C yds, \bar{z} = \frac{1}{s}\int_C zds$, or in vector form we write $\langle\bar{x}, \bar{y}, \bar{z}\rangle = \frac{1}{s}\int_C \langle x, y, z\rangle\, ds$.

Center of Mass: If the density of an object varies, then the heavier parts should contribute more to the average value than the lighter parts. This is accomplished by replacing s and ds with m and dm in the centroid formula. The center of mass of an object is $\langle\bar{x}, \bar{y}, \bar{z}\rangle = \frac{1}{m}\int_C \langle x, y, z\rangle\, dm = \frac{1}{m}\int_C \langle x, y, z\rangle\, dm$.

First Moments of mass: The quantities $M_{yz} = \int_C xdm, M_{xz} = \int_C ydm$, and $M_{xy} = \int_C zdm$ which appear in the formula for center of mass are called the first moments of mass about the yz plane, about the xz plane, and about the xy plane, respectively. The moment of a point with mas m about a plane is the product of its distance from the plane and its mass. The moment integration formulas are just a sum of moments about a plane. To find the first moment of an object about the plane $x = c$, the distance to the plane $x = c$ is $x - c$, so the moment is $M_{x=c} = \int_C (x - c)dm$ (times the distance $x - c$ by the mass dm). The center of mass is the value $\bar{x}$ such that the moment about the plane $x = \bar{x}$ is zero, or $\int_C (x - \bar{x})dm = 0$. Since $\bar{x}$ is a constant, this becomes $0 = \int_C sdm - \bar{x}\int_C dm$ or $\bar{x}m = \int xdm = m_{yz} = M_{x=0}$. Center of mass can be written in terms of moments by the formulas $\bar{x} = M_{yz}/m, \bar{y} = M_{xz}/m, \bar{z} = M_{xy}/m$. There are various applications of moments in statistics, physics, and mathematics, however the language used in each field is slightly different. Wikipedia has a wealth of information about this topic.

Moments of Inertia (second moments): Kinetic energy of a moving object is $KE = \dfrac{1}{2}mv^2$, where m is the mass and v is the velocity. A particle moving rotationally about some axis with radius r, has position function $r\theta$. The velocity is $v = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} = r\omega$, where ω is rotational velocity. The kinetic energy of a rotating particle is thus $KE = \dfrac{1}{2}m(r\omega)^2 = \dfrac{1}{2}mr^2\omega^2$. Since an object which is rotating has particles at different radii, we divide the object into little pieces with mass Δm and rotate each little piece around the axis of rotation, giving $\Delta KE = \dfrac{1}{2}\Delta mr^2\omega^2$. Summing gives us $KE \approx \sum \dfrac{1}{2}\Delta mr^2\omega^2 = \dfrac{1}{2}\omega^2 \sum r^2\Delta m$. Taking a limit gives $KE = \dfrac{1}{2}\omega^2 \displaystyle\int r^2 dm$, where r is the radius of rotation. The quantity $I = \int r^2 dm$ is called a moment of inertia about an axis of rotation (or a second moment). We can then write kinetic energy as $KE = \frac{1}{2}v^2 m = \frac{1}{2}\omega^2 I$. Notice that a moment of inertia takes the place of mass in rotational kinetic energy.

In 3D, the radius of rotation about an axis is the distance to the axis. About the x-axis the radius of rotation is $\sqrt{y^2 + z^2}$, about the y-axis the radius of rotation is $\sqrt{x^2 + z^2}$, and about the z-axis the radius of rotation is $\sqrt{x^2 + y^2}$ (you just leave off x when finding distance to x axis, leave off y when finding distance to y-axis, and leave off z when finding distance to z axis). This gives the formulas for the moments of inertia for a curve C with density δ to be $I_x = \int_C (y^2 + z^2)\delta ds$, $I_y = \int_C (x^2 + z^2)\delta ds$, and $I_z = \int_C (x^2 + y^2)\delta ds$ (we squared each distance, hence the square roots disappear). The single formula $I = \int r^2 dm$

describes all of these formulas. You can find the moment of inertia about any line. All you have to do is find a formula for the distance from a point on the curve to the axis of rotation.

Radii of Gyration: The radius of gyration R about an axis is a positive radius R such that if we replace r^2 with R^2 in the moment of inertia equation, we would have $\int R^2 dm = \int r^2 dm$. This means $R^2 m = I$ or $R = \sqrt{I/m}$. The radius of gyration about the x-axis is $R_x = \sqrt{I_x/m}$, and similarly $R_y = \sqrt{I_y/m}$ and $R_z = \sqrt{I_z/m}$. The radius of gyration about an axis is a rotational center of mass. It is used in studying energy, and is often used to simplify complex problems. Center of mass was found similarly, where we replaced x with $\bar{x}$ to obtain the equality $\int \bar{x} dm = \int x dm$ or $\bar{x} = \int x dm/m$.

8.5.3 Examples

Here an example of each idea with a single curve. When you create your lesson plan, I suggest that you focus on one particular space curve, density, and function. Then show how all the formulas are developed from there. The point to the next page is to show you how each formula is put together from the basic pieces x, y, z, ds, δ, and dm.

Consider the elliptical helix $C : \vec{r}(t) = \langle 3\sin t, 4\cos t, t\rangle$ for $0 \leq t \leq 2\pi$. The arc length differential is

$$ds = |\vec{r}'(t)|dt = \sqrt{(3\cos t)^2 + (-4\sin t)^2 + 1^2}dt = \sqrt{9\cos^2 t + 16\sin^2 t + 1}dt.$$

Arc length is then

$$s = \int_C ds = \int_0^{2\pi} \sqrt{9\cos^2 t + 16\sin^2 t + 1}dt.$$

The centroid of this curve is found using the integral formulas (which are too ugly to bother solving by hand). One example is

$$\bar{y} == \frac{1}{s}\int ds = \frac{\int_C y ds}{\int_C ds} = \frac{\int_0^{2\pi}(4\cos t)\sqrt{9\cos^2 t + 16\sin^2 t + 1}dt}{\int_0^{2\pi}\sqrt{9\cos^2 t + 16\sin^2 t + 1}dt}$$

. If the curve represents a wire in space with density given by $\delta(x, y, z) = x^2 + y^2 z$ kg/L, then $dm = \delta ds$ and we can calculate the centroid by replace ds with $dm = \delta ds$). We get the formulas $m = \int dm = \int_C \delta ds = \int_0^{2\pi}((3\sin t)^2 + (4\cos t)^2(t))\sqrt{9\cos^2 t + 16\sin^2 t + 1}dt$ and

$$\bar{y} = \frac{1}{m}\int dm = \frac{\int y dm}{\int dm} = \frac{\int_C y\delta ds}{\int_C \delta ds} = \frac{\int_0^{2\pi}(4\cos t)((3\sin t)^2 + (4\cos t)^2(t))\sqrt{9\cos^2 t + 16\sin^2 t + 1}dt}{\int_0^{2\pi}((3\sin t)^2 + (4\cos t)^2(t))\sqrt{9\cos^2 t + 16\sin^2 t + 1}dt}$$

The first moments of mass about the xz plane is the numerator of the previous formula. The moment of inertia about the x axis is

$$I_x = \int rad^2 dm = \int y^2 + z^2 dm = \int_0^{2\pi}[(4\cos t)^2 + (t)^2]((3\sin t)^2 + (4\cos t)^2(t))\sqrt{9\cos^2 t + 16\sin^2 t + 1}dt$$

with radius of gyration $R_x = \sqrt{I_x/m}$.

8.6 Gradients, Potentials, and the Fundamental Theorem

When the output dimension of a function is one, such as $f : \mathbb{R}^3 \to \mathbb{R}^1$, then the derivative in vector form is called the gradient and written $\nabla f = \langle f_x, f_y, f_z\rangle$. If a vector field $\vec{F} = \langle M, N, P\rangle$ is the gradient of some function f (so that $\nabla f = \vec{F}$), then we say that the vector field $\vec{F}$ is a gradient field, and the function f is called a potential for $\vec{F}$. The potential of a vector fields appears in differential equations, engineering, physics, and probably many other places of which I am not aware.

"Test for a gradient field": If a vector field is a gradient field (meaning there is an f with $\nabla f = \vec{F}$), and the potential f is twice continuously differentiable, then the second order mixed partial derivatives must be equal, namely $f_{xy} = f_{yx}, f_{xz} = f_{zx}$ and $f_{yz} = f_{zy}$. So if $\vec{F} = \langle M, N, P\rangle$ is a gradient field, then since $M = f_x, N = f_y, P = f_z$ we must have $M_y = N_x, M_z = P_x, N_z = P_y$. If these partial derivatives do not agree, then the vector field cannot be a gradient field. It can be shown that if a vector field is continuously differentiable on the entire plane, then $M_y = N_x, M_z = P_x, N_z = P_y$ implies that $\vec{F}$ is a gradient field. This gives us a way of checking if a vector field is a gradient field.

Differential Forms (exact forms correspond to gradient fields): A differential form is an expression of the form $Mdx + Ndy + Pdz$ (the thing that shows up in our work and flux integrals). The differential of a function $f(x, y, z)$ is the expression $df = Dfd\vec{x} = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = f_x dx + f_y dy + f_z dz$. If a differential form is the differential of a function f, then the differential form is said to be **exact**. The function f is called a potential for the differential form. Notice that $Mdx+Ndy+Pdz$ is exact if and only if $\vec{F} = \langle M, N, P\rangle$ is a gradient field. The differential form $xdx + zdy + ydz$ is exact because the potential $f = x^2/2 + yz$ satisfies $d(x^2/2 + yz) = xdx + zdy + ydz$, or $\nabla f = \langle x, y, z\rangle$. Notice that $M_y = 0 = N_x, M_z = 0 = P_x, N_z = 1 = P_y$. The differential form $-ydx + xdy$ is not exact because $M_y = -1$ is not equal to $N_x = 1$.

Finding a potential (undoing a total derivative): To find a potential, we simultaneously solve the differential equations $f_x = M, f_y = N$, and $f_z = P$. We have to find a function f which is a solution of all three integrals $\int Mdx, \int Ndy$, and $\int Pdz$. Essentially, we have to learn how to undo finding the total derivative. The following examples illustrate two method of doing so. The first method matches the book, the second is quicker route which simplifies the general idea.

Method 1: Consider the vector field $\vec{F} = \left\langle 2xy + x, x^2 - 3z, -3y + z^2\right\rangle$. Since $M_y = 2x = N_x, M_z = 0 = P_x$, and $N_z = -3 = P_y$, the field $\vec{F}$ has a potential. First integrate $\int Mdx$ to get $\int 2xy + xdx = x^2y + x^2/2 + A(y, z) = f$, where A is a constant with respect to x, which means that A may actually be a function of y and z. Then differentiate with respect to y to obtain $f_y = \frac{\partial}{\partial y}[x^2y + x^2/2 + A(y, z)] = x^2 + A_y(y, z)$. Since $N = f_y$, we have $x^2 - 3z = x^2 + A_y(y, z)$, or $A_y(y, z) = -3z$. Now integrate A_y with respect to y to obtain $A(y, z) = \int -3zdy = -3yz + B(z)$, where B is a constant with respect to y, which means that B may actually be a function of z. The partial derivative $\frac{\partial}{\partial z}[-3yz + B(z)] = -3y + B_z(z)$ should equal $P = -3y$, so we have $-3y+B_z(z) = -3y+z^2$, or $B_z(z) = z^2$. Integration yields $B = \int z^2dz = z^3/3+C$ for some constant C. Hence a potential is $f = x^2y + x^2/2 - 3yz + z^3/3 + C$ for any constant C.

Method 2: As an alternative approach, integrate all three functions simultaneously, ignoring the constants, to get $\int Mdx = x^2y+x^2/2, \int Ndy = x^2y+-3yz$, and $\int Pdz = -3yz + z^3/3$. Provided a potential exists, then the function f is formed by summing these integrals, ignoring duplicated terms. Since x^2y and $-3yz$ appear in multiple integrals (are duplicated terms), we include them once in the sum to obtain for a potential $f = x^2y + x^2/2 - 3yz + z^3/3$. This method will work if a potential exists. It is easy to check that f is a potential by computing $\nabla f = \langle 2xy + x, x^2 - 3z, -3y + z^2 \rangle$, which should equal $\vec{F}$.

Method 2 - second example: As another example, consider the vector field

$$\vec{F}(x, y, z) = \left\langle xy + yz + 1, \frac{1}{2}x^2 + xz - 3z, xy - 3y \right\rangle.$$

The test for a conservative vector fields shows $M_y = x + z = N_x, M_z = y = P_x, N_z = x - 3 = P_y$, which means $\vec{F}$ is conservative. A potential is found by integrating

$$\int xy+yz+1dx = \frac{1}{2}x^2y+xyz+x, \int \frac{1}{2}x^2+xz-3zdy = \frac{1}{2}x^2y+xyz-3yz, \int xy-3ydz = xyz-3yz.$$

The term xyz appears in all three, $\frac{1}{2}x^2y$ appears in the first and second, and $-3yz$ appears in the last two. A potential is found by summing the terms (not repeating duplicates) to obtain $f(x, y, z) = \frac{1}{2}x^2y + xyz + x - 3yz$.

When using the second approach, any time a term contains more than one variable, that term will appear in more than one integral. For the vector field $\vec{F} = \langle yz, xz, yz \rangle$, the integral $\int M dx = xyz$ contains x, y, and z, so the integral with respect to y and z must also contain this term. We compute $\int N dy = xyz, \int P dz = yz^2/2$, and notice that the same term appears in the y integral, but not in the z integral. Because $\int M dx = xyz$ contains all three variables but does not appear as a term in the third integral, we should be able to show using the "test for a conservative vector field" that there is no potential. Notice that $P_x = 0$ and $M_z = y$ are not equal, which means that $\vec{F}$ is not a gradient field. To prove a vector field does not have a potential, you must show that either $M_x \neq N_y, M_z \neq P_x$, or $N_z \neq P_y$. It is insufficient to say "There is no potential because I could not find one."

8.6.1 The Fundamental Theorem of Line Integrals - Why potentials are useful

When a vector field is a gradient field, there is a simple way to compute the work done by $\vec{F}$ along a curve $\vec{r}(t), a \leq t \leq b$. First find a potential f for $\vec{F}$, meaning $\vec{F} = \nabla f$. Let $A = \vec{r}(a)$ be your starting point and $\vec{r}(b)$ be your ending point. The work done is simply the difference in the potential from A to B, or integral form we write $W = \int dW = \int_a^b \vec{F} \cdot \vec{r}'(t)dt = \int_C M dx + N dy + P dx = f(B) - f(A)$. In other words, when a vector fields is conservative the work done is the difference in the potential. The reason for the word "potential" has to do precisely with the fact that differences in potential convert "potential energy" to work (another measure of energy). When a vector field has a potential, work done depends only on the initial and terminal point, not on the path chosen and we say that work done is independent of path.

Conservative Vector Field: If work done by a vector field $\vec{F}$ is independent of the path of motion, we say that $\vec{F}$ is a conservative vector field. If $\vec{F}$ is a gradient field, then it is conservative. The converse is true under appropriate conditions. Notice that if the start and end points are the same, then the work done by a conservative vector field is 0.

Example: Let $\vec{F} = \langle x + y, x + y \rangle$ and C is the upper semicircular curve in the plane starting at $(1, 0)$ and ending at $(-1, 0)$, followed by the straight line segment from $(-1, 0)$ to $(0, 0)$. A potential for $\vec{F}$ is $f(x, y) = \frac{1}{2}x^2 + xy + \frac{1}{2}y^2$. The flow of $\vec{F}$ along C is $\int_C \vec{F} \cdot d\vec{r} = f(0, 0) - f(1, 0) = -\frac{1}{2}$ by the fundamental theorem of line integrals. Without the fundamental theorem of line integrals, we write $r_1(t) = \langle \cos t, \sin t \rangle, 0 \leq t \leq \pi$ and $r_2(t) = \langle t, 0 \rangle, -1 \leq t \leq 0$, and then instead compute $\int_C \vec{F} \cdot d\vec{r} = \int_0^\pi \langle \cos t + \sin t, \cos t + \sin t\rangle \cdot \langle -\sin t, \cos t\rangle \, dt + \int_{-1}^0 \langle t+0, t+0\rangle \cdot \langle 1, 0\rangle \, dt = 0 - \frac{1}{2}$ as well.

The remainder of this document deals with explaining exactly why the fundamental theorem of line integrals works. Essentially it is an extension of the fundamental theorem of calculus. Since we will be seeing this theorem again in multiple different ways throughout the course, I have included the following in our lecture notes.

Change Density and the Fundamental Theorem of Calculus

The derivative of f is a density which measures change in f per unit length (so we could say the derivative is a "change density"). We write $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{(x+h)-x}$. The Fundamental Theorem of Calculus $f(b) - f(a) = \int_a^b f' dx$ says that you can find the total change in a function $f(b) - f(a)$ by adding up little changes dy which are the change density f' times length dx.

Begin by breaking the interval $[a, b]$ up into little pieces $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ of width $\Delta x_i = x_i - x_{i-1}$. The total change $f(b) - f(a)$ in f is calculated by adding up little changes $\Delta y_i = f(x_i) - f(x_{i-1})$. If we multiply and divide by Δx_i, then each little change is $\Delta y_i = \frac{f(x_i)-f(x_{i-1})}{\Delta x_i}\Delta x_i = \frac{\Delta y_i}{\Delta x_i}\Delta x_i \approx \frac{dy}{dx}\Delta x_i$, which is approximately the density $\frac{dy}{dx}$ times a length dx. The mean value theorem is the theoretical tool which allows us to remove the approximately equal and write $\frac{\Delta y_i}{\Delta x_i}\Delta x_i = \frac{dy}{dx}(c_i)\Delta x_i$ for some c_i between x_{i-1} and x_i. Summing the approximate changes and taking a limit as $\Delta x_i \to 0$ gives the fundamental theorem of calculus. Notationally, this is all summarized as

$$\begin{aligned} f(b) - f(a) &= f(b) - f(x_{n-1}) + f(x_{n-1}) - f(x_{n-2}) + \cdots + f(x_2) - f(x_1) + f(x_1) - f(x_a) \\ &= \sum f(x_i) - f(x_{i-1}) = \sum \frac{f(x_i) - f(x_{i-1})}{\Delta x_i}\Delta x_i = \sum \left(\frac{\Delta y_i}{\Delta x_i}\right)\Delta x_i \approx \sum \frac{dy}{dx}\Delta x_i \end{aligned}$$

Taking limits gives us $f(b) - f(a) = \int_a^b f'(x)dx$, the fundamental theorem of calculus.

Proving The Fundamental Theorem of Line Integrals

Let $f(x, y, z)$ be a function. Let C be a smooth space curve, parametrized by $\vec{r}(t), a \leq t \leq b$. Let $A = \vec{r}(a)$ and $B = \vec{r}(b)$ be the start and end points of the curve. The total change in f from A to B is $f(B) - f(A) = f(\vec{r}(b)) - f(\vec{r}(a))$. The fundamental theorem of line integrals states that

$$f(B) - f(A) = \int_C d(f \circ \vec{r}) = \int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f \cdot \vec{r}\,' dt.$$

The change density is a change df per unit length ds, where we use arc length for length because motion is along a curve. The change density can be written $\frac{df}{ds} = \frac{d(f \circ r)}{dt}\frac{dt}{ds}$. The chain rule gives $\frac{df}{ds} = Df D\vec{r}\frac{dt}{ds} = \nabla f \cdot \vec{r}\,' \frac{dt}{ds}$. Multiplication by ds gives $df = \frac{df}{ds}ds = \nabla f \cdot \vec{r}\,' \frac{dt}{ds}ds = \nabla f \cdot \vec{r}\,' dt$.

We compute total change as follows. Begin by breaking the interval $[a, b]$ up into little pieces $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ Total change in f is then

$$\begin{aligned} f(B) - f(A) &= f(\vec{r}(b)) - f(\vec{r}(t_{n-1})) + f(\vec{r}(t_{n-1})) - f(\vec{r}(t_{n-2})) + \cdots + f(\vec{r}(t_1)) - f(\vec{r}(t_a)) \\ &= \sum f(\vec{r}(t_i)) - f(\vec{r}(t_{i-1})) = \sum \Delta(f \circ \vec{r}) = \sum \frac{f(\vec{r}(t_i)) - f(\vec{r}(t_{i-1}))}{\Delta t_i}\Delta t_i \\ &= \sum \left(\frac{\Delta(f \circ \vec{r})}{\Delta t_i}\right)\Delta t_i = \sum D(f \circ \vec{r})(t)\Delta t_i = \sum Df D\vec{r}\Delta t_i = \sum \nabla f \cdot \vec{r}\,' \Delta t_i \end{aligned}$$

Taking limits gives us the fundamental theorem of line integrals: $f(B) - f(A) = \int_C \nabla f \cdot d\vec{r}$.

Chapter 9

Optimization

After completing this chapter, you should be able to do the following:

1. Explain the properties of the gradient, its relation to level curves and level surfaces, and how it can be used to find directional derivatives.
2. Find equations of tangent planes using the gradient and level surfaces. Use the derivative (tangent planes) to approximate functions, and use this in real world application problems.
3. Explain the second derivative test in terms of eigenvalues. Use the second derivative test to optimize functions of several variables.
4. Use Lagrange multipliers to optimize a function subject to constraints.

9.1 Preparation and Homework Suggestions

	Preparation Problems (11th ed)				Preparation Problems (12th ed)			
Day 1	14.5:13	14.5:17	14.6:1	14.6:13	14.5:15	14.5:19	14.6:1	14.6:13
Day 2	14.7:3	14.7:21	14.5:31-32	14.6:50a	14.7:2	14.7:16	14.5:35-36	14.6:54a
Day 3	14.8:5	14.8:15	14.8:27	14.8:33	14.8:5	14.8:15	14.8:27	14.8:33

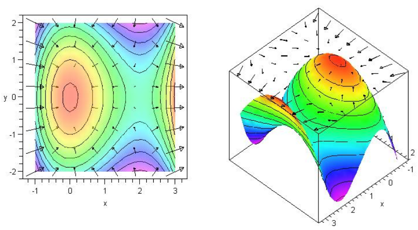
The following homework problems line up with the topics we will discuss in class. Please do enough of each type of problem to master the material.

Topic (11th ed.)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Directional Derivatives and the Gradient	14.5	1-22	23-32	33-36	
Tangent Planes and approximation	14.6	1-22	23-24, 47-58, 60-63	59	
2nd Derivative Test (use eigenvalues)	14.7	1-38	39-44, 49-52,	45-48, 53-64	65-70
Lagrange Multipliers	14.8	1-32	33-40	41-44	45-50

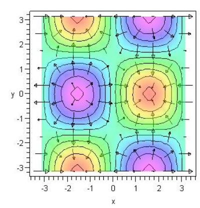
Topic (12th ed.)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Directional Derivatives and the Gradient	14.5	1-24	25-36	37-40	
Tangent Planes and approximation	14.6	1-22	23-24, 31-32, 49-62, 64-67	63	
2nd Derivative Test (use eigenvalues)	14.7	1-38	39-44, 49-60,	45-48, 61-68	69-74
Lagrange Multipliers	14.8	1-32	33-40	41-44	45-50

9.2 The Gradient

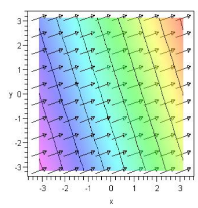
For a function $f : \mathbb{R}^n \to \mathbb{R}$ with one output, the derivative Df when written as a vector is called the gradient of f and written ∇f. If C is a level curve $c = f(x, y)$ of the function $f(x, y)$, and $\vec{r}(t)$ is a parametrization of the curve in the plane, then the composition $f(\vec{r}(t))$ equals the constant c. The chain rule for derivatives then shows that $DfD\vec{r} = 0$, or $\nabla f \cdot \vec{r}' = 0$. This means that the gradient is orthogonal to the direction vector of a tangent line to the level curve, i.e. the gradient is normal to level curves. The gradient of a function $f(x, y, z)$ is normal to level surfaces. The pictures below illustrate this idea for several functions. The first two pictures show both a 2D and 3D plot of the same function. The next two pictures give contour plots and gradient field plots of two functions. The last plot is a 3D plot of several level surfaces. The gradient vectors are normal to the level surfaces.



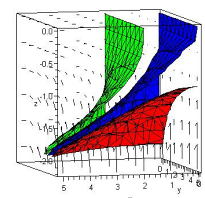
$f = x^3 - 3x^2 - y^2 + 2$



$f = \sin x \cos y$



$f = 3x + y$



$f = x^2 + yz^3$

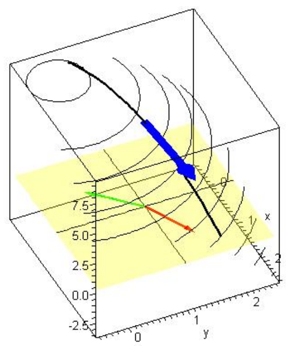
For a function $z = f(x, y)$, the partial derivatives f_x and f_y give the slope of the function in the x and y directions, respectively. Let $\vec{u} = \langle u_1, u_2\rangle$ be a unit vector (representing any direction), and (a, b) a point in the domain of f. The directional derivative of f in the direction of $\vec{u}$ at (a, b) is

$$D_{\vec{u}}f(a, b) = \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} = Df(a, b)\vec{u} = \vec{\nabla} f(a, b) \cdot \vec{u}.$$

Geometrically, the directional derivative is the slope of a tangent line to the surface at $(a, b, f(a, b))$ which lies in a vertical plane containing the vector $\langle u_1, u_2, 0\rangle$. The intersection of this vertical plane with the surface is the space curve $\vec{r}(t) = \langle a + u_1t, b + u_2t, f(a + u_1t, b + u_2t)\rangle$, which passes through the point $(a, b, f(a, b))$ at $t = 0$. The chain rule gives the derivative of $\vec{r}$ at $t = 0$ as $r'(0) = \langle u_1, u_2, Df(r(0))Dr(0)\rangle = \langle u_1, u_2, f_x(a, b)u_1 + f_y(a, b)u_2\rangle = \langle u_1, u_2, \nabla f(a, b) \cdot \vec{u}\rangle$. One unit increase in the $\vec{u}$ direction gives a rise in the z direction of $\nabla f(a, b) \cdot \vec{u}$ units. Hence we see that the directional derivative is $D_{\vec{u}}f(a, b) = \nabla f(a, b) \cdot \vec{u}$.

Recall the differential notation $dy = f'dx$ from 112. This can be extended to all dimensions as $d\vec{y} = D\vec{f}d\vec{x}$, or $d(\text{outputs}) = Dfd(\text{inputs})$. For a function $z = f(x, y)$, we have $dz = \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$. If $\vec{u} = \langle u_1, u_2\rangle$ is a unit vector, then a change in the inputs $dx = u_1$ and $dy = u_2$ gives a change in the output dz as the directional derivative in the direction of $\vec{u}$. We again have $D_{\vec{u}}f(a, b) = Df(a, b)\vec{u} = \nabla f(a, b) \cdot \vec{u}$. Interpretations about the gradient come immediately when you use the differential notation $d(\text{outputs}) = Dfd(\text{inputs})$.

Since $D_{\vec{u}}f(a, b) = \nabla f(a, b) \cdot \vec{u} = |\nabla f(a, b)||\vec{u}| \cos\theta$ for θ the angle between $\nabla f(a, b)$ and $\vec{u}$, we see that the directional derivative is greastest when $\cos\theta = 1$ or $\theta = 0$. The direction of greatest increase is given by the gradient. The direction of greatest decrease is opposite the gradient.



The function $f(x,y) = 9 - x^2 - y^2$ has deriviative $Df = \begin{bmatrix}-2x & -2y\end{bmatrix}$, i.e. $\nabla f = \langle -2x, -2y\rangle$ The directional derivative in the $\vec{u} = \langle 1, 0\rangle$ direction at any point (x, y) is $D_{\vec{u}}f(x,y) = \langle -2x, -2y\rangle \cdot \langle 1, 0\rangle = -2x$, or the derivative in the x direction. The directional derivative in the direction $\langle 2, 1\rangle$ (which as a unit vector is $\frac{1}{\sqrt{5}}\langle 2, 1\rangle$) is $D_{\langle 2,1\rangle}f(x,y) = \langle -2x, -2y\rangle \cdot \frac{1}{\sqrt{5}}\langle 2, 1\rangle = \frac{1}{\sqrt{5}}(-4x-2y)$. At the point $(1, 1)$, we have $\nabla f(1,1) = \langle -2, -2\rangle$ and $D_{\langle 2,1\rangle}f(1,1) = \langle -2, -2\rangle \cdot \frac{1}{\sqrt{5}}\langle 2, 1\rangle = -\frac{6}{\sqrt{5}}$. The direction of greatest increase at $(1, 1)$ is in the $\langle -2, -2\rangle$ direction. This is illustrated in the picture to the left. The xy plane is shaded in the picture. The direction of the gradient points back to the origin (it is a 2D vector). The direction of u points away from the origin (hence the dot product is negative). Several level curves of f are drawn in 3D with their height included. The space curve shown is a curve in the surface. A tangent vector to that curve is also drawn. The change in height of that tangent vector is the directional derivative, as the change in the xy direction is 1 unit.

9.3 Tangent planes and Approximation

Since the gradient is normal to level surfaces, the gradient $\nabla f(x, y, z)$ of function $f(x, y, z)$ can be used to find the tangent plane to level surfaces. For example, the hyperboloid of one sheet $1 = x^2 + y^2 - z^2$ is the level surface $f = 1$ of the function $f = x^2 + y^2 - z^2$. The point $(1, -2, 2)$ is on this hyperboloid, so the gradient $\nabla f(x, y, z) = \langle 2x, 2y, -2z\rangle$ evaluated at $(1, -2, 2)$ gives the vector $\nabla f(1, -2, 2) = \langle 2, -4, -4\rangle$, which is a normal vector for the tangent plane to the hyperboloid at $(1, -2, 2)$. An equation of the tangent plane is thus $2(x - 1) - 4(y + 2) - 4(z - 2) = 0$.

If a surface can be written in the form $z = f(x, y)$, then the function $g(x, y, z) = z - f(x, y)$ has as its gradient $\nabla g = \langle -f_x, -f_y, 1\rangle$. Hence, a normal vector to the tangent plane of a surface $z = f(x, y)$ is $\vec{n} = \langle -f_x, -f_y, 1\rangle$, which we already discovered as $\vec{n} = \langle 1, 0, f_x\rangle \times \langle 0, 1, f_y\rangle$.

The differential $dy = f'dx$ says a little change in y can be approximated by multiplying the derivative by a little change in x. For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, the **differential** is $d\vec{y} = D\vec{f}d\vec{x}$, where $D\vec{f}$ is the derivative, $d\vec{y}$ is an m dimensional vector of changes in the outputs, and $d\vec{x}$ is an n dimensional vector of changes in the inputs. We estimate changes in our outputs by multiplying the derivative by changes in our inputs. For a function $z = f(x, y)$, this gives $dz = f_x dx + f_y dy$. If we write our change in inputs as $\langle dx, dy\rangle = \vec{u}ds$ for a unit vector $\vec{u}$ with magnitude ds, then a change in z is approximately $dz = Df\vec{u}ds = D_{\vec{u}}f ds$, or the product of the directional derivative and ds.

The differential formula $d\vec{y} = D\vec{f}d\vec{x}$ can be used to connect ideas about tangent lines, tangent planes, and approximation in all dimensions. The following table summarizes how to use this notation in various settings.

Function	$d\vec{y}$	Df	$d\vec{x}$	$\vec{x}=\vec{c}$	Tangent space (line, plane, etc.) $d\vec{y}=D\vec{f}(\vec{c})d\vec{x}$ $\vec{y}-\vec{f}(\vec{c})=D\vec{f}(\vec{c})(\vec{x}-\vec{c})$
$y=f(x)$	dy	f'	dx	$x=c$	$y-f(c)=f'(c)(x-c)$
$\vec{r}(t)=\langle x,y,z\rangle$	$\begin{bmatrix}dx\\dy\\dz\end{bmatrix}$	$\begin{bmatrix}x'(t)\\y'(t)\\z'(t)\end{bmatrix}$	dt	$t=c$	$\begin{bmatrix}x-x(c)\\y-y(c)\\z-z(c)\end{bmatrix}=\begin{bmatrix}x'(c)\\y'(c)\\z'(c)\end{bmatrix}(t-c)$
$z=f(x,y)$	dz	$\begin{bmatrix}f_x & f_y\end{bmatrix}$	$\begin{bmatrix}dx\\dy\end{bmatrix}$	$(x,y)=(a,b)$	$z-f(a,b)=\begin{bmatrix}f_x(a,b) & f_y(a,b)\end{bmatrix}\begin{bmatrix}x-a\\y-b\end{bmatrix}$
$\vec{r}(u,v)=\langle x,y,z\rangle$	$\begin{bmatrix}dx\\dy\\dz\end{bmatrix}$	$\begin{bmatrix}x_u & x_v\\y_u & y_v\\z_u & z_v\end{bmatrix}$	$\begin{bmatrix}du\\dv\end{bmatrix}$	$(u,v)=(a,b)$	$\begin{bmatrix}x-x(a,b)\\y-y(a,b)\\z-z(a,b)\end{bmatrix}=\begin{bmatrix}x_u & x_v\\y_u & y_v\\z_u & z_v\end{bmatrix}\begin{bmatrix}u-a\\v-b\end{bmatrix}$

A change $d\vec{y}$ in output is the output variable $\vec{y}$ minus its value $\vec{f}(\vec{c})$ at the input $\vec{c}$. A change $d\vec{x}$ in the input is the variable $\vec{x}$ minus its value at $\vec{c}$. Tangent planes can be found in all dimensions using this notation.

We can use differential notation to estimate changes in a function. If the temperature at a point in the plane is given by $T(x,y)=x^2-xy-y^2$ degrees Fahrenheit, and a particle is at $(2,3)$, estimate the change in temperature if the particle moves about .1 units in the direction of $\langle 3,4\rangle$. The gradient of T is $\nabla T=\langle 2x-y,-x-2y\rangle$, which at $(2,3)$ is $\nabla T(2,3)=\langle 1,-8\rangle$. The change in inputs is $\langle dx,dy\rangle=.1\frac{\langle 3,4\rangle}{|\langle 3,4\rangle|}=\frac{.1}{5}\langle 3,4\rangle=\langle .06,.08\rangle$. We calculate the change in temperature (the output) as $dT=DT(2,3)\begin{bmatrix}dx\\dy\end{bmatrix}=T_xdx+T_ydy=(1)(.06)+(-8)(.08)=-.58$. So the temperature will decrease a little over half a degree.

A rectangle is rather wide and short. When you measure the edges of the rectangle to estimate the area, which edge must be measured more precisely to not affect the area of the triangle? Area is $A(h,w)=hw$, so $dA=\begin{bmatrix}A_h & A_w\end{bmatrix}\begin{bmatrix}dh\\dw\end{bmatrix}=\begin{bmatrix}w & h\end{bmatrix}\begin{bmatrix}dh\\dw\end{bmatrix}=wdh+hdw$. Since w is large, a small change dh in the measurement of the height will have a much larger affect on the change in area. Hence, you must be more precise when measuring the shorter height.

The total resistance R in a circuit with two parallel resistors with resistance R_1 and R_2 is given by the formula $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$. Solving for R and taking derivatives, one can show that $dR=\begin{bmatrix}\frac{R^2}{R_1^2} & \frac{R^2}{R_2^2}\end{bmatrix}\begin{bmatrix}dR_1\\dR_2\end{bmatrix}$. If R_1 changes from 10 to 9.9, and R_2 changes from 20 to 20.2, would you expect a positive or negative change in the total resistance R? We have $dR_1=-.1$ and $dR_2=.2$, and $\frac{1}{R}=\frac{3}{20}$, so $R=20/3$ and $dR=\begin{bmatrix}\frac{(20/3)^2}{10^2} & \frac{(20/3)^2}{20^2}\end{bmatrix}\begin{bmatrix}-.1\\.2\end{bmatrix}=(2/3)^2(-.1)+(1/3)^2(.2)=-4/90+2/90=-2/90<0$ which is negative. Manufactures of circuit boards have to account for variations in the resistance of resistors. Differentials allow them to estimate total changes in resistance due to possible changes in each resistor.

9.3.1 Local Linearization

The textbook discusses the "Local linearization" of a function as $L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$, which is equivalent to $L(x,y)-f(a,b)=$

df, or $L = f + dz$ (just add the function to the change in the function to get an approximate value for the function at (x, y)). The error $dz = L(x, y) - f(a, b)$ in approximating f is also called $E(x, y)$ in the book. You can estimate how much error there is by the formula $|E| \leq \frac{1}{2}M(|dx| + |dy|)^2$, where M is any upper bound for the values $|f_{xx}|, |f_{yy}|$, and $|f_{xy}|$. The value of this formula is that it tells you how far off the change dz could be from the real change $\Delta z = f(x + dx, y + dy) - f(x, y)$ in output.

9.4 The Second Derivative Test

The first derivative test breaks down in higher dimensions, because there are more than 2 ways to approach a point of the domain. All that can be said is that at an extreme value, the gradient is zero (as the tangent plane should be horizontal). In higher dimensions, there are three classifications of critical points: maximum, minimum, saddle point (a point where the tangent plane is horizontal, but in some directions you increase and in other directions you decrease). The second derivative test does not break down. To understand the second derivative test, we need to learn about eigenvalues and eigenvectors of a matrix.

If $z = f(x, y)$, then $Df(x, y) = \begin{bmatrix} f_x & f_y \end{bmatrix}$ is a vector field with two inputs and two outputs. Its derivative $D^2f(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ is a 2×2 square matrix called the Hessian of f. This matrix is symmetric, in that the upper right and lower left are the same, if f is twice continuously differentiable.

An eigenvector is a vector (direction), such that multiplication by the matrix is the same as multiplying the vector by a scalar (that scalar is called an eigenvalue). Notationally this is written $A\vec{v} = \lambda\vec{v}$, where λ is the eigenvalue and $\vec{v}$ is an eigenvector. Eigenvalues can be found by subtracting λ from each of the entries on the diagonal of a square matrix, and then asking for which values of λ the determinant equals zero (solve $det(A - \lambda I) = 0$). In linear algebra, you will learn much more about eigenvectors and eigenvalues. For example, the eigenvalues of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are found as follows. Write $\det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0$. Compute $(2-\lambda)(2-\lambda)-(1)(1) = 0$ or $4-4\lambda+\lambda^2-1 = 0$. Hence $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$ or the eigenvalues are $\lambda = 1, 3$.

The eigenvalues of D^2f give the "directional" second derivative in the direction of a corresponding eigenvector. The largest eigenvalue is the largest possible value of the second derivative in any direction. The smallest eigenvalue is the smallest possible value of the second derivative in any direction. The **second derivative test** follows: If the eigenvalues are all positive at a critical point, then in every direction the function is concave upwards, which means that the function has a minimum at that critical point. If all the eigenvalues are negative, then in every direction the function is concave downwards, and function has a maximum there. If there is a positive eigenvalue and a negative eigenvalue, the function has a saddle point there. If either the largest or smallest eigenvalue is zero, then the second derivative test fails.

For the function $f(x, y) = x^2-6xy+y^2$, the gradient is $Df = \begin{bmatrix} 2x - 6y & -6x + 2y \end{bmatrix}$, which is zero only at $x = 0, y = 0$ (solve the system of equations $2x - 6y = 0, -6x+2y = 0$). The Hessian is $D^2f = \begin{bmatrix} 2 & -6 \\ -6 & 2 \end{bmatrix}$. The eigenvalues are found by solving $0 = \det \begin{bmatrix} 2-\lambda & -6 \\ -6 & 2-\lambda \end{bmatrix} = (2-\lambda)^2-36 = 4-4\lambda+\lambda^2-36 = (\lambda-8)(\lambda+4)$, so $\lambda = 8, -4$ are the eigenvalues. Since there is a positive eigenvalue and a negative eigenvalue, the function is concave upwards in one direction and concave downwards in another direction, so there is a saddle point at the origin.

For the function $f(x,y) = x^3 - 3x + y^2 - 4y$, the gradient is $Df = \begin{bmatrix} 3x^2 - 3 & 2y - 4 \end{bmatrix}$, which is zero at $x = 1, y = 2$ or $x = -1, y = 2$. Hence there are two critical points. The Hessian is $D^2 f = \begin{bmatrix} 6x & 0 \\ 0 & 2 \end{bmatrix}$. Since there are two critical points, we have th find the eigenvalues of two matricies. When $x = -1, y = 2$, the eigenvalues of $\begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix}$ are $\lambda = -6, 2$. Since one is positive and one is negative, there is a saddle point at $(-1, 2)$. When $x = 1, y = 2$, the eigenvalues of $\begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$ are $\lambda = 6, 2$. Since both are positive, there is a minimum at $(-1, 2)$ (as in every direction the function is concave upwards).

We can use these ideas to find the dimensions of the rectangular prism of maximum volume that is located above the xy plane, and below the paraboloid $z = 9 - x^2 - y^2$. We want to maximize the function $V(x,y) = (2x)(2y)(9 - x^2 - y^2)$. The gradient is zero at $x = 3/2, y = 3/2$, and the Hessian at that critical point is $\begin{bmatrix} -54 & -18 \\ -18 & -54 \end{bmatrix}$. The eigenvalues are $\lambda = -36, -72$, which are both negative, which means we have found a maximum. I skipped a lot of details which you can check. The maximum volume is $f(3/2, 3/2) = 81/2$ and occurs at $(3/2, 3/2)$. Since x is only half the length, the dimensions are 3 by 3 by 9/2.

If the domain of a function is restricted to a small region, then you use the second derivative test to find the optimum solutions on the interior of the domain. You use the first derivative test on the boundary of the domain to find the optimum solutions on the boundary. If a function is continuous on a domain that is closed (includes its boundary) and bounded, then there will always be a maximum and minimum (this is called the extreme value theorem).

9.5 Lagrange Multipliers

To optimize a function f, subject to a constraint (such as a height restriction, or a budget constraint for a business), we use a technique called Lagrange Multipliers. Let f be the function you want to optimize and $g = 0$ is the constraint. Define $L = f - \lambda g$ for some scalar lambda which will be determined. Find where all partials of L are zero. Under suitable conditions, the optimum solutions will be a solution of $\nabla L = \vec{0}$.

For example, to find the dimensions of the rectangular prism of maximum volume that is located above the xy plane, and below the paraboloid $z = 9 - x^2 - y^2$, we want to optimize the volume function $f(x,y,z) = 4xyz$. Our constraint is the height of z, which we rewrite as $g(x,y,z) = z - (9 - x^2 - y^2) = 0$. The Lagrangian is $L(x,y,z,\lambda) = 4xyz - \lambda(z - 9 + x^2 + y^2)$, and so the partials of L are $L_x = 4yz - 2\lambda x, L_y = 4xz - 2\lambda y, L_z = 4xy - \lambda, L_\lambda = -(z - 9 + x^2 + y^2)$. We now set each partial equal to zero and solve the system of equations. We solve the first three equations for λ to obtain $\lambda = \frac{4yz}{2x} = \frac{4xz}{2y} = \frac{4xy}{1}$. The equation $\frac{4yz}{2x} = \frac{4xz}{2y}$ means $y^2 = x^2$ or $x = y$ as the values for x and y must both be positive. The equation $\frac{4xz}{2y} = \frac{4xy}{1}$ means $4z = 8y^2$ or $z = 2y^2$. The we now substitute $x = y$ and $z = 2y^2$ into the equation $L_\lambda = -g(x,y,z) = 0 = -(z - 9 + x^2 + y^2)$ to obtain $2y^2 - 9 + y^2 + y^2 = 0$ or $y^2 = 9/4$ so $y = 3/2$. Hence $x = 3/2, z = 9/2$ and the dimensions are 3 by 3 by 9/2.

To find the point closest to the origin on the hyperbolic cylinder $x^2 - z^2 = 1$, we want to optimize the distance $f = \sqrt{x^2 + y^2 + z^2}$ subject to the constraint

$g = x^2 - z^2 - 1 = 0$. However, the square root in the optimization function will result in rather messy derivatives, so instead we notice that distance is minimized at the same places where distance squared is minimized, so we can use $f = x^2 + y^2 + z^2$ instead. The Lagrangian is $L = x^2 + y^2 + z^2 - \lambda(x^2 - z^2 - 1)$. The gradient is $\nabla L(x, y, z, \lambda) = \left\langle 2x - 2lambdax, 2y, 2z + 2\lambda x, -(x^2 - z^2 - 1)\right\rangle$. We now solve for when the gradient is zero to obtain $2x - 2\lambda x = 2x(1 - \lambda) = 0, 2y = 0, 2z(1 + \lambda) = 0$ for the first three equations. From the first equation we have $x = 0$ or $\lambda = 1$. However the constraint $x^2 - z^2 = 1$ (which is $L_\lambda = 0$) shows that $x \neq 0$, which means that $\lambda = 1$. The second partial tells us that $y = 0$, and the third partial shows us that $2z(2) = 0$ or $z = 0$. Since $z = 0$, we have using our constraint again that $x = \pm 1$. So there are two solutions $(-1, 0, 0)$ and $(1, 0, 0)$.

9.5.1 Why Lagrange Multipliers works - linear dependence

If the domains of f and the constraint g are 2 dimensional, then $g = 0$ represents a level curve of the function g. At a maximum of f where $g = 0$, the level curve of f which passes through the location of the maximum will have a tangent line that is parallel to the level curve of g. Hence, the gradients of f and g should be parallel, as they are normal to level curves (and since they are normal to the same tangent line, they should be parallel). Hence the gradient of f is a scalar multiple of the gradient of g. Call that scalar λ. Then $\nabla f = \lambda \nabla g$. Hence, at a maximum or minimum we will have $0 = \nabla f - \lambda \nabla g$. This is equivalent to solving $\nabla L = \vec{0}$.

If the domains of f and the constraint are 3 dimensional, then $g = 0$ represents a level surface of the function g. At a maximum of f where $g = 0$, the level surface of f which passes through the location of the maximum will have a tangent plane that is tangent to the level surface $g = 0$. Hence, the gradients of f and g should be parallel as they are both normal vectors to the same tangent plane. So the gradient of f is a scalar multiple of the gradient of g. This line of reasoning extends to all dimensions.

If you have multiple constraints, then let the Lagrangian be $L = f - \lambda_1 g_1 - \lambda_2 g_2 - \cdots$. Under suitable conditions, it can be shown that optimum solutions will satisfy the equation $\nabla L = 0$. Just keep subtracting a new variable times the next constraint for each new constraint. This works precisely because at a an optimal solution, it can be shown that the gradient of f is a linear combination of the gradients of the constraints, i.e. $\nabla f = \lambda_1 g_1 + \cdots + \nabla_n g_n$. You will learn more about this topic when you study linear algebra.

Chapter 10

Double Integrals

After completing this chapter, you should be able to do the following:

1. Explain how to setup and compute a double integrals, as well as how to interchange the bounds of integration. Use these ideas to find area and volume.
2. For planar regions, find area, mass, centroids, center of mass, moments of inertia, and radii of gyration.
3. Explain how to change coordinate systems in integration, in particular to polar coordinates. Explain what the Jacobian of a transformation is, and how to use it.
4. Explain how to use Green's theorem to compute flow and flux along and across a curve.

10.1 Preparation and Homework Suggestions

	Preparation Problems (11th ed)				Preparation Problems (12th ed)			
Day 1	15.1:21	15.1:35	15.1: 43	15.2:13	15.2:33	15.2:51	15.2: 59	15.3:17
Day 2	15.2: 21	15.2:31	15.3: 3	15.3: 23	15.6: 3	15.6:11	15.4: 9	15.3: 33
Day 3	15.7:8	15.7:9	16.4: 7	16.4:15	15.8:8	15.8:9	16.4: 7	16.4:19

The following homework problems line up with the topics we will discuss in class.

Topic (11th ed)	Sec	Basic Practice	Good Problems	Thy/App	Comp
Double Integrals	15.1	1-16, 21-50	17-20, 51-54, 57-66	55-56	67-76
Double Integral Applications	15.2	1-12, 15-18, 19-40	13, 14, 41-48, 53-56	49-52	
Polar Coordinates	15.3	1-22,23-32	33-42 (do 37 and 40 for sure)		43-46
Jacobian	15.7	1-8, 15-17	9-10,12-14, 19-22	18, 24	
Green's Theorem	16.4	1-20	21-34, 39-40	35-38	41-44

Topic 12th ed	Sec	Basic Practice	Good Problems	Thy/App	Comp
Double Integrals (rect.)	15.1	1-28			
Double Integrals	15.2	1-24,33-46	19-32, 47-56,57-68	69-84	85-94
Area, Average Value	15.3	1-22	23-25	26	
Polar Integrals	15.4	1-16	17-26, 27-36,41	37-40, 42-46	47-50
Double Integral Applications	15.6	1-20			
Jacobian	15.8	1-8, 17-19	9-10,12-16, 21-24	20, 26	
Green's Theorem	16.4	1-24	25-38, 42	39-41	43-46

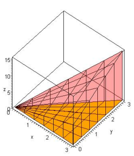
It is crucial that you do not attempt to solve every integral. For the most part, you are learning to set up integrals in high dimensions. **I would suggest that you do at least 15 problems a day or more in chapter 15 and 16**, where you spend time setting up integrals and not solving them. Check your work against the answers I provide online, as I just give the set up for each problem.

10.2 Double Integrals

The integral $\int_a^b f dx$ represents the area of the region in the xy plane above the interval $[a, b]$ along the x axis under the curve $f(x)$. We will define a double integral $\int\int_R f dA$ in such as manner to obtain for positive $f(x, y)$ the volume of the solid region in space above a region R in the xy plane under the surface $z = f(x, y)$. To start with, we will assume that the region R is a rectangle $a \leq x \leq b, c \leq y \leq d$. Slice the rectangle up into many tiny rectangles of dimensions $\Delta x_i, \Delta y_j$, making a grid for your region R. The area of each little sub rectangle in the grid is $\Delta A_{ij} = \Delta x_i \Delta y_j$. If the grid is made so that Δx_i and Δy_j are both really small, then the height of the surface $z = f(x, y)$ above a sub rectangle is approximately constant, and can be approximated by the value of f at some point $f(x_i, y_j)$ in that sub rectangle. This gives an approximation for a small piece of volume as $\Delta V_{ij} \approx f(x_i, y_j)\Delta A_{ij} = f(x_i, y_j)\Delta x_i \Delta y_j$. Add up the small pieces of volume to get total volume as $V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\Delta x_i \Delta y_j$. The double limit $V \approx \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\Delta x_i \Delta y_j$, if it exists, is called the double integral of f over R, and we write $\int\int_R f dA$. This limit will always exist if the function is continuous and bounded on R.

If a double integral over a rectangle R exists, then it can be computed using the formulas $\int\int_R f dA = \int_a^b \left(\int_c^d f dy\right) dx = \int_c^d \left(\int_a^b f dx\right) dy$. These last two integrals are called iterated integrals. First integrate the inside integral with respect to the variable inside, then integrate the result with respect to the outside variable. For example the integral $\int_0^2 \left(\int_1^4 2x + 4xy dy\right) dx = \int_0^2 \left(2xy + 2xy^2\Big|_{y=1}^{4}\right) dx = \int_0^2 36x dx = 18x^2\Big|_{x=0}^{2} = 72$ gives the same answer as the integral $\int_1^4 \left(\int_0^2 2x + 4xy dx\right) dy = \int_1^4 \left(x^2 + 2x^2 y\Big|_{x=0}^{2}\right) dy = \int_1^4 4 + 8y dy = 4y + 4y^2\Big|_{y=1}^{4} = 16 + 64 - (4 + 4) = 72$. The reason a double integral can be computed using either iterated integral has to do with looking at cross sections. Rather than slicing R into a grid using both Δx and Δy, just pick one direction in which to make slices. For example, if we cut the region R along lines parallel to the y axis, then each slice has a width Δx_i, and we pick a point x_i in each interval on the x axis. The area of the cross section of the plane $x = x_i$ under $z = f(x_i, y)$ above the xy plane is given by the integral $A = \int_c^d f(x_i, y)dy$. Multiply this area $\int_c^d f(x_i, y)dy$ by Δx_i for each i to obtain an approximation for the volume above the ith slice of R, i.e. $\Delta V_i \approx \int_c^d f(x_i, y)dy\Delta x_i$. Total volume is found by adding up the little bits of volume and taking a limit as $\Delta x_i \to 0$. This gives the volume as $V = \int_a^b \left(\int_c^d f dy\right) dx$. Since the parenthesis take up extra space, we write simply $\int\int_R f dA = \int_a^b \int_c^d f dy dx = \int_c^d \int_a^b f dx dy$.

If the region R is not a rectangle, but is bounded, and the function f is bounded for all (x, y) in the region R, then we define the double integral by placing the region R inside a rectangle which contains R and defining $f(x, y) = 0$ for all (x, y) not in R. This is how the double integral $\int\int_R f dA$ is defined theoretically. To compute the double integral, we pick bounds for the integral which describe our region. The outer bounds of the integral must be two constants. The inner bounds can be functions which involve variables of the outer bounds. The outer bounds represent two horizontal or vertical lines. The inner bounds represent two functions whose input is the outer bound variable.



The volume of the region under the plane $z = 2x + 3y$ above the region R which is bounded by the lines $y = x, y = 0, x = 3$ can be found using a double integral. If we choose y as the outer variable, and x as the inner variable, then we have to pick two constants which trap all the y values, and then pick two functions of y which trap all the x values. The constants are $0 \leq y \leq 3$, and the functions are $y \leq x \leq 3$. The corresponding double integral is $\int_0^3 \int_y^3 2x + 3y dx dy$. Alternatively, if I choose x as the outer variable and y as the inner variable, then x is between the constants 0 and 3 ($0 \leq x \leq 3$) and y is between the functions $y = 0$ and $y = x$ ($0 \leq y \leq x$). The corresponding integral is $\int_0^3 \int_0^x 2x + 3y dy dx$. Notice that the integrand $2x + 3y$ is the same in both iterated integrals, it was just the bounds that changed.

If $f = 1$, then $\int\int_R 1 dA$ is the area of the region R. Hence area (a 2 dimensional quantity) is computed by adding up little bits of area dA. For the region $a \leq x \leq b, 0 \leq y \leq f(x)$, we have $\int\int_R 1 dA = \int_a^b \int_0^{f(x)} dy dx = \int_a^b y\Big|_0^{f(x)} dx = \int_a^b f(x) dx$, which is the formula found in first semester calculus.

10.2.1 Switching the Order of Integration

Sometimes it is valuable to switch the bounds of integration by reordering the variables. This is done by describing the region R (often by constructing a graph), and then changing the way you describe the function. For example, to compute the integral $\int_0^1 \int_y^1 e^{x^2} dx dy$ we could first try to integrate the inside integral but we would fail (as $\int e^{x^2} dx$ does not have an anti derivative which is an elementary function). So instead we write the bounds as inequalities $0 \leq y \leq 1, y \leq x \leq 1$ and realize that the region describe by these inequalities is the triangle in the first quadrant under the line $y = x$ for $0 \leq x \leq 1$. Hence we can describe the region also as $0 \leq x \leq 1, 0 \leq y \leq x$ which gives the integral $\int_0^1 \int_0^x e^{x^2} dy dx$. The inner integral is now $\int_0^x e^{x^2} dy = ye^{x^2}\Big|_0^x = xe^{x^2}$. Hence we have $\int_0^1 \int_0^x e^{x^2} dy dx = \int_0^1 xe^{x^2} dx$, which is solved by letting $u = x^2, du = 2x dx, dx = \frac{du}{2x}, u(0) = 0, u(1) = 1$ and so $\int_0^1 xe^{x^2} dx = \int_{u=0}^{u=1} xe^u \frac{du}{2x} = \frac{1}{2}\int_{u=0}^{u=1} e^u du = \frac{1}{2}e^u\Big|_0^1 = \frac{1}{2}(e^1 - e^0) = \frac{1}{2}(e - 1)$.

The region R inside a circle of radius 3 above the x axis can be describe as either $-3 \leq x \leq 3, 0 \leq y \leq \sqrt{9 - x^2}$ or as $0 \leq y \leq 3, -\sqrt{9 - y^2} \leq x \leq \sqrt{9 - x^2}$. Hence the integral $\int\int_R 1 dA$ can calculated as either $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} dy dx$ or $\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} dx dy$. Both integrals give the answer $9\pi/2$ which is the area of this region.

The integral $\int_0^2 \int_{x^2}^{2x} dydx$ gives the area of the region between the curves $y = 2x$ and $y = x^2$ and can also be written as $\int_0^4 \int_{y/2}^{\sqrt{y}} dxdy$.

10.3 Physical Applications

Integrals are used in a wide variety of applications. You should notice that in most of the applications below, a formula which works for double integrals is the same as the integral which we used for line integrals. The differentials dx, ds, dA remind us which dimension we are working in. For lack of a better notation, I will use $d\square$ to represent any of these differentials when you are free to pick the needed differential. The material which follows is very similar to the work we did in the line integrals section. I start by giving a summary of the differential formulas, and then we will use them to find physical quantities.

- Area: $dA = fdx, fds, dxdy, rdrd\theta, \frac{\partial(x,y)}{\partial(u,v)}dudv$,
- Volume: $dV = fdA$,
- Average Value: $(\bar{f})\square = \int fd\square$,
- Mass: $dm = \delta d\square, \delta dx, \delta ds, \delta dA$
- Center of Mass: $(\bar{x}, \bar{y}, \bar{z})m = \int \langle x, y, z\rangle\, dm$, where $\int \bar{x}dm = \int xdm$ is a first moment of mass
- Moment of Inertia: $I = \int r^2 dm$ is a moment of inertia, and $\int R^2 dm = \int r^2 dm$ gives $R^2 m = I$ or $R = \sqrt{I/m}$

I strongly suggest that as you do each problem, start from a formula in differential notation, and then modify it so that you have the right number of integrals. If you do this, you will learn how to do calculus in all dimensions with ease.

Consider the region $R = \{(x, y)|0 \leq x \leq 3, 0 \leq y \leq x\}$ in the plane with density function $\delta(x, y) = x^2 + y^2 + 1$. Area is $A = \int_0^3 \int_0^x dydx$. Mass is $m = \int\int_R dm = \int_0^3 \int_0^x (x^2 + y^2 + 1)dydx$. We can also compute

$$\bar{y} = \frac{\int ydm}{\int dm} = \frac{\int\int_R y\delta dA}{\int\int_R \delta dA} = \frac{\int_0^3 \int_0^x y(x^2 + y^2 + 1)dydx}{\int_0^3 \int_0^x x^2 + y^2 + 1dydx}$$

$$R_x = \sqrt{\frac{\int r^2 dm}{\int dm}} = \sqrt{\frac{\int y^2 dm}{\int dm}} = \sqrt{\frac{\int_0^3 \int_0^x y^2(x^2 + y^2 + 1)dydx}{\int_0^3 \int_0^x x^2 + y^2 + 1dydx}}$$

The temperature of a metal object covering the region $R = \{(x, y)|0 \leq x \leq 3, 0 \leq y \leq x\}$ in the plane is given by $f(x, y) = x^2 + y^2 + 1$. The area of the region R is $A = \int\int_R dA = \int_0^3 \int_0^x dydx$ and the average temperature is $AV = \frac{1}{A}\displaystyle\int_0^3 \int_0^x x^2 + y^2 + 1dydx$.

Consider a metal plate occupying the region in the plane bounded by the curves $x = y^2$ and $y = x$, with density function $\delta(x, y) = xy$. The mass of the plate is $m = \int\int_R \delta dA = \int_0^1 \int_{y^2}^{y} (xy)dxdy$. The mass of the region in the plane bounded by the curves $x = 2y$ and $x = y^2$, with density $x^2 + y^2$, is $m = \int_0^2 \int_{y^2}^{2y} (x^2 + y^2)dxdy$. The center of mass is $\bar{x} = \frac{1}{m}\int_0^2 \int_{y^2}^{2y} x(x^2 + y^2)dxdy$ and $\bar{y} = \frac{1}{m}\int_0^2 \int_{y^2}^{2y} y(x^2 + y^2)dxdy$.

The centroid of many regions is geometrically obvious (such as the center of a circle, square, or rectangle). If a centroid is known, then you can use this knowledge to simplify many integrals. For example, we can compute $\int_0^4 \int_0^6 3x + 5y dy dx = 3 \int_0^4 \int_0^6 x dy dx + 5 \int_0^4 \int_0^6 y dy dx = 3\bar{x}A + 5\bar{y}A = 3 \cdot 2 \cdot 24 + 5 \cdot 3 \cdot 24$, since the centroid of the rectangle $R = [0, 4] \times [0, 6]$ is $(2, 3)$ with area 24.

10.4 Changing Coordinate Systems - Generalized u-substitution

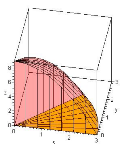
To solve the definite integral $\int_0^3 e^{2x} dx$, we make a substitution $u = 2x$ or $x = u/2$ and then notice that $dx = \frac{1}{2} du$. Under this substitution, the interval $[0, 3]$ transforms to the interval $[0, 6]$ and the integral becomes $\int_0^3 e^{2x} dx = \int_0^6 e^u \frac{1}{2} du$. Notationally we can write this as $\int_{C_x} f(x)dx = \int_{C_u} f(x(u)) \frac{dx}{du} du$, where C_x and C_u are the integration bounds in terms of x and u respectively, and $x(u) = u/2$, so $\frac{dx}{du} = \frac{1}{2}$. Similarly, the substitution $x = \tan\theta$ gives $\frac{dx}{d\theta} = \sec^2\theta$ and allows us to compute $\int_0^1 \frac{1}{1+x^2} dx = \int_0^{\pi/4} \frac{1}{1+\tan^2\theta} \sec^2\theta d\theta = \int_0^{\pi/4} d\theta = \pi/4$, since $\tan 0 = 0$ and $\tan(\pi/4) = 1$.

To generalize this to higher dimensions we have to (1) change the bounds of integration, and (2) replace $\frac{dx}{du}$ with the generalized version in high dimensions. To change the bounds in a new coordinate system requires that we describe the exact same region using different coordinates. We will illustrate this with examples. The term $\frac{dx}{du}$ in higher dimensions is called the Jacobian of the transformation, and is the absolute value of the determinant of the derivative of the change of coordinates.

10.4.1 Polar Coordinates

Polar coordinates are defined by $x = r\cos\theta, y = r\sin\theta$, which we can represent using function notation as $T(r, \theta) = \langle r\cos\theta, r\sin\theta\rangle$. Recall that the derivative of this transformation is $DT(r, \theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$. The derivative measures small changes in x, y based on small changes in r, θ. Determinants calculate area (up to a plus or minus sign), so the determinant of the derivative measures the change in area which occurs when you change from one coordinate system to another. To get rid of the plus or minus sign, we take the absolute value of the determinant of the derivative and write $\frac{\partial(x,y)}{\partial(r,\theta)} = |\det(DT(r, \theta))| = |r\cos^2\theta + r\sin^2\theta| = |r|$. This quantity is called the Jacobian of the transformation from x, y coordinates to r, θ coordinates. As long as $r > 0$, we can drop the absolute value and we have that the Jacobian is $\frac{\partial(x,y)}{\partial(r,\theta)} = r$. So if we wish to change from x, y to r, θ coordinates, we use the formula $\int\int_{R_{xy}} f(x, y)dxdy = \int\int_{R_{r\theta}} f(T(r, \theta)) \frac{\partial(x,y)}{\partial(r,\theta)} drd\theta = \int\int_{R_{r\theta}} f(T(r, \theta)) r dr d\theta$. In differential form, we can abbreviate this as $dA = dxdy = rdrd\theta$. In other words, after you change all the x and y terms to r and θ, you have to multiply the entire expression by r.

The region inside a circle of radius a in the plane is easily described in polar coordinates as $0 \le r \le a, 0 \le \theta \le 2\pi$. As a rectangular integral, we can find the area of a circle by writing $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 1 dy dx$, which is a rather difficult integral to compute. However if we change to polar coordinates, then we compute $\int_0^a \int_0^{2\pi} r d\theta dr$ or $\int_0^{2\pi} \int_0^a r dr d\theta = \int_0^{2\pi} r^2/2\big|_0^a d\theta = 2\pi a^2/2 = \pi a^2$ rather easily. When regions of integration involve circles or curves that are easily described in polar coordinates, often a change from rectangular to polar coordinates is extremely useful.

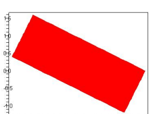


The volume of the region in space in the first octant which is below the paraboloid $z = 9 - x^2 - y^2$, above the xy-plane, and satisfying $x \geq y$, can be found using the iterated double integral $\int\int_R 1dA = \int_0^{3/\sqrt{2}} \int_y^{\sqrt{9-y^2}} 9 - x^2 - y^2 dxdy$. The region R in the xy plane is described in polar coordinates more simply as $0 \leq \theta \leq \pi/4, 0 \leq r \leq 3$. Changing to polar coordinates we have $z = 9 - (x^2 + y^2) = 9 - r^2$ and then we compute the volume as $V = \int_0^{\pi/4} \int_0^3 (9 - r^2) r dr d\theta$, which is a much easier integral to compute.

10.4.2 General Change of coordinates

In general, the Jacobian of a transformation T is the absolute value of the determinant of the derivative of the transformation, often written $\frac{\partial(x,y)}{\partial(u,v)}$. Polar coordinates is one of many standard coordinate systems which people use. Some problems are simple if you make the right substitution and very difficult if you don't. One advantage of working in higher dimensions is that you have the ability to be creative and find the right change of coordinates to solve a problem. Here are a few examples.

To find the area inside of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, use the change of coordinates $x = au, y = bv$. The equation of the ellipse becomes $u^2 + v^2 = 1$ in the uv coordinate system. The Jacobian of this transformation is $\left|\det \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right| = ab$, so we have $A = \int\int_R dxdy = \int\int_{R_{uv}} ab dudv = ab \int\int_{R_{uv}} dudv$. Rather than actually compute the integral, recall that $\int\int_{R_{uv}} dudv$ is the area of the region in the uv plane, which is the area inside a circle of radius 1, or π. Hence the area of an ellipse is $ab\pi$. If you want a difficult challenge, try computing this integral without this change of coordinates.



If R is the region in the plane bounded by the curves $x+2y = 1, x+2y = 4, 2x - y = 0, 2x - y = 8$, and we need to compute the integral $\int\int_R xdxdy$, then creating bounds for the integral is a rather ugly mess. Instead, we perform the change of coordinates $u = x + 2y, v = 2x - y$ (notice that for the first time we have u and v solved in terms of x and y). This change of coordinates transforms us to a box $[1, 4] \times [0, 8]$ in the uv-plane. To find the Jacobian of this transformation, we need to first solve for x and y in terms of u and v. To eliminate y, we add $u+2v$ and get $5x$. Similarly, eliminating x gives $2u-v = 5y$, so $x = \frac{u+2v}{5}$ and $y = \frac{2u-v}{5}$. The Jacobian of this transformation is $\left|\det \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}\right| = |-1/25 - 4/25| = |-1/5| = 1/5$ (don't forget the absolute value). Hence we have $\int\int_{R_{xy}} xdxdy = \int_1^4 \int_0^8 \frac{u+2v}{5} \left(\frac{1}{5}\right) dvdu$, which is now a fairly simple integral to compute. The right change of coordinates can simplify a problem. Which change of coordinates to use is suggested to you in the homework on the first few problems, and then you are given a few where you are to choose your own.

In the previous problem, to find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$, we first solved for x and y in terms of u and v. The "Inverse Function Theorem" implies that $\frac{\partial(x,y)}{\partial(u,v)} = 1/\left(\frac{\partial(u,v)}{\partial(x,y)}\right)$. In other words, we could have found the derivative of the transformation $T(x, y) = \langle u = x + 2y, v = 2x - y\rangle$ as $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, and then after noticing that the absolute value of the determinant is $|-5|$, we invert this to obtain $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{5}$. So you can either solve for x and y first, and then take the determinant, or you can take the determinant and then invert the result. You get the same answer either way.

10.5 Green's Theorem

10.5.1 Circulation Density

Recall that circulation of $\vec{F}$ along a simple closed curve C is $\oint_C \vec{F} d\vec{r}$. We now define circulation density for a vector field $\vec{F}(x,y) = \langle M, N\rangle$ in the plane. At the point (x,y) in the plane, create a circle C_a of radius a centered at (x,y), where the area inside of C_a is given by A_a. The quotient $\frac{1}{A_a}\oint_{C_a} \vec{F} d\vec{r}$ is circulation per area. The limit $\lim_{a\to 0} \frac{1}{A_a}\oint_{C_a} \vec{F} d\vec{r}$ is called the circulation density of $\vec{F}$ at (x,y). It can be shown that $\lim_{a\to 0} \frac{1}{A_a}\oint_{C_a} \vec{F} d\vec{r} = N_x - M_y$, and that C_a can be replaced with a square of side lengths a centered at (x,y) with interior area A_a or any collection of curves C_a which "shrink nicely" to (x,y). We have $N_x - M_y$ is circulation per unit area.

As an example, for the vector field $\vec{F} = \langle -y, x\rangle$, the circulation along a circle of radius a is $2\pi a^2$. Division by the area πa^2 and taking a limit gives $\lim_{a\to 0} \frac{1}{\pi a^2} 2\pi a^2 = 2$ which equals $N_x - M_y = 1 - (-1) = 2$.

10.5.2 Green's Theorem - Circulation Version

The idea of circulation density is used to show that circulation along C can be computed by adding up circulation density times dA along R, the interior of C. This result is called Green's Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \int\int_R N_x - M_y dA.$$

The circulation of the vector field $\vec{F} = \langle y^2, x\rangle$ along the circle of radius 2 is by Green's Theorem $\int\int_R 1 - 2y dA = A - 2\bar{y}A = \pi(2)^2 - 2(0)\pi(2)^2 = 4\pi$ where A is the area inside the circle. The circulation of the vector field $\vec{F} = \langle -y, 0\rangle$ along edge of the triangle with vertices $(0,0), (4,0), (0,3)$ is by Green's Theorem $\int_C \vec{F}\cdot d\vec{r} = \int\int_R 1 dA = \frac{1}{2} 3\cdot 4 = 6$, or just the area inside of the curve. In fact, the area inside R is $\int_C \vec{F} \cdot d\vec{r} = \int_C -y dx = \int\int_R dA$ for any curve C, which means that area can be computed using a line integral. A planimeter is a drafting tool which uses Green's theorem to calculate area of any planar region.

10.5.3 Flux Density - Divergence $\nabla \cdot \vec{F}$

For a vector field $\vec{F}(x,y) = \langle M, N\rangle$, we define the flux density, or divergence of $\vec{F}$, as follows. Recall that flux of $\vec{F}$ across C is $\oint_C \vec{F}\cdot\vec{n} ds = \oint_C M dy - N dx$ where $\vec{n}$ is the unit outward pointing normal vector and C is oriented counter clockwise. At the point (x,y) in the plane, create a circle C_a of radius a centered at (x,y), where the area inside of C_a is given by A_a. The quotient $\frac{1}{A_a}\oint_{C_a} \vec{F}\cdot\vec{n} ds$ is a flux per area. The limit $\lim_{a\to 0} \frac{1}{A_a}\oint_{C_a} \vec{F}\cdot\vec{n} ds$ is called the flux density of $\vec{F}$ at (x,y), or divergence of $\vec{F}$. It can be shown that $\lim_{a\to 0} \frac{1}{A_a}\oint_{C_a} \vec{F}\cdot\vec{n} ds = M_x + N_y$, and we write $\text{div}\vec{F} = M_x + N_y = \nabla \cdot \vec{F}$. Just as with circulation density, it can also be shown that C_a can be replaced with a square of side length a centered at (x,y) with interior area A_a, or any collection of curves C_a which "shrink nicely" to (x,y), and you still obtain the divergence (flux per unit area) as $M_x + N_y$. The notation $\text{div}\vec{F} = \nabla\cdot\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\rangle \cdot \langle M, N\rangle = \frac{\partial}{\partial x} M + \frac{\partial}{\partial y} N$ gives a convenient way to remember divergence.

As an example, we compute the flux density at $(0,0)$ for the vector field $\vec{F} = \langle x, y\rangle$. A circle of radius a has unit normal vector $\vec{n} = \frac{\langle x,y\rangle}{|\langle x,y\rangle|}$, so the flux is $\int_C \langle x, y\rangle \cdot \frac{\langle x,y\rangle}{|\langle x,y\rangle|} ds = \int_C \sqrt{x^2+y^2} ds = a\int_C ds = a2\pi a$, since the circumference of a circle is $2\pi a$. The area inside a circle of radius a is πa^2. Hence $\lim_{a\to 0} \frac{1}{\pi a^2} 2\pi a^2 = 2$, which equals $\text{div}\vec{F} = M_x + N_y = 1 + 1 = 2$.

10.5.4 Green's Theorem - Flux version

Flux in the plane can be computed by instead adding up little bits of flux, which are found by multiplying a flux density by area. This gives the flux version of Green's Theorem

$$\int_C M dy - N dx = \int_C \vec{F} \cdot \vec{n} ds = \int\int_R \text{div}\vec{F} dA = \int\int_R (M_x + N_y)\, dA$$

for C a simple closed curve with interior R and outward normal $\vec{n}$, where C is oriented counter clockwise.

For the vector field $\vec{F} = \langle x, y\rangle$, we have $\text{div}\vec{F} = 2$. The flux of $\vec{F}$ across the curve which forms the boundary of the quarter circle of radius 3 in the first quadrant (it has 3 curves which form the boundary C) is simply $\int_C M dy - N dx = \int_C \vec{F} \cdot \vec{n} ds = \int\int_R 2dA = 2A = 2\frac{9\pi}{4} = \frac{9\pi}{2}$. Alternatively this integral requires setting up 3 integrals. The integrals along the axes are zero because $\vec{F}\cdot\vec{n} = 0$. The non-zero integral is found using the parameterization $\vec{r}(t) = \langle 3\cos t, 3\sin t\rangle$, and is $\int_0^{\pi/2} M dy - N dx = \int_0^{\pi/2} (3\cos t)(3\cos t) - (3\sin t)(-3\sin t) dt = \int_0^{\pi/2} 9 dt = \frac{9\pi}{2}$.

10.6 Proofs of Green's Theorems

10.6.1 Proof of the Circulation version of Green's Theorem

To see why Green's theorem is true, the first step will be to add zero. Let $\vec{F}(x, y) = \langle M, N\rangle$ be a vector field in the plane. Let C be a simple closed smooth space curve, whose interior is R. The circulation of $\vec{F}$ along C is given by $\int_C \vec{F} \cdot d\vec{r}$. Begin by breaking the region R into little rectangular pieces with boundary C_{ij} oriented counterclockwise and interior with area ΔA_{ij}. If the rectangular region touches the curve C, then the boundary contains a small portion of C. Notice that when we add together the circulation along adjoining rectangles, we calculate the flow along the common boundary twice, once with each orientation. By our "adding zero" trick, the sum of the circulation along adjacent rectangles equals the circulation along the exterior of the union of the rectangles. All of the interior integrals disappear, which means we can calculate total circulation by adding up the circulation along each C_{ij}.

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \sum\sum \int_{C_{ij}} \vec{F} \cdot d\vec{r} = \sum\sum \frac{1}{\Delta A_{ij}} \int_{C_{ij}} \vec{F} \cdot d\vec{r}\Delta A_{ij} \\ &= \sum\sum \frac{\text{circulation along } C_{ij}}{\Delta A_{ij}} \Delta A_{ij} \approx \sum\sum (N_x - M_y)\Delta A_{ij}\end{aligned}$$

Taking limits gives us Green's Theorem $\int_C \vec{F} \cdot d\vec{r} = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int\int_R (N_x - M_y)\, dA$.

10.6.2 Proof of the Flux version of Green's Theorem

We now discuss why Green's theorem (the flux version) works. Let $\vec{F}(x,y) = \langle M, N\rangle$ be a vector field in the plane. Let C be a simple closed smooth space curve (oriented counterclockwise), whose interior is R. The flux of $\vec{F}$ across C is given by $\int_C \vec{F}\cdot\vec{n}ds$, where $\vec{n}$ is the unit outward normal to C. To find total flux, we add up little bits of flux. Break the region R into little rectangular pieces with boundary C_{ij} oriented counterclockwise. If the rectangular region touches the curve C, then the boundary contains a small portion of C. When we add together the flux across adjoining rectangles, we calculate the flux along the common boundary twice, once with each orientation. The sum of the flux across adjacent rectangles is the same as the flux across the exterior of the union of the rectangles. This means that the total flux equals the sum of the flux across all the little rectangles. We now write

$$\begin{aligned}\int_C \vec{F}\cdot\vec{n}ds &= \sum\sum\int_{C_{ij}}\vec{F}\cdot\vec{n}ds = \sum\sum\frac{1}{\Delta A_{ij}}\int_{C_{ij}}\vec{F}\cdot\vec{n}ds\Delta A_{ij}\\ &= \sum\sum\frac{\text{flux across } C_{ij}}{\Delta A_{ij}}\Delta A_{ij} \approx \sum\sum(M_x+N_y)\Delta A_{ij}\end{aligned}$$

Taking limits gives us the flux version of Green's Theorem $\int_C \vec{F}\cdot\vec{n}ds = \int\int_R (M_x+N_y)\,dA$.

Chapter 11

Surface Integrals

After completing this chapter, you should be able to do the following:

1. Explain how to setup surface integrals, as well as how to compute surface area.
2. Use surface integrals to compute flux across a surface.
3. Apply the principles studied in the line integrals section on average value, density, centroids, center of mass, first moments of mass, moments of inertia, and radii of gyration to surfaces. Summarize your results using differential notation.
4. Describe Stokes's Theorem and how it relates to circulation.

11.1 Preparation and Homework Suggestions

Here are the preparation problems.

	Preparation Problems (11th and 12th ed)			
Day 1	16.5:9	16.5:33	16.6:19	16.6:41
Day 2	16.5:31	16.6:45	16.7:3	16.7:17

The following homework problems line up with the topics we will discuss in class. Do as many of each type as you can. You will want to get into the Good and Theory problems to obtain an A in the class.

Topic	Sec	Basic Practice	Good Problems	Thy/App	Comp
Surface Integrals	16.5	1-12 , 19-32, 33-38,39-44,	13-18		
Param. Surface Integrals	16.6	17-26, 35-44, 45-48	27-34		
Stokes's Theorem	16.7	1-6, 13-18, 19	7-10, 20-23, 26	11-12, 24-25	

Topic	Sec	Basic Practice	Good Problems	Thy/App	Comp
Surface Integrals	16.5	17-26, 37-54	55-56		
More Surface Integrals	16.6	1-18, 19-28, 43-46	29-42, 47-48		
Stokes's Theorem	16.7	1-6, 13-18, 19	7-10, 20-23, 26	11-12, 24-25	

As a general rule of thumb, do not solve every integral. The most important part is making sure you understand how to set up integrals, as well as why the integration formulas we use actually work. As you set up an integral, take a moment to ask yourself how you would solve it using u-substitution or integration by parts, or make a note that neither of those methods would work.

11.2 Surface Area and Surface Integrals

Recall that the line integral $\int_C 1ds$ gives arc length of a curve C and is an extension of the single integral $\int_a^b 1dx$ which gives length of $[a,b]$. We will define the integral $\int\int_S 1d\sigma$ to give surface area of a surface S, which is an extension of the double integral $\int\int_R 1dA$ which gives area of a region R.

We start by finding the surface area of the surface $S : z = f(x,y)$ for (x,y) in some region R. The idea is to add up little pieces of surface area. Divide R into a tiny grid where each little rectangle has dimensions Δx and Δy. The portion of the surface above a tiny rectangle is approximately a parallelogram. The vectors $\langle 1,0,f_x\rangle \Delta x$ and $\langle 0,1,f_y\rangle \Delta x$ represent the edges of the parallelogram. The area of the parallelogram is the magnitude of their cross product, which is $|\langle -f_x,-f_y,1\rangle \Delta x\Delta y| = \sqrt{f_x^2+f_y^2+1}$. A little piece of surface area is approximated using the formula $d\sigma = \sqrt{f_x^2+f_y^2+1}dA$. Surface area is found by adding up little bits of surface area, hence $\sigma = \int\int_S d\sigma = \int\int_R \sqrt{f_x^2+f_y^2+1}dA$ where S stands for the surface, and R stands for the shadow of the surface in the xy-plane.

If the surface is given by the parametrization $S : \vec{r}(u,v) = \langle x,y,z\rangle$ for (u,v) in some region R, then let Δu and Δv be small changes in the inputs u and v (placing a grid on the region R). A portion of the surface corresponding to a tiny change in u and v is approximately a parallelogram, whose edges are given by the vectors $\vec{r}_u\Delta u$ and $\vec{r}_v\Delta v$. The magnitude of the cross product of these two vectors is in differential notation $d\sigma = |\vec{r}_u \times \vec{r}_v|dudv$. Total surface area is found by adding up little bits of surface area, giving the integral formula $\sigma = \int\int_S d\sigma = \int\int_R |\vec{r}_u \times \vec{r}_v|dudv$. Notice that if the surface is $z = f(x,y)$, then a parametrization is $\vec{r}(x,y) = \langle x,y,f(x,y)\rangle$ and so $|\vec{r}_x \times \vec{r}_y|dxdy = |\langle 1,0,f_x\rangle \times \langle 0,1,f_y\rangle|dxdy = \sqrt{f_x^2+f_y^2+1}dA$. for a surface $y = f(x,z)$, the parametrization $\vec{r}(x,z) = \langle x,f(x,z),z\rangle$ gives $d\sigma = \sqrt{f_x^2+1+f_z^2}dxdz$. Likewise $d\sigma = \sqrt{1+f_y^2+f_z^2}dydz$ for a surface $x = f(y,z)$.

Consider the surface $S : z = 9 - x^2 - y^2$ for $x^2+y^2 \leq 3$. We compute $\sqrt{f_x^2+f_y^2+1} = \sqrt{4x^2+4y^2+1}$. Surface area is $\sigma = \int\int_S d\sigma = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{4x^2+4y^2+1}dydx$. Converting to polar coordinates gives $\int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}rdrd\theta$, which is an integral we can compute (letting $u = 4r^2+1$). The same surface can be parametrized as $\vec{r}(r,\theta) = \left\langle r\cos\theta, r\sin\theta, 9-r^2\right\rangle$ for $0 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$. We calculate $|\vec{r}_r \times \vec{r}_\theta| = |\langle \cos\theta,\sin\theta,-2r\rangle \times \langle -r\sin\theta, r\cos\theta, 0\rangle| = \left|\left\langle 2r^2\cos\theta, 2r^2\sin\theta, r\cos^2\theta + r\sin^2\theta\right\rangle\right| = \left|\left\langle 2r^2\cos\theta, 2r^2\sin\theta, r\right\rangle\right| = \sqrt{4r^4+r^2} = r\sqrt{4r^2+1}$. Hence we have $d\sigma = r\sqrt{4r^2+1}drd\theta$ as before, which means $\sigma = \int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}rdrd\theta$. Notice that you get the same answer in two different ways.

If a surface consists of many different pieces (a cube has 6 faces), then a surface integral over such a surface is the sum of the integrals over each of the surfaces. To integrate the function $g(x,y,z) = yz$ over the surface of the wedge in the first octant bounded by the coordinate planes and the planes $x = 2$ and $y + z = 1$, we notice that the surface has 5 parts. The portions are $S_1 : x = 0$ for $0 \leq y \leq 1, 0 \leq z \leq 1-y$, $S_2 : x = 2$ for for $0 \leq y \leq 1, 0 \leq z \leq 1-y$, $S_3 : y = 0$

for $0 \le x \le 2, 0 \le z \le 1$, $S_4 : z = 0$ for $0 \le x \le 2, 0 \le y \le 1$, and $S_4 : z = 1 - y$ for $0 \le x \le 2, 0 \le y \le 1$. Hence to find $\int\int_S g d\sigma$, we must evaluate all 5 integrals. We compute $d\sigma_1 = \sqrt{1+0+0}dzdy, d\sigma_2 = \sqrt{1+0+0}dzdy, d\sigma_3 = \sqrt{0+1+0}dzdx, d\sigma_4 = \sqrt{0+0+1}dydx, d\sigma_5 = \sqrt{0+(-1)^2+1}dydx$, and so

$$\begin{array}{lllllll} & \int\int_{S_1} g d\sigma & +\int\int_{S_2} g d\sigma & +\int\int_{S_3} g d\sigma & +\int\int_{S_4} g d\sigma & +\int\int_{S_5} g d\sigma \\ = & \int_0^1\int_0^{1-y} yzdzdy & +\int_0^1\int_0^{1-y} yzdzdy & +\int_0^2\int_0^1 (0)zdzdx & +\int_0^2\int_0^1 y(0)dydx & +\int_0^2\int_0^1 y(1-y)\sqrt{2}dydx \\ = & \int_0^1\int_0^{1-y} yzdzdy & +\int_0^1\int_0^{1-y} yzdzdy & +0 & +0 & +\int_0^2\int_0^1 y(1-y)\sqrt{2}dydx \\ = & 1/24 & +1/24 & +0 & +0 & +\sqrt{2}/3 \end{array}$$

To match the book, the book states that if $\vec{F}(x, y, z)$ is a function whose level surface is S, and we project the surface onto some planar region R with normal vector $\vec{p}$, then $d\sigma = \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|}dA$. For functions of the form $z = f(x, y)$, let $F = z - f(x, y)$ and $\vec{p} =< 0, 0, 1 >$which gives $\frac{|\nabla F|}{|\nabla F \cdot \vec{p}|} = \sqrt{f_x^2 + f_y^2 + 1}$. So the ideas in the book can just be replaced with parametrizing the surface.

11.2.1 Flux across a surface

Recall that in the plane, the flux of $\vec{F}(x, y)$ across a curve C is a measure of how much fluid flows across the curve in the outward direction $\vec{n}$, and we write Flux $= \int_C \vec{F} \cdot \vec{n} ds$ for a unit vector $\vec{n}$. The flux of $\vec{F}(x, y, z)$ across an orientable surface S with continuous unit normal vector $\vec{n}(x, y, z)$ is the surface integral Flux $= \int\int_S \vec{F} \cdot \vec{n} d\sigma$. If the surface S is a closed surface, then it is customary to use the outward pointing normal vector $\vec{n}$ to perform computations. If a surface is parametrized by $\vec{r}(u, v)$, then $\vec{n}(x, y, z) = \pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$, where the plus or minus is chosen to make the vector point out of the surface. Since $d\sigma = |\vec{r}_u \times \vec{r}_v|dudv$, flux is $\int\int_S \vec{F} \cdot \vec{n} d\sigma = \pm \int\int_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}|\vec{r}_u \times \vec{r}_v|dudv = \pm \int\int_S \vec{F} \cdot \vec{r}_u \times \vec{r}_v dudv$, and we never have to look at the magnitude of the cross product. Some surfaces, such as a Mobius strip, do not have a continuous unit normal $\vec{n}$, and are called nonorientable. Flux across such a surface is not defined.

We now find the flux of $\vec{F} =< x + y, y, z >$ across the surface of the cube in the first quadrant bounded by $x = 2, y = 3, z = 5$. We will use the outer normal vector. The cube has 6 surfaces, so we have to compute the flux across all 6 surfaces. The surfaces, normal vectors, and vector field along each surface are found in the table below:

$\vec{r}(u, v)$	$\vec{n}$	$\vec{F}(\vec{r}(u, v))$	$\vec{F} \cdot \vec{n}$	$d\sigma$	Flux
$\langle 0, y, z\rangle$, $0 \le y \le 3, 0 \le z \le 5$	$\langle -1, 0, 0\rangle$	$\vec{F}(0, y, z) = \langle y, y, z\rangle$	$-y$	$1dydz$	$\int_0^3\int_0^5 -ydzdy$
$\langle 2, y, z\rangle$, $0 \le y \le 3, 0 \le z \le 5$	$\langle 1, 0, 0\rangle$	$\vec{F}(2, y, z) = \langle 2 + y, y, z\rangle$	$2 + y$	$1dydz$	$\int_0^3\int_0^5 2 + ydzdy$
$\langle x, 0, z\rangle$, $0 \le x \le 2, 0 \le z \le 5$	$\langle 0, -1, 0\rangle$	$\vec{F}(x, 0, z) = \langle x, 0, z\rangle$	0	$1dxdz$	0
$\langle x, 3, z\rangle$, $0 \le x \le 2, 0 \le z \le 5$	$\langle 0, 1, 0\rangle$	$\vec{F}(x, 3, z) = \langle x + 3, 3, z\rangle$	3	$1dxdz$	$\int_0^2\int_0^5 3dzdx$
$\langle x, y, 0\rangle$, $0 \le x \le 2, 0 \le y \le 3$	$\langle 0, 0, -1\rangle$	$\vec{F}(x, y, 0) = \langle x + y, y, 0\rangle$	0	$1dxdy$	0
$\langle x, y, 5\rangle$, $0 \le x \le 2, 0 \le y \le 3$	$\langle 0, 0, 1\rangle$	$\vec{F}(x, y, 5) = \langle x, y, 5\rangle$	5	$1dxdy$	$\int_0^2\int_0^3 5dydx$

The sum of the first two integrals is $\int_0^3\int_0^5 2dzdy = 2(3)(5) = 30$. We compute $\int_0^2\int_0^5 3dzdx = 3(2)(5) = 30$ and $\int_0^2\int_0^3 5dydx = 5(2)(3) = 30$. The flux of $\vec{F}$ across this surface is the sum 90.

To find the flux of $\vec{F} = \frac{\langle -x, -y, -z\rangle}{(x^2+y^2+z^2)^{-3/2}}$ across the unit sphere (using the outward normal to orient the sphere - so that flow out is seen as positive). We should get a negative number since the vector field has all arrows pointing in. I will first show the general procedure for doing this integral, and then show you a short cut only because of symmetry. Parametrize the sphere using $\vec{r}(\theta,\phi) = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$ for $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$. We compute (with a little patience) $\vec{r}_\theta \times \vec{r}_\phi = \left\langle -\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi\right\rangle = -\sin\phi\,\langle x,y,z\rangle$. Since this vector points into the sphere, we multiply by -1 and compute flux across S as $\int_0^{2\pi}\int_0^{\pi} \frac{\langle -x,-y,-z\rangle}{(x^2+y^2+z^2)^{-3/2}} \cdot \sin\phi\,\langle x,y,z\rangle\, d\phi d\theta = \int_0^{2\pi}\int_0^{\pi} -\sin\phi(x^2+y^2+z^2)^{-1/2} d\phi d\theta = \int_0^{2\pi}\int_0^{\pi} -\sin\phi d\phi d\theta = -4\pi$. Alternatively, you could notice that due to symmetry the unit normal vector is $\vec{n} = \frac{\langle x,y,z\rangle}{\sqrt{x^2+y^2+z^2}} = \langle x,y,z\rangle$ since the radius is 1 and the normal vector has to point radially. Hence $\vec{F}\cdot\vec{n} = -(x^2+y^2+z^2) = -1$. So the flux is $\int\int_S -1 d\sigma = -\sigma = -4\pi$, since the surface area of a sphere of radius a is $4\pi a^2$. Notice that in this example I bypassed parametrizing the surface by finding the unit normal vector through inspection, and used known facts about surface area. Flux, as seen in these last two problems, leads to Gauss's Law, Ampere's Law, Faraday's Law, and many other quantities of interest in electromagnetism.

11.3 Physical Applications

Integrals are used in a wide variety of applications. You should notice that in most of the applications below, a formula which works for a surface integral is the same as the double integral and line integral formulas. The differentials $dx, ds, dA, d\sigma$ remind us which type of integral to use. For lack of a better notation, I will use $d\Box$ to represent any of these differentials when you are free to pick the needed differential. I start by giving a summary of the differential formulas, and then we will use them to find some physical quantities.

- Area: $dA = fdx, fds, dxdy, rdrd\theta, \frac{\partial(x,y)}{\partial(u,v)}dudv$,
- Surface Area: $|\vec{r}_u \times \vec{r}_v|dudv = d\sigma = \sqrt{f_x^2+f_y^2+1}dxdy = \sqrt{f_x^2+1+f_z^2}dxdz = \sqrt{1+f_y^2+f_z^2}dydz$
- Volume: $dV = fdA = fd\sigma$,
- Flux: $d\text{Flux} = \vec{F}\cdot\vec{n}d\sigma, \vec{F}\cdot\vec{n}ds$ where $\vec{n}$ is a normal vector to either a surface S or a curve C.
- Average Value: $(\bar{f})\Box = \int f d\Box$,
- Mass: $dm = \delta d\Box, \delta dx, \delta ds, \delta dA, \delta d\sigma$
- Center of Mass: $(\bar{x},\bar{y},\bar{z})m = \int \langle x,y,z\rangle\, dm$, where $\int \bar{x}dm = \int xdm$ is a first moment of mass
- Moment of Inertia: $I = \int r^2 dm$ is a moment of inertia, and $\int R^2 dm = \int r^2 dm$ gives $R^2 m = I$ or $R = \sqrt{I/m}$

I strongly suggest that as you do each problem, start from a formula in differential notation, and then modify it so that you have the right number of integrals. If you do this, you will learn how to do calculus in all dimensions with ease.

11.3.1 Examples

The temperature at each point in space on the surface of a sphere of radius 3 is given by $T(x, y, z) = \sin(xy + z)$. The average temperature on the sphere is given by the surface integral $AV = \frac{1}{\sigma}\int\int_S f d\sigma$. A parametrization of the surface is $\vec{r}(\theta, \phi) = \langle 3cos\theta \sin\phi, 3\sin\theta\sin\phi, 3\cos\phi\rangle$ for $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. We have $T(\theta, \phi) = \sin((3\cos\theta\sin\phi)(3\sin\theta\sin\phi) + 3\cos\phi)$, and the surface area differential is $d\sigma = |\vec{r}_\theta \times \vec{r}_\phi| = 9\sin\phi$. The surface area is $\sigma = \int_0^{2\pi}\int_0^{\pi} 9\sin\phi d\phi d\theta$ and the average temperature on the surface is $AV = \frac{1}{\sigma}\int_0^{2\pi}\int_0^{\pi}\sin((3\cos\theta\sin\phi)(3\sin\theta\sin\phi) + 3\cos\phi)9\sin\phi d\phi d\theta$.

Consider the surface which is the upper hemisphere of radius 3. A parametrization of the surface is $\vec{r}(\theta, \phi) = \langle 3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi\rangle$ for $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$. The surface area differential is $d\sigma = |\vec{r}_\theta \times \vec{r}_\phi| d\theta d\phi = 9\sin\phi d\theta d\phi$. The surface area is $\sigma = \int_0^{2\pi}\int_0^{\pi/2} 9\sin\phi d\phi d\theta$. If the density is $\delta(x, y, z) = z^2$, then we have

$$\bar{y} = \frac{\int y dm}{\int dm} = \frac{\int\int_S y\delta d\sigma}{\int\int_S \delta d\sigma} = \frac{\int_0^{2\pi}\int_0^{\pi/2}(3\sin\theta\sin\phi)(3\cos\phi)^2(9\sin\phi)d\phi d\theta}{\int_0^{2\pi}\int_0^{\pi/2}(3\cos\phi)^2(9\sin\phi)d\phi d\theta}$$

$$R_y = \sqrt{\frac{\int r^2 dm}{\int dm}} = \sqrt{\frac{\int x^2 + z^2 dm}{\int dm}} = \sqrt{\frac{\int_0^{2\pi}\int_0^{\pi/2}[(3\cos\theta\sin\phi)^2 + (3\cos\phi)^2](3\cos\phi)^2(9\sin\phi)d\phi d\theta}{\int_0^{2\pi}\int_0^{\pi/2}(3\cos\phi)^2(9\sin\phi)d\phi d\theta}}$$

11.4 Stokes's Theorem and Circulation Density

11.4.1 Curl $\nabla \times \vec{F}$

Green's Theorem generalizes to surfaces, but we first have to discuss circulation density of $\vec{F} = \langle M, N, P\rangle$ about a line with unit direction vector $\vec{n}$. At a point (x, y, z) in space, create a plane through (x, y, z) with unit normal vector $\vec{n}$. In that plane, create a circle C_a of radius a centered at (x, y, z), where σ_a is the surface area of the region inside C_a on the plane. The quotient $\frac{1}{\sigma_a}\oint_C \vec{F}d\vec{r}$ is a ratio of circulation per surface area. The limit $\lim_{a\to 0}\frac{1}{\sigma_a}\oint_{C_a}\vec{F}d\vec{r}$ is called the circulation density of $\vec{F}$ at (x, y, z) about $\vec{n}$. It can be shown that $\lim_{a\to 0}\frac{1}{\sigma_a}\oint_{C_a}\vec{F}d\vec{r} = \langle P_y - N_z, M_z - P_x, N_x - M_y\rangle\cdot\vec{n}$, and so the circulation density of $\vec{F}$ about $\vec{n}$ is $\langle P_y - N_z, M_z - P_x, N_x - M_y\rangle\cdot\vec{n}$. We call the vector $\langle P_y - N_z, M_z - P_x, N_x - M_y\rangle$ the curl of $\vec{F}$ and write $\text{curl}(\vec{F}) = \nabla\times\vec{F} = \langle P_y - N_z, M_z - P_x, N_x - M_y\rangle$. The notation $\nabla\times\vec{F}$ gives a convenient way to remember the formula, as

$$\text{curl}\vec{F} = \nabla\times\vec{F} = \det\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ M & N & P\end{bmatrix} = \left\langle\frac{\partial}{\partial y}P - \frac{\partial}{\partial z}N, \frac{\partial}{\partial z}M - \frac{\partial}{\partial x}P, \frac{\partial}{\partial x}N - \frac{\partial}{\partial y}M\right\rangle$$

Circulation density of $\vec{F}$ at (x, y, z) about $\vec{n}$, written $\text{curl}\vec{F}\cdot\vec{n}$ is largest when $\vec{n}$ points in the direction of $\text{curl}F$. This means that the curl of a vector field is the direction of the greatest circulation density, and the magnitude of the curl is the circulation density in that direction.

11.4.2 Stokes's Theorem

A vector field in the plane $\langle M, N\rangle$ can be extended to 3D by writing $\vec{F} = \langle M, N, 0\rangle$. The curl is $\text{curl}\vec{F} = \langle 0, 0, N_x - M_y\rangle$. A normal vector to a surface S

in the xy plane is $\vec{n} = \langle 0,0,1\rangle$, Green's Theorem can be written as $\int_C \vec{F}\cdot d\vec{r} = \int\int_S \text{curl}\cdot\vec{n}d\sigma = \int\int_R (N_x - M_y)\,dA$, where C is a simple closed curve in the plane and $S = R$ is the region in the xy-plane inside C. Stokes's Theorem states that

$$\int_C \vec{F}\cdot d\vec{r} = \int\int_S \text{curl}\cdot\vec{n}d\sigma$$

where C can be any simple closed space curve which is the boundary of a surface S, where $\vec{n}$ is a continuous unit normal vector to the surface S and C is oriented compatibly with $\vec{n}$ by the right hand rule.

Let $\vec{F} = \langle x+y, y+z, x+z\rangle$. Let C be the space curve which travels around a triangle in space hitting the vertices $(2,0,0),(0,3,0),(0,0,6)$ in this order. The planar surface $S : \frac{x}{2}+\frac{y}{3}+\frac{z}{6} = 1$ or $z = 6-3x-2y$ for $R : 0 \le x \le 2, 0 \le y \le \frac{6-3x}{2}$ has as its boundary the curve C. The surface S has an upward pointing normal vector and downward pointing normal vector. Since C is oriented to go counterclockwise around S when seen from above, we will use the upward pointing normal vector $\vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1\rangle = \langle 3,2,1\rangle$. We calculate $\text{curl}\vec{F} = \langle -1,-1,-1\rangle$, and so Stokes's Theorem states $\int_C \vec{F}\cdot d\vec{r} = \int\int_S \text{curl}\vec{F}\cdot \frac{\vec{r}_x\times\vec{r}_y}{|\vec{r}_x\times\vec{r}_y|}d\sigma = \int\int_S \langle -1,-1,-1\rangle \cdot \frac{\langle 3,2,1\rangle}{|\langle 3,2,1\rangle|}|\langle 3,2,1\rangle|dA = \int\int_R -6dA = -6A = -6\cdot\frac{1}{2}\cdot 2\cdot 3 = -18$. Notice that it was not necessary to compute the magnitude $|\vec{r}_u \times \vec{r}_v|$. Alternatively, you can parametrize all three edges and compute three line integrals.

Let $\vec{F} = \left\langle z, y^2, x^2\right\rangle$, so that $\text{curl}\vec{F} = \langle 0, 1-2x, 0\rangle$. The parametric surface $S : \vec{s}(u,v) = \langle u\cos v, u\sin v, u\rangle$ for $0 \le u \le 1, 0 \le v \le 2\pi$ is a cone. The curve $C : \vec{c}(t) = \langle \cos t, \sin t, 1\rangle$ is a circle of radius one at height $z = 1$ and forms the boundary of the surface S. If we choose as our normal vector to S the outward normal (away from the z axis), then to use Stokes's theorem we must orient C counterclockwise as seen from below the surface. This meas that we must reverse the direction we move along C. A normal vector to S is $\vec{r}_u \times \vec{r}_v = \langle -u\cos v, -u\sin v, u\rangle$, but to make it point outward we change the sign. Stokes's Theorem states that $\int_{2\pi}^0 \left\langle 1, \sin^2 t, \cos^2 t\right\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt = \int_0^{2\pi}\int_0^1 \langle 0, 1-2(u\cos v), 0\rangle \cdot \langle u\cos v, u\sin v, -u\rangle\, dudv = \int_0^{2\pi}\int_0^1 (1-2(u\cos v))(u\sin v)dudv = 0$. Both integrals evaluate to zero. Alternatively, you can use Stokes theorem in another way by realizing that the surface $S_2 : \vec{r}(u,v) = \langle u\cos v, u\sin v, 1\rangle$ for $0 \le u \le 1, 0 \le v \le 2\pi$ has the same boundary (use the flat top of the cone at height 1 instead of the sides of the cone). The normal vector we need to match the orientation on C is $\langle 0,0,-1\rangle$. Hence Stokes theorem gives (using a different S with the same boundary) $\int_{2\pi}^0 \left\langle 1, \sin^2 t, \cos^2 t\right\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt = \int_0^{2\pi}\int_0^1 \langle 0, 1-2(u\cos v), 0\rangle \cdot \langle 0,0,-1\rangle\, dudv = \int_0^{2\pi}\int_0^1 0dudv = 0$. This last integral is by far the simplest to compute, as it is the integral of zero. One of the benefits of Stokes's Theorem is that you can pick a new surface to simplify a problem.

11.4.3 Why Stokes's Theorem Works

We now discuss why this theorem works. Let $\vec{F}(x,y,z) = \langle M,N,P\rangle$ be a vector field in space. Let C be a simple closed space curve, which is the boundary of a surface S with orientation $\vec{n}$. Orient C compatibly with $\vec{n}$ by the right hand rule. The circulation of $\vec{F}$ along C is given by $\int_C \vec{F}\cdot d\vec{r}$. As with all integral problems, we show that in order to find total circulation, we add up little bits of circulation. We will use an "adding zero" trick. Breaking the surface S into little parallelogramish pieces with boundary C_{ij} oriented counterclockwise about $\vec{n}$. If a parallelogramish piece touches the curve C, then the boundary contains a small portion of C. When we add together the circulations along adjoining pieces of the surface, we calculate the flow along the common boundary twice, once with each orientation. Our "adding zero" trick gives the sum of the circulation along adjacent pieces of the surface to be zero. Hence total circulation is found by adding up little bits of circulation. If we divide and multiply by the surface area enclosed by each little piece, then each little bit of circulation is a circulation density (about $\vec{n}$) times surface area. Notationally we write

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \sum\sum \int_{C_{ij}} \vec{F}\cdot d\vec{r} = \sum\sum \frac{1}{\Delta\sigma_{ij}} \int_{C_{ij}} \vec{F}\cdot d\vec{r}\Delta\sigma_{ij} \\ &= \sum\sum \frac{\text{circulation about } \vec{n} \text{ along } C_{ij}}{\Delta\sigma_{ij}}\Delta\sigma_{ij} \approx \sum\sum \operatorname{curl}(\vec{F})\cdot\vec{n}\Delta\sigma_{ij}\end{aligned}$$

Taking limits gives us Stokes's Theorem $\int_C \vec{F}\cdot d\vec{r} = \int\int_S \operatorname{curl}(\vec{F})\cdot\vec{n}d\sigma$.

Chapter 12

Triple Integrals

After completing this chapter, you should be able to do the following:

1. Explain how to setup and compute triple integrals, as well as how to interchange the bounds of integration. Use these ideas to find area and volume.
2. Explain how to change coordinate systems in integration, with an emphasis on cylindrical, and spherical coordinates. Explain what the Jacobian of a transformation is, and how to use it.
3. Use triple integrals to find physical quantities such as center of mass, radii of gyration, etc. for solid regions.
4. Explain how to use the Divergence theorem to compute the flux of a vector fields out of a closed surface.

12.1 Preparation and Homework Suggestions

	Preparation Problems (11th ed)				Preparation Problems (12th ed)			
Day 1	15.4:3	15.4:21	15.4:24	15.5:5	15.5:3	15.5:21	15.5:24	15.6:23
Day 2	15.4:29	15.5:9	15.6:11	15.6:31	15.5:29	15.6:27	15.7:11	15.7:31
Day 3	15.6: 56	15.6:80	16.8:6	16.8:13	15.7: 56	15.7:78	16.8:6	16.8:13

The following homework problems line up with the topics we will discuss in class.

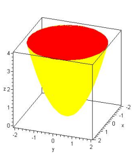
Topic	Sec	Basic Practice	Good Problems	Thy/App	Comp
Triple Integrals	15.4	1-36	41-44, 47-48	45-56	49-52
Triple Integral Applications	15.5	1-18	19, 20, 26-28	21-25	
Cylindrical and Spherical	15.6	1-10, 13, 15-30, 33-38, 43-62	11, 12, 31, 32, 39-41	86-90	
Triple Integral Applications	15.6	63-85			
Jacobian	15.7	16,17,19,20	21-24		
Divergence Theorem	16.8	1-16	17, 21-26	27, 28-32	

Topic	Sec	Basic Practice	Good Problems	Thy/App	Comp
Triple integrals	15.5	1-16, 21-32	17-20, 33-44	45-48	49-52
Triple Integral Applications	15.6	21-34		35-38	
Cylindrical and Spherical	15.7	1-10, 13, 15-30, 33-38, 43-62	11, 12, 31, 32, 39-41	83-86	
Triple Integral Applications	15.7	63-82			
Jacobian	15.8	18,19,21,22	23-26		
Divergence Theorem	16.8	1-16	17, 21-26	27, 28-32	

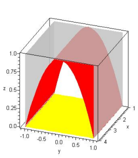
It is crucial that you do not attempt to solve every single integral. For the most part, you are learning to set up integrals in high dimensions. **I would suggest that you do at least 15 problems a day or more in chapter 15**, where you spend time setting up integrals and not solving them. Check your work against the answers I provide online, as there I just give the set up for each problem.

12.2 Triple Integrals

The triple integral of a function $f(x, y, z)$ over a solid region D in space, written $\int\int\int_D f dV$, is defined similarly as a double integral. We break a region in space up into little boxes of dimensions $\Delta x, \Delta y, \Delta z$, and then take a limit of the sum $\sum\sum\sum f\Delta x\Delta y\Delta z$ to get the triple integral. If the function is $f = 1$, then this integral gives the volume of the region D in space. Since there are 6 ways to order x, y, and z, there are 6 different iterated integrals which can be used to compute a triple integral. Sometimes one order to use is easier than another. Which one to use will come with practice. Make sure you tackle the problems in the text which ask you to write an integral in 6 different ways. You have to pick an outside, middle, and inner variable in any case. The outside variable is bounded between two constants, representing two parallel planes which bound the region D. The middle variable is bounded between two functions of the outer variable. These two middle functions represent two cylinders which bound the solid region D (not circular cylinders, but generic cylinders which are parallel to the inner variable axis). The inner variable is bounded between two functions of the outer and middle variables, and represent two surfaces which bound region D.



The region above the paraboloid $z = x^2 + y^2$ below the plane $z = 4$ has x values between -2 and 2 (the outer bounds). The y values lie in the cylinder $x^2 + y^2 = 4$ or $y = -\sqrt{4 - x^2}$ and $y = \sqrt{4 - x^2}$. The z values lie between the surfaces $z = x^2 + y^2$ and $z = 4$. So we have for our bounds the inequalities $-2 \leq x \leq 2, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, x^2 + y^2 \leq z \leq 4$ (planes, cylinders, surfaces) which can be used to find the volume of this region by the triple integral $V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} 1dzdydx$. This integral is rather complicated to solve by hand as written, but using an appropriate high dimensional change of coordinates, this integral becomes rather trivial. Spend most of your time learning to set up integrals rather than solving them.



The region D below the cylinder $z = 1 - y^2$ above the xy plane and between the planes $x = 1, x = 4$ can be found as follows. If we choose x, y, z, as our outer, middle, inner bounds, then $1 \leq x \leq 4, -1 \leq y \leq 1, 0 \leq z \leq 1 - y^2$. If we choose instead the order z, x, y, then we have $0 \leq z \leq 1, 1 \leq x \leq 4, -\sqrt{1 - z} \leq y \leq \sqrt{1 - z}$ (just solve for y in the equation $z = 1 - y^2$ to find the bounds for y). Hence the volume of this region can be written two ways as $V = \int_1^4\int_{-1}^{1}\int_0^{1-y^2} 1dzdydx =$

$\int_0^1 \int_{-1}^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} 1dydxdz$.

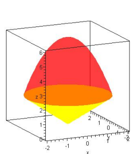
We currently have two ways of finding volume, either $\int\int_R fdA$ or $\int\int\int_D 1dV$. If we use the differential notation $dA = fdx = dxdy$ and $dV = fdA = fdxdy = dxdydz$, then we can use this notation to remind us of how to find area and volume in any manner. The formula $dA = fdx$ reminds us that little pieces of area can be found by taking a height times a little change in x and using a single integral. The formula $dA = dxdy$ reminds us that area can also be found using a double integral as area is little changes in width times little changes in length. Similarly $dV = fdA = fdxdy$ reminds us that volume can be found using a double integral, or $dV = dxdydz$ says that volume can be found using a triple integral. The shell method $dV = 2\pi xf(x)dx$ and disk method $dV = \pi(f(x))^2dx$ can also be summarized in differential notation, and since there is only dx you only use a single integral. If you can remember how to compute a little piece of volume dV using any of these formulas, then total volume is found by adding up all the little pieces. The number of integrals used depends on the number of differentials dx, dy, dz. If you use a single integral to represent one, two, or three integrals, then you can write area as "add up little bits of area" $A = \int_R dA$, and volume as "add up little bits of volume" $V = \int_D dV$, where the number of integrals is determined by the differential formula you choose for dA or dV.

12.3 Changing Coordinate Systems - Generalized u-substitution

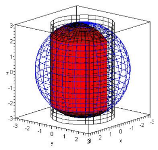
Just as we did with polar coordinates in 2, we can compute a Jacobian for any change of coordinates in 3D. We will focus on cylindrical and spherical coordinate systems. Remember that the Jacobian of a transformation is the absolute value of the determinant of the derivative of the transformation.

12.3.1 Cylindrical Coordinates

Cylindrical coordinates are defined by $x = r\cos\theta, y = r\sin\theta, z = z$, which we can represent using function notation as $T(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$. The derivative of this transformation is $DT(r,\theta,z) = \begin{bmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$. The Jacobian of the transformation is $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = |\det(DT(r,\theta,z))| = |r|$. Provided that $r \geq 0$, we have the differential formula $dV = rdrd\theta dz$, or in integral form the equation $\int\int\int_{D_{xyz}} f(x,y,z)dxdydz = \int\int\int_{D_{r\theta z}} f(T(r,\theta,z))\frac{\partial(x,y,z)}{\partial(r,\theta,z)}drd\theta dz = \int\int\int_{D_{r\theta z}} f(T(r,\theta,z))(r)drd\theta dz$. In summary, converting to cylindrical coordinates requires that we multiply by r. Cylindrical coordinates are extremely useful for problems which involve cylinders, paraboloids, and cones.



We now find the volume of the region D in space which is above the cone $z = \sqrt{x^2+y^2}$ and below the paraboloid $z = 6 - x^2 - y^2$. Switching to cylindrical coordinates, these equations become $z = r$ for the cone and $z = 6 - r^2$ for the paraboloid. The surfaces intersect when $r = 6 - r^2$, or $r^2 + r - 6 = (r+3)(r-2) = 0$, or $r = 2, -3$. Since $z > 0$, we know that the intersection we seek is at $r = 2$. So the region D in space is the set of points above $z = r$, below $z = 6 - r^2$, and inside the cylinder $r = 2$. We can describe this region using the inequalities $0 \leq \theta \leq 2\pi, 0 \leq r \leq 2, r \leq z \leq 6 - r^2$. This gives us the volume as $V = \int_0^{2\pi} \int_0^2 \int_r^{6-r^2} (r)dzdrd\theta$.

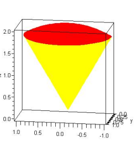


The integral $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} dzdydx$ represents the volume of the region in space satisfying the inequalities $0 \leq x \leq 2, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}, -\sqrt{9-x^2-y^2} \leq z \leq \sqrt{9-x^2-y^2}$, or the region in space with positive x values, inside the cylinder $x^2+y^2=4$, and inside the sphere $x^2+y^2+z^2=9$. Since this problem involves a cylinder, we will write the integral in cylindrical coordinates. The region is described using $-\pi/2 \leq \theta \leq \pi/2, 0 \leq r \leq 2, -\sqrt{9-r^2} \leq z \leq \sqrt{9-r^2}$, which means that we can rewrite the integral as $V = \int_{-\pi/2}^{\pi/2} \int_0^2 \int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} (r) dzdrd\theta$. The volume of the region outside the cylinder of radius 2, inside the sphere of radius 3, with positive x values is given by either $V = \int_{-\pi/2}^{\pi/2} \int_2^3 \int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} (r) dzdrd\theta$ or $V = \int_{-\pi/2}^{\pi/2} \int_{-\sqrt{5}}^{\sqrt{5}} \int_2^{\sqrt{9-z^2}} (r) drdzd\theta$ (since the cylinder intersects the sphere at a height of $\sqrt{5}$).

12.3.2 Spherical Coordinates

Spherical coordinates are defined by $T(\rho,\theta,\phi) = \langle \rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi \rangle$. The Jacobian of this transformation is $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = |\det(DT(\rho,\theta,\phi)| = |-\rho^2\sin\phi|$ (verify this as homework). Using differential notation, we write $dV = dxdydz = |\rho^2\sin\phi| d\rho d\theta d\phi$, so that whenever you convert to spherical coordinates from rectangular, you have to multiply by $|\rho^2\sin\phi|$ inside the integral, or just $\rho^2\sin\phi$ if you require $0 \leq \phi \leq \pi$. Problems which involve cones and spheres often have simple integrals in spherical coordinates.

The volume inside a sphere of radius a is $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dzdydx$. This integral is rather difficult to solve by hand (if it is even possible). However the region inside the sphere is describe in spherical coordinates as $0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi, 0 \leq \rho \leq a$. Hence the volume is found using $V = \int_0^\pi \int_0^{2\pi} \int_0^a \rho^2 \sin\phi d\rho d\theta d\phi$. This integral can be evaluated using elementary techniques and gives the volume $V = \frac{4}{3}\pi a^3$.



The integral $\int_0^\pi \int_0^1 \int_{\sqrt{3}r}^{\sqrt{4-r^2}} rdzdrd\theta$ represents the region in space with $y \geq 0, r \leq 1$, above the cone $z = \sqrt{3}r$, and below the sphere $z^2 = 4 - r^2$. The cone and sphere intersect precisely when $r = 1$. In rectangular coordinates, the integral is $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{3(x^2+y^2)}}^{\sqrt{4-x^2-y^2}} dzdydx$. In spherical coordinates, the cone is given by the equation $\phi = \pi/6$ (draw a right triangle with hypotenuse 2 and one side length 1), and the sphere is given by the equation $\rho = 2$. The integral becomes $\int_0^\pi \int_0^{\pi/6} \int_0^2 \rho^2 \sin\phi d\rho d\phi d\theta$.

12.4 Physical Applications

Integrals are used in a wide variety of applications. Notice that in all of the applications we have done this semester, a formula which works for double integrals will also work for single, line, surface, and triple integrals. The differentials $dx, ds, dA, d\sigma, dV$ remind us which dimension we are working in. For lack of a better notation, I will use $d\square$ to represent any of these differentials when you are free to pick the needed differential. The line integrals section contains the details for each of the physical quantities we compute.

12.4.1 The Short Version

Everything we learned about physical applications this semester can be summarized in the following table list, where recall that $d\square$ stands for any of $dx, ds, dA, d\sigma, dV$.

- Area: $dA = f dx, f ds, dxdy, rdrd\theta, \frac{\partial(x,y)}{\partial(u,v)} dudv,$
- Volume: $dV = f dA, dxdydz, rdrd\theta dz, \rho^2 \sin\phi d\rho d\theta d\phi, \frac{\partial(x,y,z)}{\partial(u,v,w)} dudvdw,$
- Surface Area: $d\sigma = |\vec{r}_u \times \vec{r}_v| dudv = \sqrt{f_x^2 + f_y^2 + 1} dxdy$
- Average Value: $\bar{f}\square = \int f d\square$, and $dm = \delta d\square, \delta dx, \delta ds, \delta dA, \delta d\sigma, \delta dV$
- Center of Mass: $(\bar{x}, \bar{y}, \bar{z})m = \int (x,y,z) dm$, where $\int \bar{x} dm = \int x dm$ is a first moment of mass
- Moment of Inertia: $I = \int rad^2 dm$ is a moment of inertia, and $\int R^2 dm = \int r^2 dm$ gives $R^2 m = I$ or $R = \sqrt{I/m}$. Recall that $rad^2 = y^2 + z^2$ for I_x, $rad^2 = x^2 + z^2$ for I_y, and $rad^2 = x^2 + y^2$ for I_z.
- Flux: $d\text{Flux} = \vec{F} \cdot \vec{n} d\sigma$ or $\vec{F} \cdot \vec{n} ds$ where $\vec{n}$ is a normal vector to either a surface S or a curve C. Also Green's theorem and the Divergence theorem are $d\text{Flux} = (M_x + N_y)dA = (M_x + N_y + P_z)dV = \text{div}(\vec{F})dV = \nabla \cdot \vec{F} dV$, where $\nabla \cdot \vec{F}$ is flux density.
- Circulation: $d\text{Circ} = \vec{F} \cdot \vec{T} ds = M dx + N dy$. Green's theorem and Stokes's Theorem are $d\text{Circ} = (N_x - M_y)dA = \text{curl}(\vec{F}) \cdot \vec{n} d\sigma = \text{curl}(\vec{F}) \cdot |\vec{r}_u \times \vec{r}_v| dudv$, where $\text{curl}(\vec{F}) = \nabla \times \vec{F}$, and $\nabla \times \vec{F} \cdot \vec{n}$ is circulation density about $\vec{n}$.

I strongly suggest that as you do each problem, start from a formula in differential notation, and then modify it so that you have the right number of integrals. If you do this, you will learn how to do calculus in all dimensions with ease.

12.4.2 Examples

Consider the region in space inside the cylinder $r = 4$ bounded above by the cone $z = r$ and below by the xy-plane. Using cylindrical coordinates, we have $dV = rdzdrd\theta$. The volume is $\int_0^{2\pi} \int_0^4 \int_0^r rdzdrd\theta$. If the density is $\delta(x,y,z) = x^2 + z$, then we have

$$\bar{y} = \frac{\int y dm}{\int dm} = \frac{\int\int\int_D y\delta dV}{\int\int\int_D \delta dV} = \frac{\int_0^{2\pi}\int_0^4\int_0^r r\sin\theta((r\cos\theta)^2 + z) rdzdrd\theta}{\int_0^{2\pi}\int_0^4\int_0^r ((r\cos\theta)^2 + z) rdzdrd\theta}$$

$$R_z = \sqrt{\frac{\int r^2 dm}{\int dm}} = \sqrt{\frac{\int x^2 + y^2 dm}{\int dm}} = \sqrt{\frac{\int_0^{2\pi}\int_0^4\int_0^r ((r\cos\theta)^2 + (r\sin\theta)^2)((r\cos\theta)^2 + z) rdzdrd\theta}{\int_0^{2\pi}\int_0^4\int_0^r ((r\cos\theta)^2 + z) rdzdrd\theta}}$$

The temperature at each point in space of a solid covering the region D (the upper portion of the sphere of radius 4 centered at the origin) is given by $T(x,y,z) = \sin(xy + z)$. The volume is $V = \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} dzdydx$ and the average temperature is $AV = \frac{1}{V}\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} \sin(xy + z)dzdydx$.

12.5 Differentials

Differentials can be used to summarize most of the work we do in integration. In this unit we have discussed the area differential

$$dA = fdx = dxdy = rdrd\theta = \frac{\partial(x,y)}{\partial(u,v)}dudv$$

and volume differential

$$dV = fdA = dxdydz = rdrd\theta dz = \rho^2 \sin\phi d\rho d\theta d\phi = \frac{\partial(x,y,z)}{\partial(u,v,w)}dudvdw.$$

Differentials can be thought of as a little piece of whatever they represent. To find a total, we add up little pieces. To find total area we add up little pieces of area $A = \int_R dA$. To find total volume we add up little pieces of volume $V = \int_D dV$. In any case our choice for dA or dV will tell us whether we use a single, double, or triple integral to do the calculations.

12.6 Flux Density - Divergence $\nabla \cdot \vec{F}$

For a vector field $\vec{F}(x,y) = \langle M, N\rangle$, we define the flux density, or divergence of $\vec{F}$, as follows. Recall that flux of $\vec{F}$ across C is $\oint_C \vec{F}\cdot\vec{n}ds = \oint_C Mdy - Ndx$ where $\vec{n}$ is the unit outward pointing normal vector and C is oriented counter clockwise. At the point (x,y) in the plane, create a circle C_a of radius a centered at (x,y), where the area inside of C_a is given by A_a. The quotient $\frac{1}{A_a}\oint_{C_a}\vec{F}\cdot\vec{n}ds$ is a flux per area. The limit $\lim_{a\to 0}\frac{1}{A_a}\oint_{C_a}\vec{F}\cdot\vec{n}ds$ is called the flux density of $\vec{F}$ at (x,y), or divergence of $\vec{F}$. It can be shown that $\lim_{a\to 0}\frac{1}{A_a}\oint_{C_a}\vec{F}\cdot\vec{n}ds = M_x + N_y$, and we write div$\vec{F} = M_x + N_y = \nabla\cdot\vec{F}$. Just as with circulation density, it can also be shown that C_a can be replaced with a square of side length a centered at (x,y) with interior area A_a, or any collection of curves C_a which "shrink nicely" to (x,y), and you still obtain the divergence (flux per unit area) as $M_x + N_y$. The notation div$\vec{F} = \nabla\cdot\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\rangle \cdot \langle M, N\rangle = \frac{\partial}{\partial x}M + \frac{\partial}{\partial y}N$ gives a convenient way to remember divergence.

In 3D, the flux of $\vec{F}$ across S, $\int\int_S \vec{F}\cdot\vec{n}d\sigma$, is a measure of flow across S where $\vec{n}$ is a continuous unit normal vector to S. Flux density at (x,y,z) is found by creating a sphere S_a of radius a centered at (x,y,z) with interior volume V_a and outward normal vector $\vec{n}$, and considering the quotient flux per volume $\frac{1}{V_a}\int\int_{S_a}\vec{F}\cdot\vec{n}d\sigma$. The limit $\lim_{a\to 0}\frac{1}{V_a}\int\int_{S_a}\vec{F}\cdot\vec{n}d\sigma$ is the divergence of $\vec{F}$ at (x,y,z) and equals div$\vec{F}(x,y,z) = \nabla\cdot\vec{F} = M_x + N_y + P_z$.

As an example, we compute the flux density at $(0,0,0)$ for the vector field $\vec{F} = \langle x,y,z\rangle$. A sphere of radius a has unit normal vector $\vec{n} = \frac{\langle x,y,z\rangle}{|\langle x,y,z\rangle|}$, so the flux is $\int\int_S \langle x,y,z\rangle\cdot\frac{\langle x,y,z\rangle}{|\langle x,y,z\rangle|}d\sigma = \int\int_S \sqrt{x^2+y^2+z^2}d\sigma = a\int\int_S d\sigma = a4\pi a^2$, since the surface area of a sphere is $4\pi a^2$. The volume inside a sphere of radius a is $\frac{4}{3}\pi a^3$. Hence $\lim_{a\to 0}\frac{3}{4\pi a^3}4\pi a^3 = 3$, which equals div$\vec{F} = M_x + N_y + P_z = 1 + 1 + 1 = 3$.

12.6.1 The Divergence Theorem

The divergence theorem states that flux in space can be computed by instead adding up little bits of flux, which are found by multiplying a flux density by volume. This gives the Divergence theorem

$$\int\int_S \vec{F}\cdot\vec{n}d\sigma = \int\int\int_D \nabla\cdot\vec{F}dV = \int\int\int_D (M_x + N_y + P_z)\,dV$$

for S a closed surface with interior D and outward normal $\vec{n}$.

Let S be the surface of the rectangular box in the first quadrant bounded by the plane $x = 0, x = 1, y = 0, y = 2, z = 0, z = 3$ with outward normal $\vec{n}$. Let $\vec{F}$ be the vector field $\left\langle x+3y^2, y^2-4x, 2z+xy \right\rangle$, hence the divergence is $M_x + N_y + P_z = 1 + 2y + 2 = 3 + 2y$. The flux of $\vec{F}$ across the surface S would require 6 integrals. The divergence theorem gives $\int\int_S \vec{F} \cdot \vec{n} d\sigma = \int\int\int_D 3 + 3y dV = 3V + 3\bar{y}V = 3 \cdot 6 + 3 \cdot 1 \cdot 6 = 36$.

Now consider the vector field $\vec{F} = \langle yz, -xz, 3xz \rangle$. Let D be the solid region in space inside the cylinder of radius 4, above the plane $z = 0$, and below the paraboloid $z = x^2+y^2$. The surface S consists of 3 portions. The flux of $\vec{F}$ across S is found using the divergence theorem as $\int\int\int_D \nabla \cdot \vec{F} dV = \int\int\int_D -3x dV = -3\bar{x}V = 3 \cdot 0 \cdot V = 0$. If $\bar{x} \neq 0$, then the bounds of integration could have been $0 \leq \theta \leq 2\pi, 0 \leq r \leq 4, 0 \leq z \leq r^2$, and $dV = r dz dr d\theta$.

We now discuss why the Divergence theorem is true. Let $\vec{F}(x, y, z) = \langle M, N, P \rangle$ be a vector field in space. Let S be a simple closed smooth surface (oriented with the outward unit normal vector $\vec{n}$), whose interior is the solid region D. The flux of $\vec{F}$ across S is given by $\int\int_S \vec{F} \cdot \vec{n} d\sigma$. The idea is that in order to find total flux, we add up little bits of flux. Break the region D into little cubical pieces with boundary S_{ijk}, each oriented using the outward normal. If the cubical region touches the surface S, then the boundary contains a small portion of S. Notice that when we add together the flux across adjoining cubes, we calculate the flux across the common boundary twice, once with each orientation. By our "adding zero" trick, the sum of the flux across adjacent cubes is the same as the flux across the exterior of the union of the cubes, as the interior sums cancel. We can calculate total flux by adding up the flux along each S_{ijk}.

$$\begin{aligned}\int\int_S \vec{F} \cdot \vec{n} d\sigma &= \sum\sum\sum \int\int_{S_{ijk}} \vec{F} \cdot \vec{n} d\sigma = \sum\sum\sum \frac{1}{\Delta V_{ijk}} \int\int_{S_{ijk}} \vec{F} \cdot \vec{n} d\sigma \Delta V_{ijk} \\ &= \sum\sum\sum \frac{\text{flux across } S_{ijk}}{\Delta V_{ijk}} \Delta V_{ijk} \approx \sum\sum\sum (M_x + N_y + P_z) \Delta V_{ijk}\end{aligned}$$

Taking limits gives us the Divergence theorem $\int\int_S \vec{F} \cdot \vec{n} d\sigma = \int\int\int_D (M_x + N_y + P_z) \, dV$.